Coloring graphs on surfaces

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Introduction

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2005 : the proof is verified by Gonthier using Coq (a formal proof management system).

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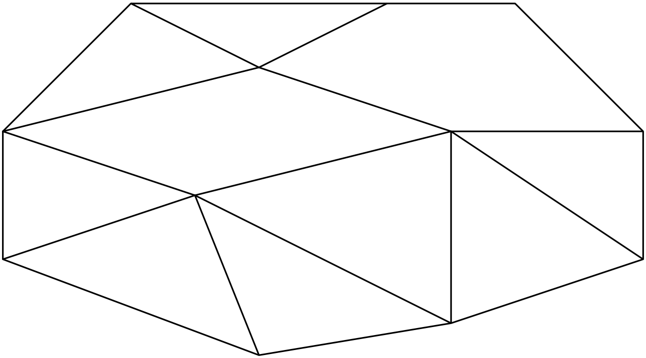

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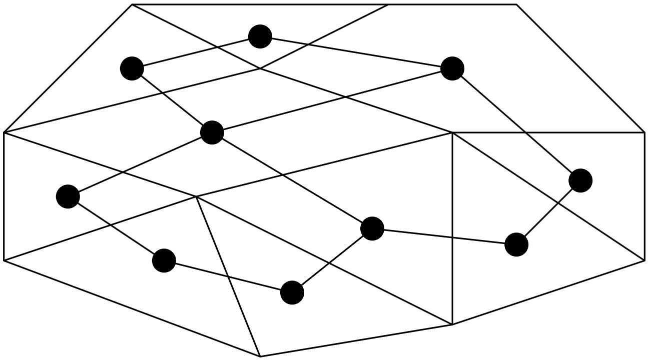

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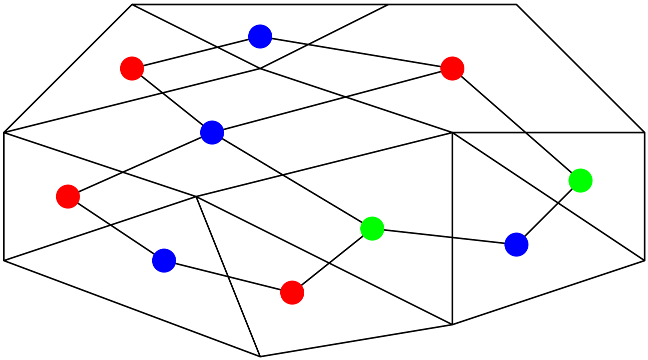

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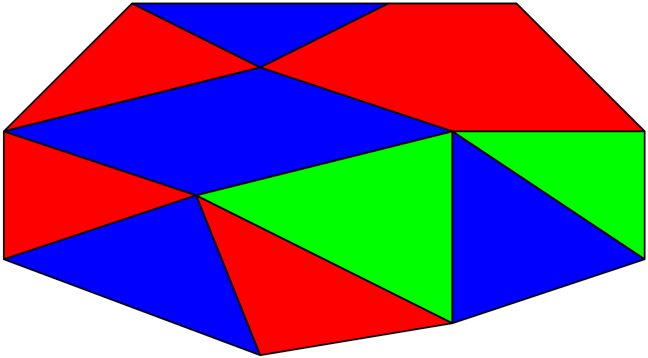

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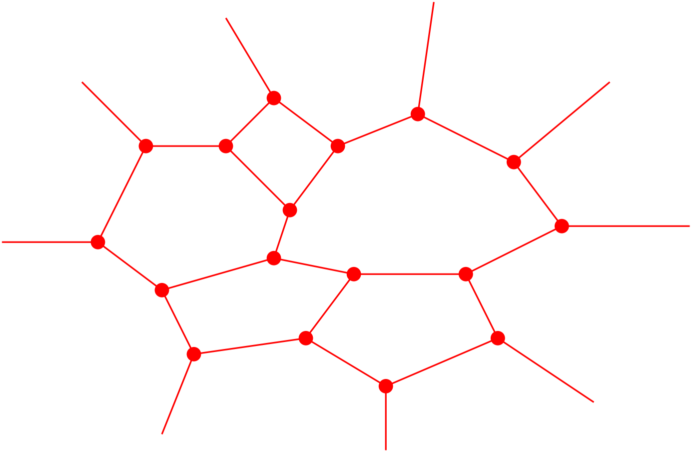
- For every planar graph G, $\chi(G) \leq 4$.

