43ème École de Printemps d'Informatique Théorique (ÉPIT)

Graphes et surfaces : algorithmique, combinatoire et topologie

Speaker: Dimitrios Thilikos

CIRM, Marseille, May 09–13, 2016

Lectures:

- Monday 09/05/2016 - 11:45–12:30 (45'): **22** pages

  Complexity, Graphs, P vs. NP, NP-Hardness, Fixed-parameter tractability
- Monday 09/05/2016 - 16:00–17:30 (90'): **42** pages

  treewidth, dynamic programming
- Tuesday 10/05/2016 - 16:00–17:00 (60'): **26** pages

  Sphere-cut decompositions
- Thursday 10/05/2016 - 11:00–12:30 (90'): **31** pages

  Bidimensionality and subexponential algorithms
- Friday 10/05/2016 - 09:00–10:30 (90'): **14**+**20**=**34** pages

  Bidimensionality and Kernels + Irrelevant vertex technique

155 pages, **375** Minutes in total, 145 seconds per page (2.42 minutes per page)

Part 1, Monday 09/05/2016 - 11:45–12:30 (45')

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A set of YES-instances $\Pi \subseteq \Sigma^*$ where Σ is an alphabet, typically $\Sigma = \{0, 1\}$

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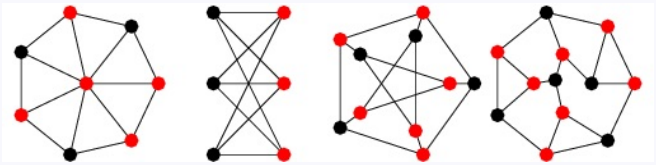

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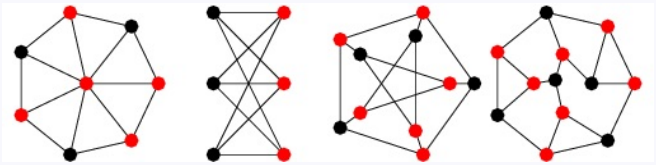

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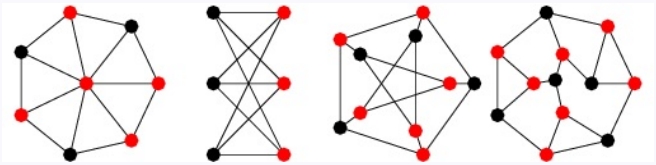

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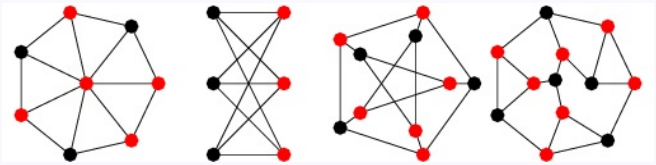

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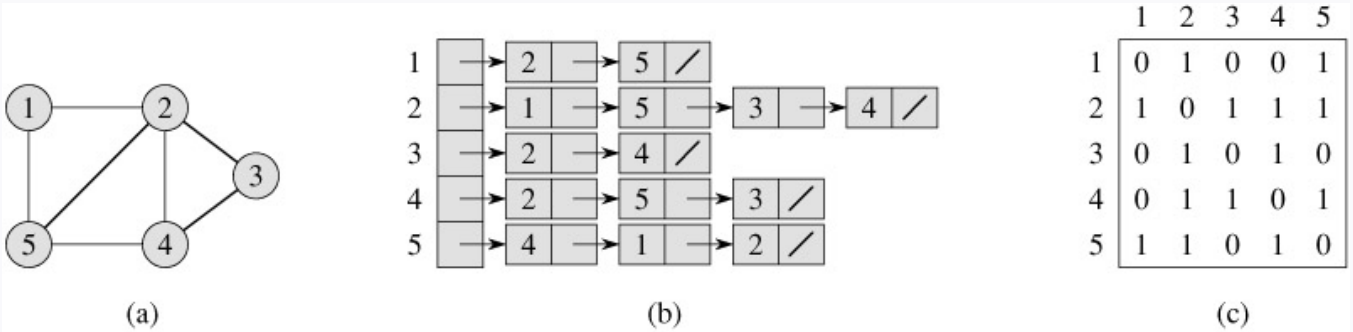
Algorithm for VERTEX COVER: a ~~procedure~~ than receives as input $x = \langle G, k \rangle$ and outputs whether $x \in \Pi_{\textbf{vc}}$