Duality



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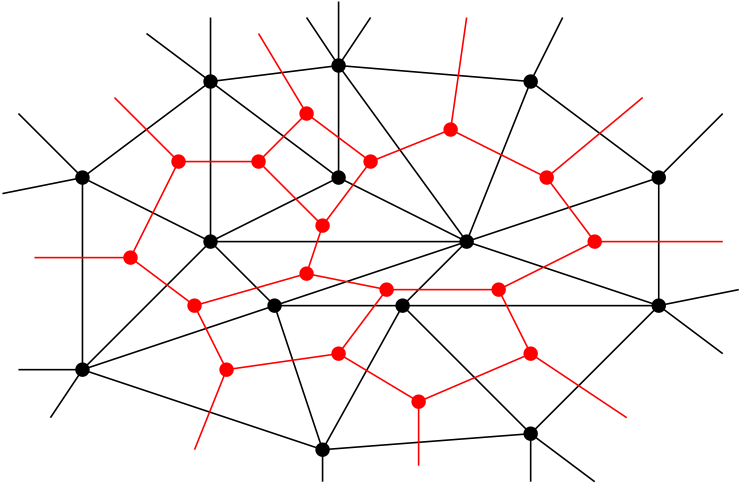

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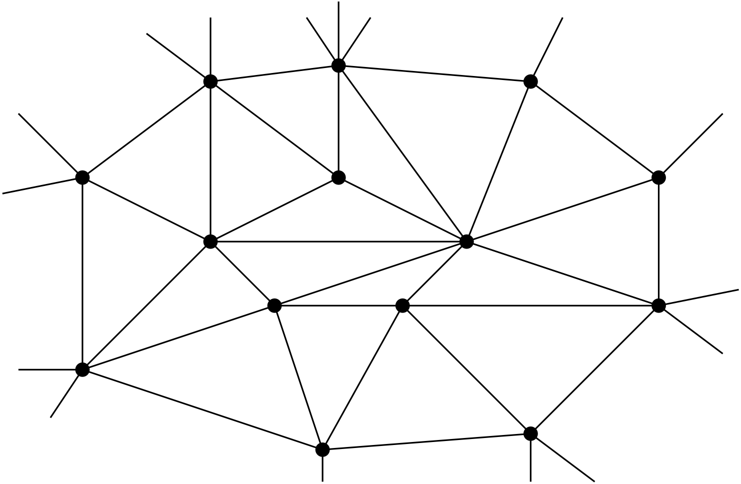

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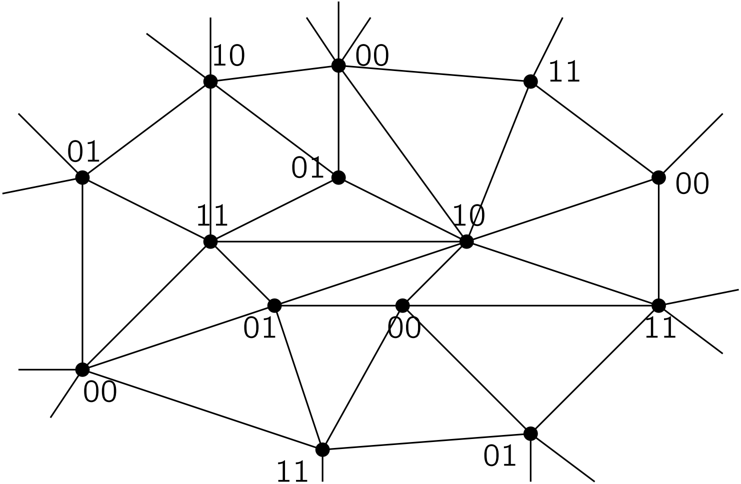