Data structures

Data structures for graphs: Adjacency list, Adjacency matrix



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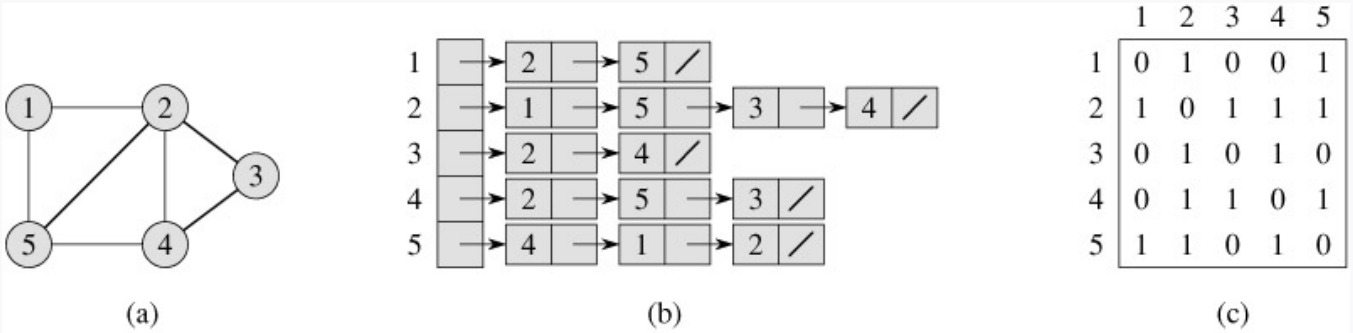

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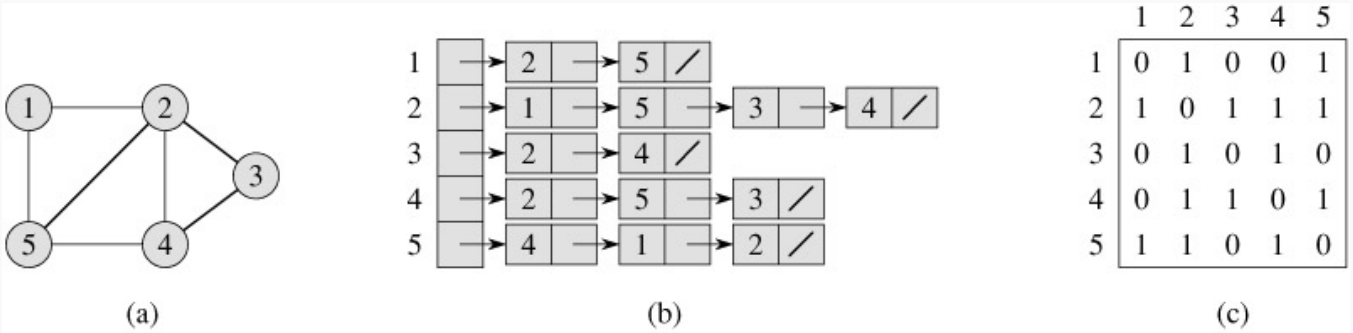

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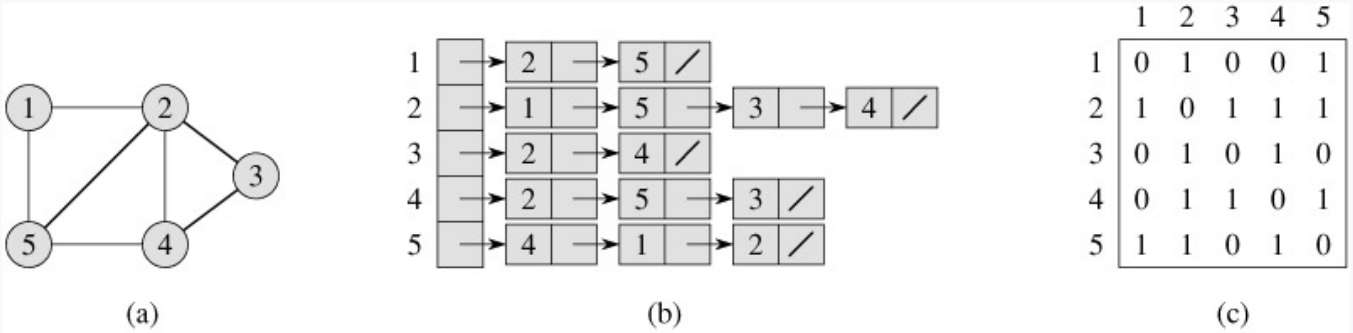

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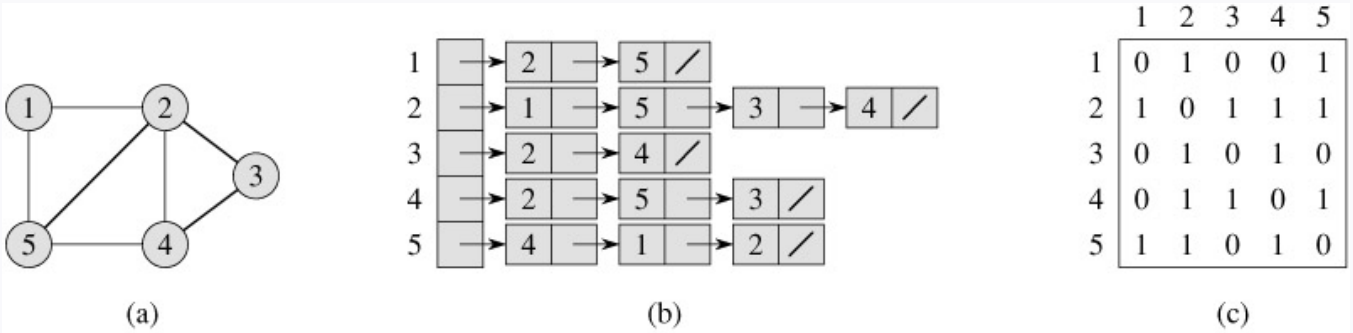

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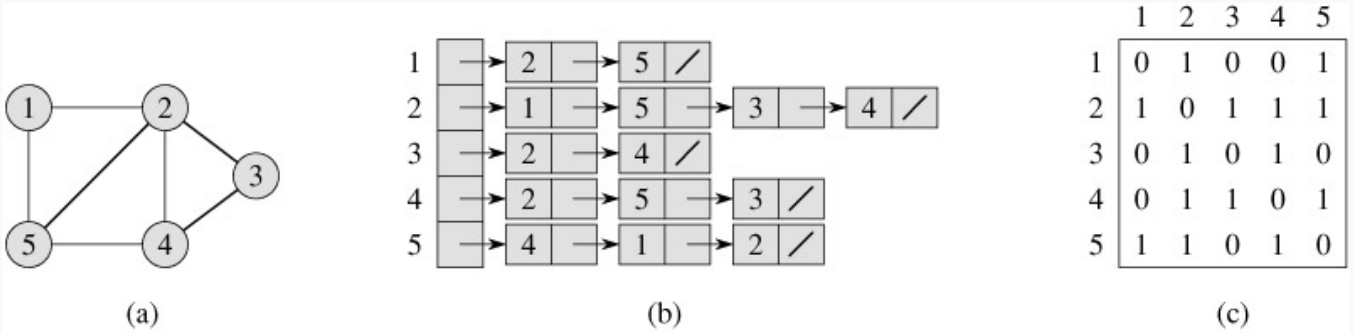

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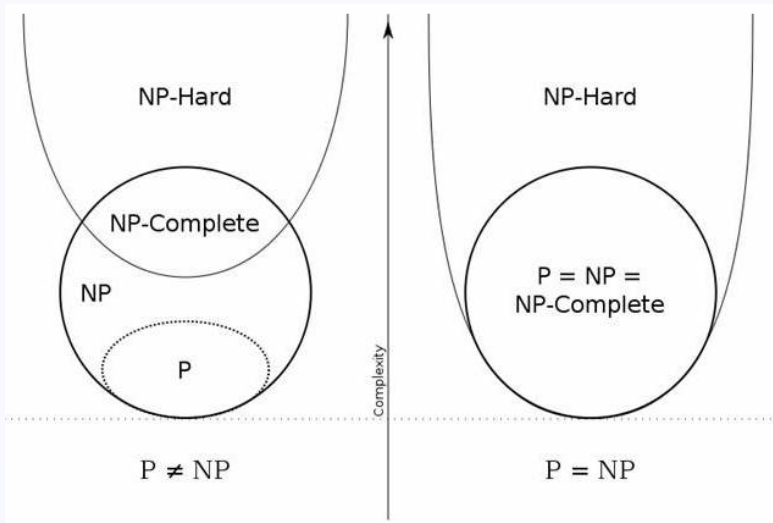

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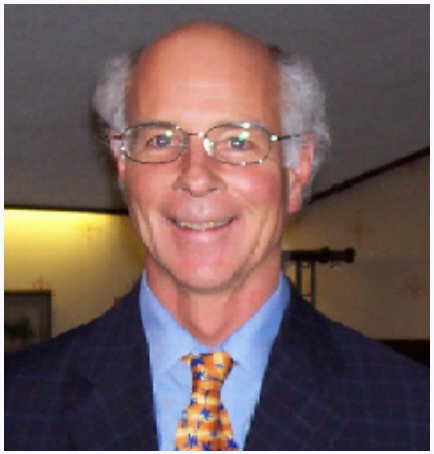
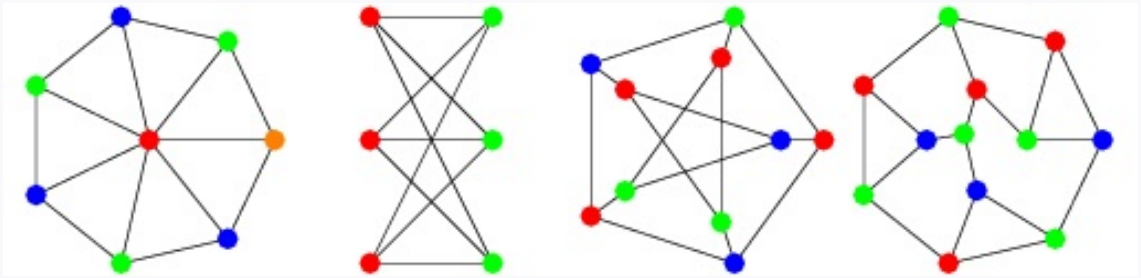


Parameterized complexity was introduced by Mike Fellows and Rod Downey proposed a way to refine the above landscape!