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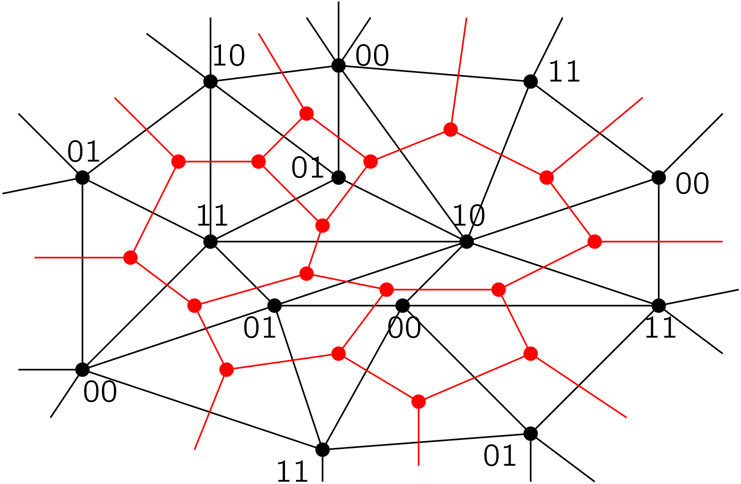

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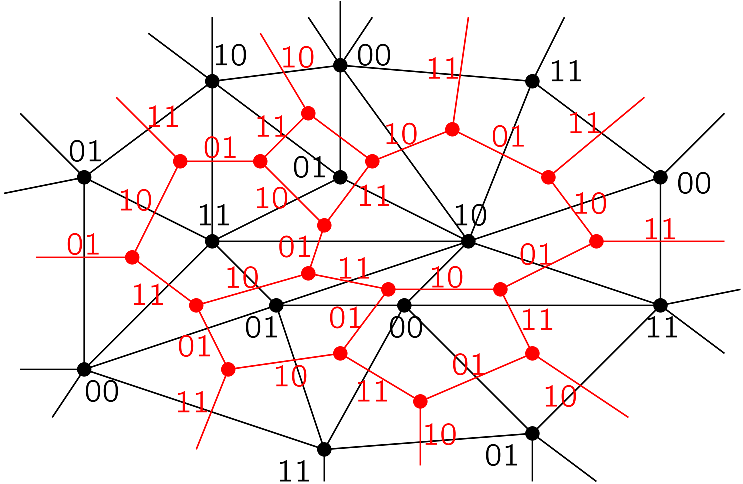
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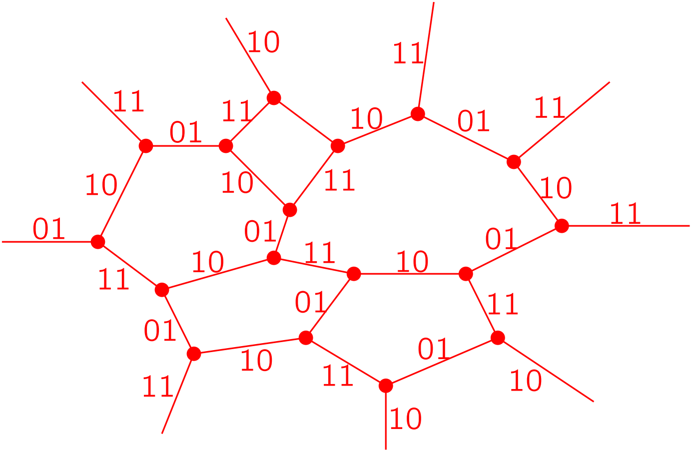
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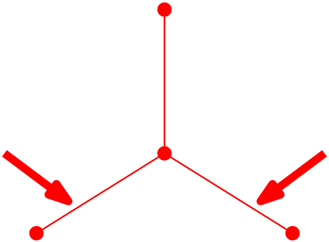