Three NP-complete problems



VERTEX COLORING

Instance: A graph G and an integer $k \geq 0$.

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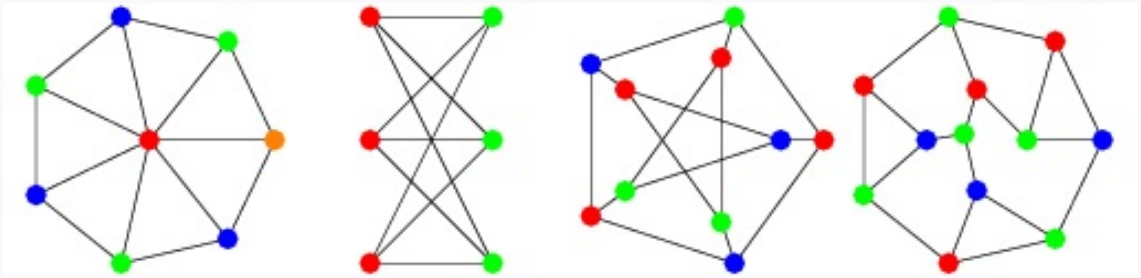

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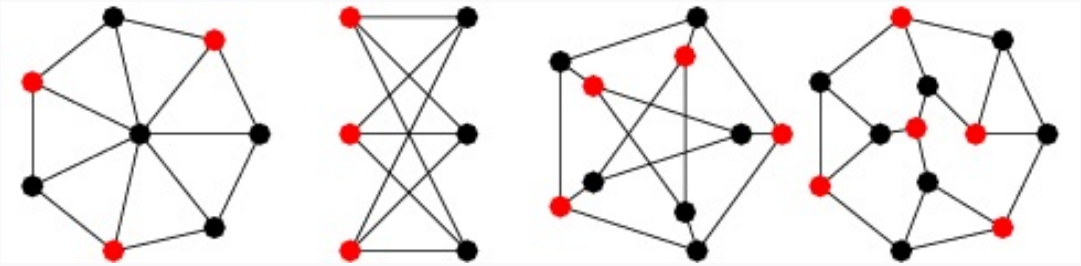

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Question: $\exists S \in V(G) : |S| \geq k \quad \wedge \quad \forall e \in E(G)\ |e \cap S| \leq 1$?

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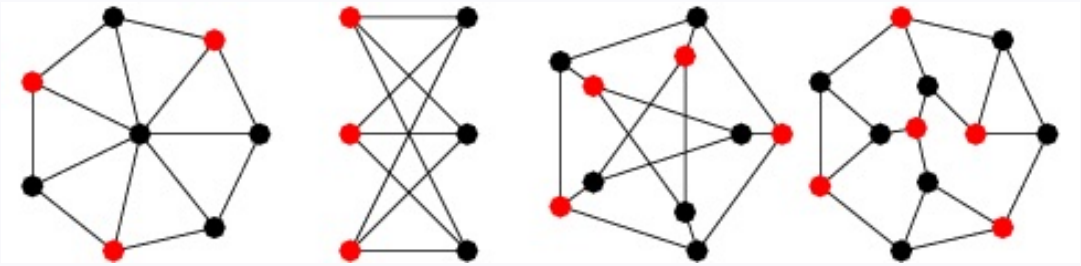

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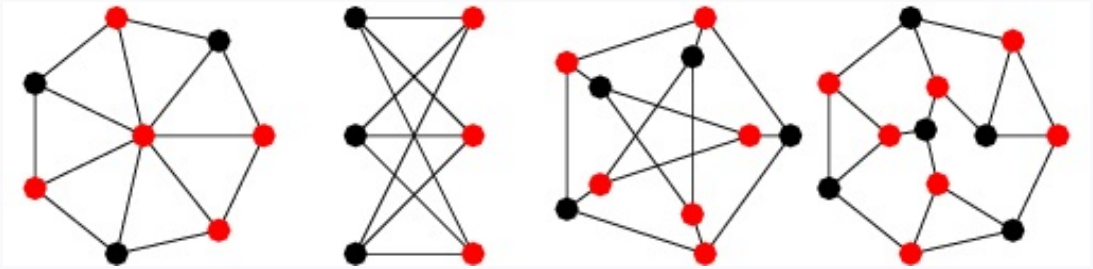

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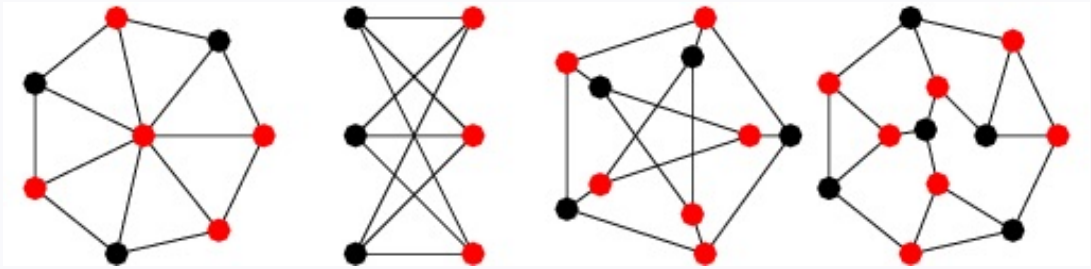

Vertex Cover

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It can easily be solved in $O(2^k \cdot n)$ steps

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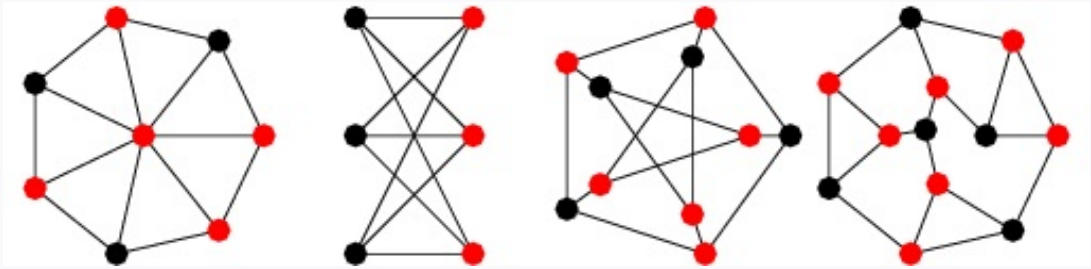

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It can be solved in $O(1.2738^k + k \cdot n)$ steps [Chen, Kanj, Xia, 2010]

Comparisons

Summary:

Vertex Coloring	$O(n^2 \cdot k^n)$	
Independent Set	$O(n^{k+1})$	
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Different Interleavings between the parameter k and the main part n of the input.

Comparison between $O(2^k \cdot n)$ and $O(n^{k+1})$

	$n = 50$	$n = 100$	$n = 150$
$k = 2$	625	2.500	5.625
$k = 3$	15.625	125.000	421.875
$k = 5$	390.625	6.250.000	31.640.623
$k = 10$	$1,9 \times 10^{12}$	$9,8 \times 10^{14}$	$3,7 \times 10^{16}$
$k = 20$	$1,8 \times 10^{26}$	$9,5 \times 10^{31}$	$2,1 \times 10^{35}$

The ratio $\frac{n^{k+1}}{2^k \cdot n}$ for several values of n and k.

How the parameters appear?

VLIf design: In VLSI chip construction, the number of circuit layers is no more than 10. While the problem is, in general, NP-complete, when we fix the number of layers, it becomes tractable.

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Computational Biology: in general, many problems in DNA chain reconstruction are intractable. In the majority of the cases, real instances have special properties (e.g., bounded treewidth or pathwidth – by 11) that facilitate the design of efficient algortihms.

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Robotics: The number of degrees of freedom in motion planning problems are not more than 10. While these problems are NP-complete in general, the become tractable taking into account this natural restriction.

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Compilers: One of the main tasks of a compiler for the language ML is the compatibility checking of type declarations of the program. It is known that the general problem is EXP-complete. However, in real cases, the implementations work well as there is an algorithm with complexity $O(2^k \cdot n)$, where n is the size of the program and k is the depth of its type declarations. As, normally, $k \leq 10$, the problem can be considered tractable.

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Examples

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We can do the same with all the problems that have some integer in their instances, such as Vertex Coloring and Vertex Cover.

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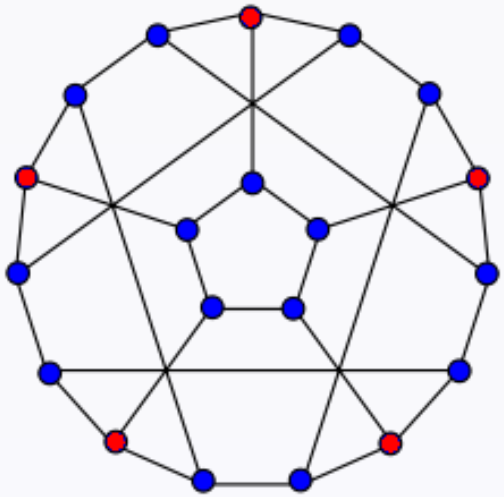
Other parameterizations of the above problems can be defined as

$\kappa(G, k) = \Delta(G)$ or

$\kappa(G, k) = \mathbf{genus}(G)$

$\kappa(G, k) = \Delta(G) + k$

Some parameterized problems



p-Dominating Set

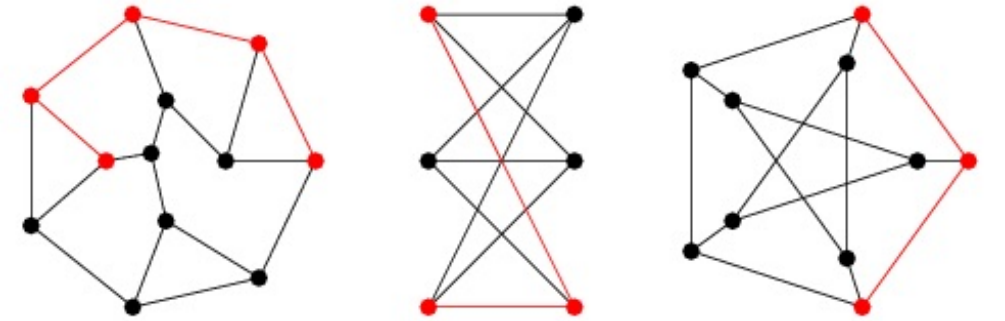
Instance: A graph G and an integer $k \geq 0$