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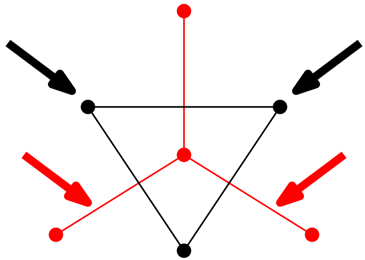

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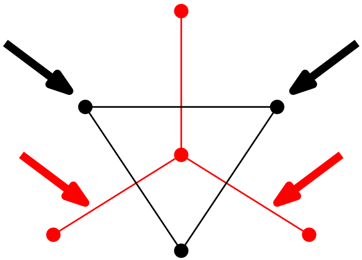

The Four color theorem

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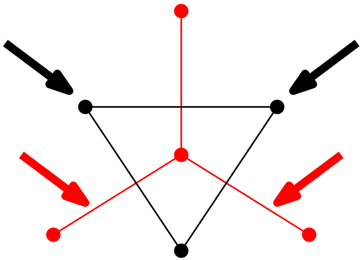

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The edges of every 2-edge-connected cubic planar graph can be colored with 3 colors (i.e. partitioned into 3 perfect matchings).

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As a consequence, any planar graph contains a vertex of degree at most 5.

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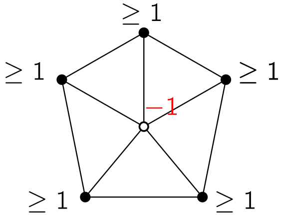

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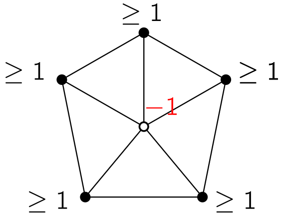

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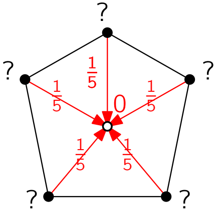

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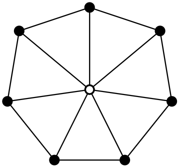

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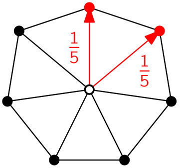

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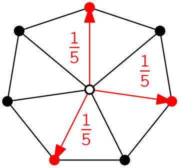

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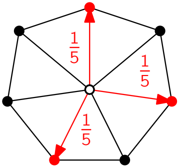

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Its final charge is at least $d - 6 - \frac{d}{2} \cdot \frac{1}{5} \geq 0$

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As a consequence, $\delta \leq \frac{1}{2}(5 + \sqrt{1 + 24g})$.

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For any surface Σ of Euler genus g, except the Klein bottle, the complete graph on $\lfloor \frac{1}{2}(7 + \sqrt{1 + 24g}) \rfloor$ can be embedded in Σ.

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For the Klein bottle, Heawood's formula gives a bound of 7, whereas it can be proved that every graph embedded on the Klein bottle has chromatic number at most 6 (and this is best possible, since K_6 vertices can be embedded in this surface).

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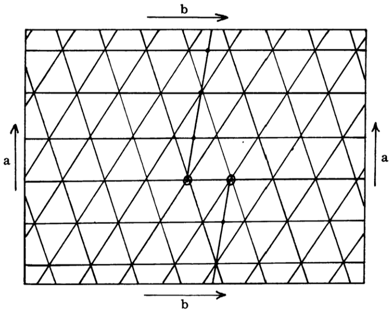

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Youngs (1996). Any non-bipartite quadrangulation G of the projective plane satisfies $\chi(G) = 4$.