Parameter: k,

Question:

$\exists S \in V(G) : |S| \leq k \quad \wedge \quad \forall v \in V(G) - S \; \exists u \in S \; \{v, u\} \in E(G)$?

Some parameterized problems



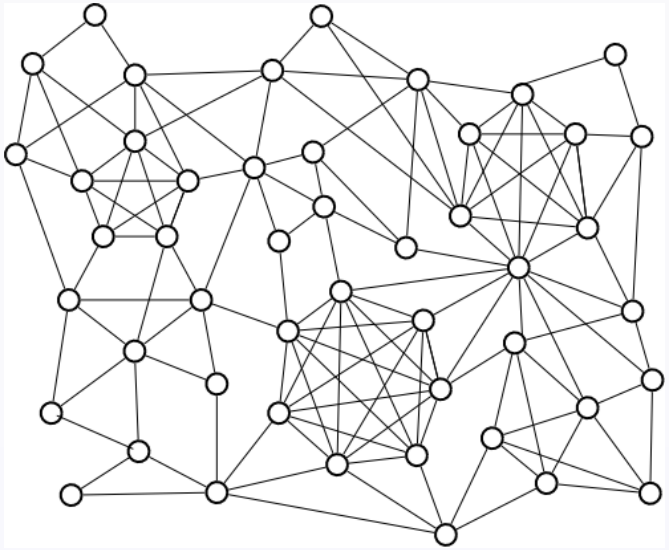
p-PATH

Instance: A graph G and an integer $k \geq 0$.

Parameter: k

Question: Does G contain a path of length k?

Some parameterized problems



p-Clique

Instance: A graph G and an integer $k \geq 0$.

Parameter: k,

Question: $\exists S \in V(G) : |S| \leq k \quad \wedge \quad \forall v, u \in S \ \{v, u\} \in E(G)$?

More parameterized problems

p-Steiner Tree

Instance: A graph G, $S \subseteq V(G)$, $k \in \mathbb{N}$.

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Question: $\exists R \in V(G) : |R| \leq k$, $R \cap S = \emptyset$, $G[S \cup R]$ is connected?

Here $\kappa(G, S, k) = k$

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- The function f is called *parameterized dependence* of the running time of the FPT-algoritm

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[Bounded Search Tree Method]

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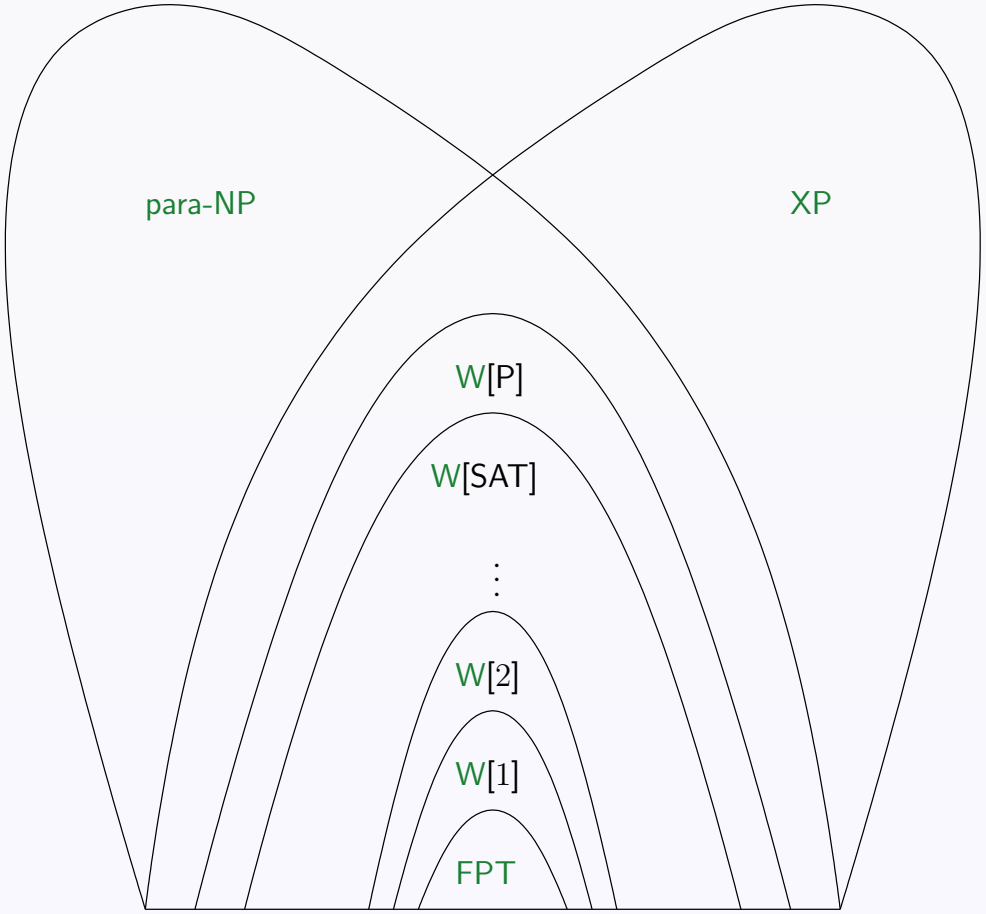
Therefore, p-Vertex Cover$\in 2^{O(k)}$-FPT.

Panorama of Parameterized complexity classes



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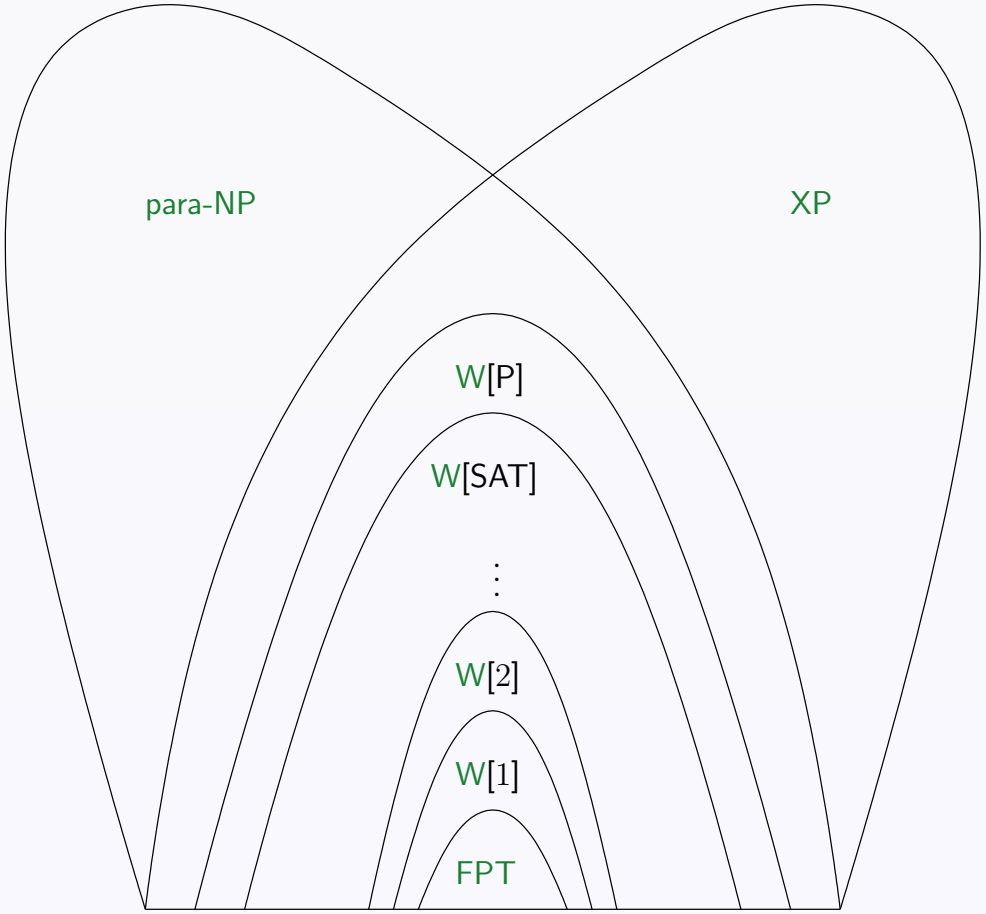