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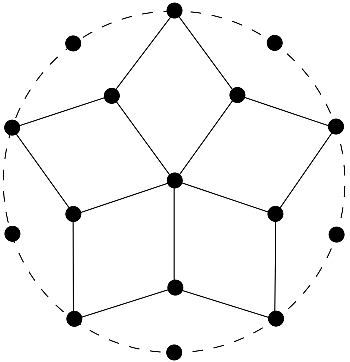

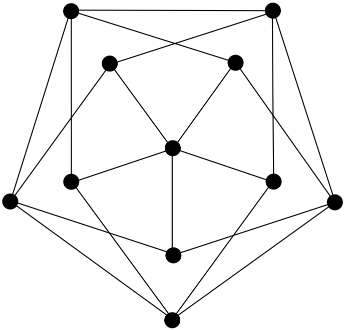

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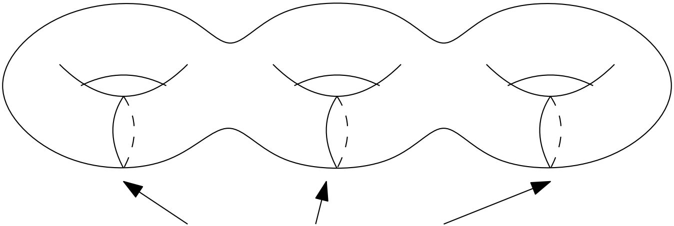

g even cycles, far apart, such that after cutting along them, the resulting graph is planar

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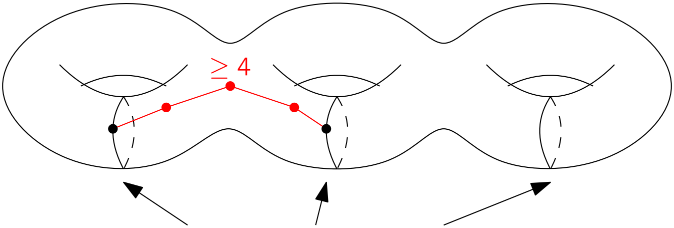

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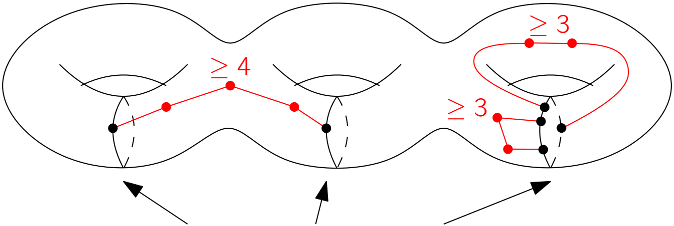

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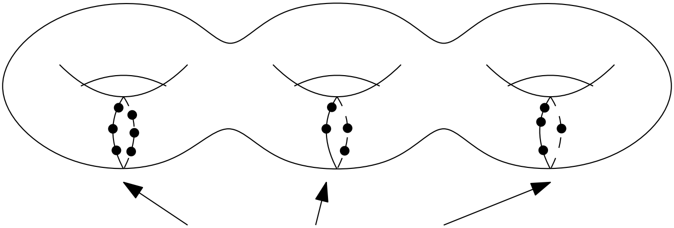

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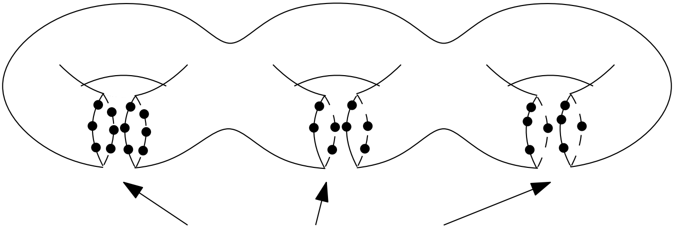

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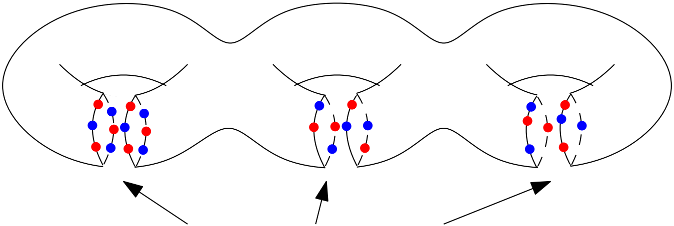

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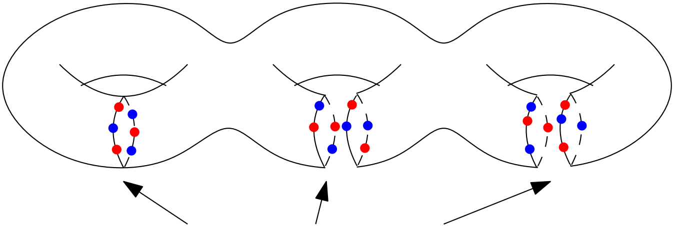

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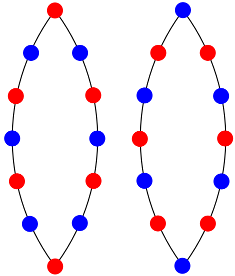

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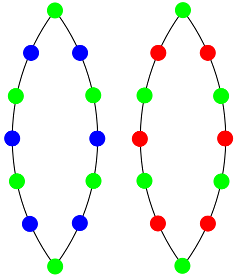

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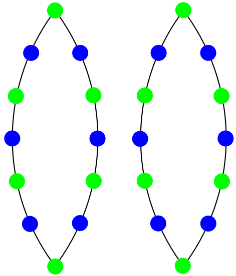

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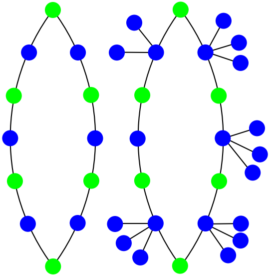

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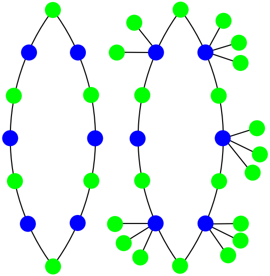

Some consequences

Lemma (Folklore)

Assume that some property $\mathcal{P}$ holds for locally planar graphs. Then there is a function f such that for any graph of Euler genus g, at most $f(g)$ vertices can be removed so that the resulting graph satisfies property $\mathcal{P}$.

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As a consequence of the result of Thomassen, in any graph embedded on some surface of bounded Euler genus, a constant number of vertices can be removed so that the resulting graph is 5-colorable.

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As a consequence of the result of Thomassen, in any graph embedded on some surface of bounded Euler genus, a constant number of vertices can be removed so that the resulting graph is 5-colorable.

Problem (Albertson 1981)

Is there a function f, such that any graph embedded on a surface of Euler genus g can be made 4-colorable by removing at most $f(g)$ vertices?

One more question

The Four color theorem

Planar graphs are 4-colorable.

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Theorem (Thomassen 1993)

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Question (Robertson 1992)

Is it true that the edges of every 2-edge-connected cubic locally planar graph can be colored with 3 colors (i.e. partitioned into 3 perfect matchings)?

Bonus: List-coloring of planar graphs

Theorem (Thomassen 1995)

If G is planar, and any vertex is given an arbitrary list of 5 colors, then G has a coloring in which each vertex receives a color from its list.

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Stronger version (for the induction)

If G is planar, and vertices have arbitrary lists of size

- 1 for two adjacent vertices of the outerface
- 3 for the other vertices of the outerface
- 5 for the remaining vertices

then G has a coloring in which each vertex receives a color from its list.