p-Vertex Cover: FPT

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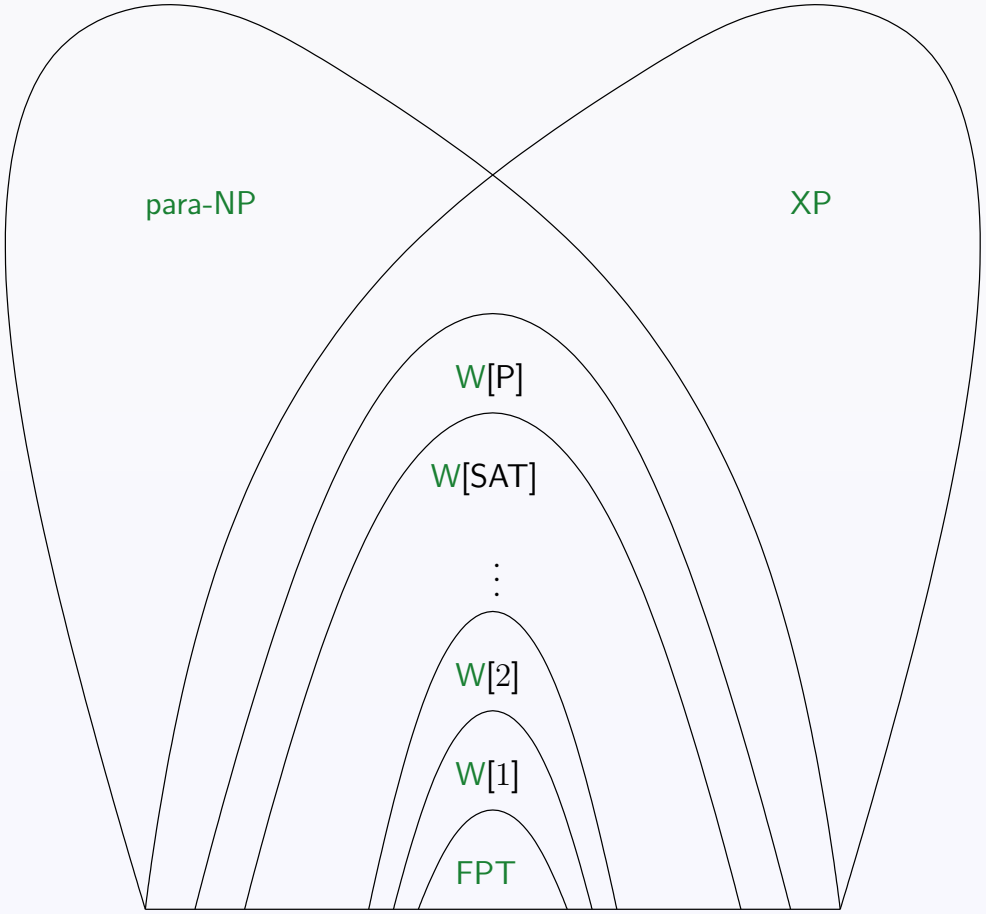

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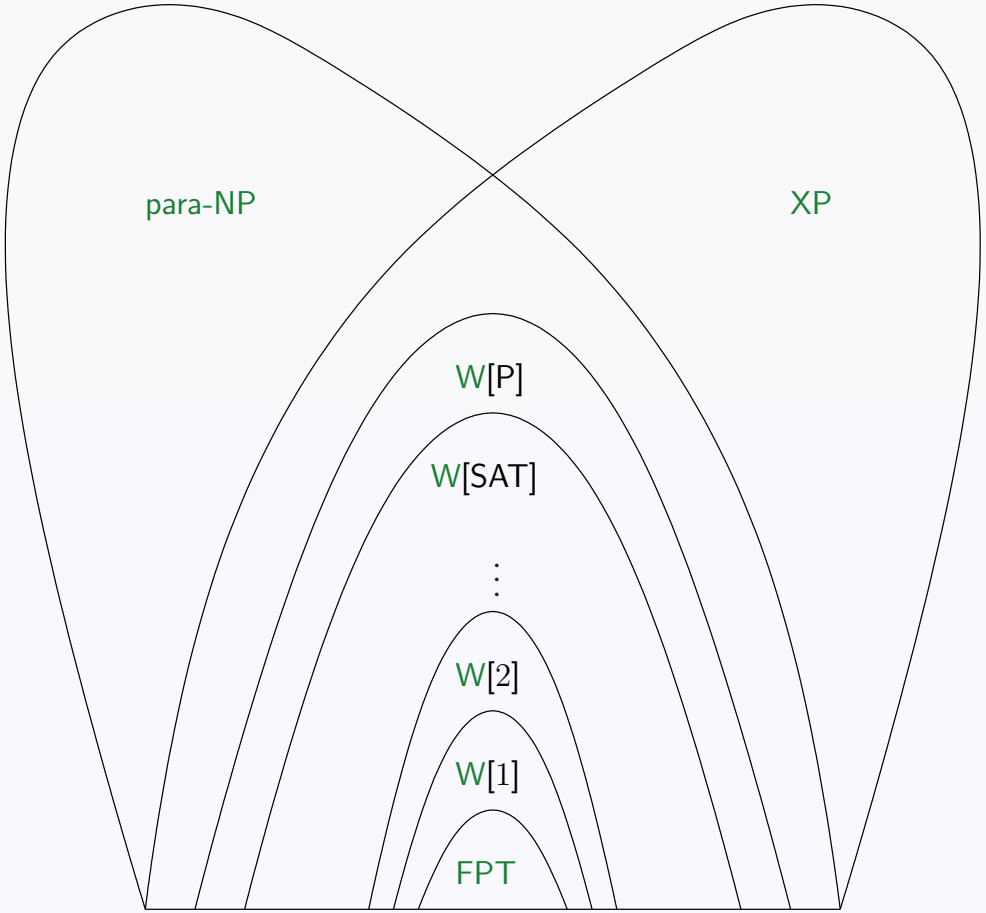

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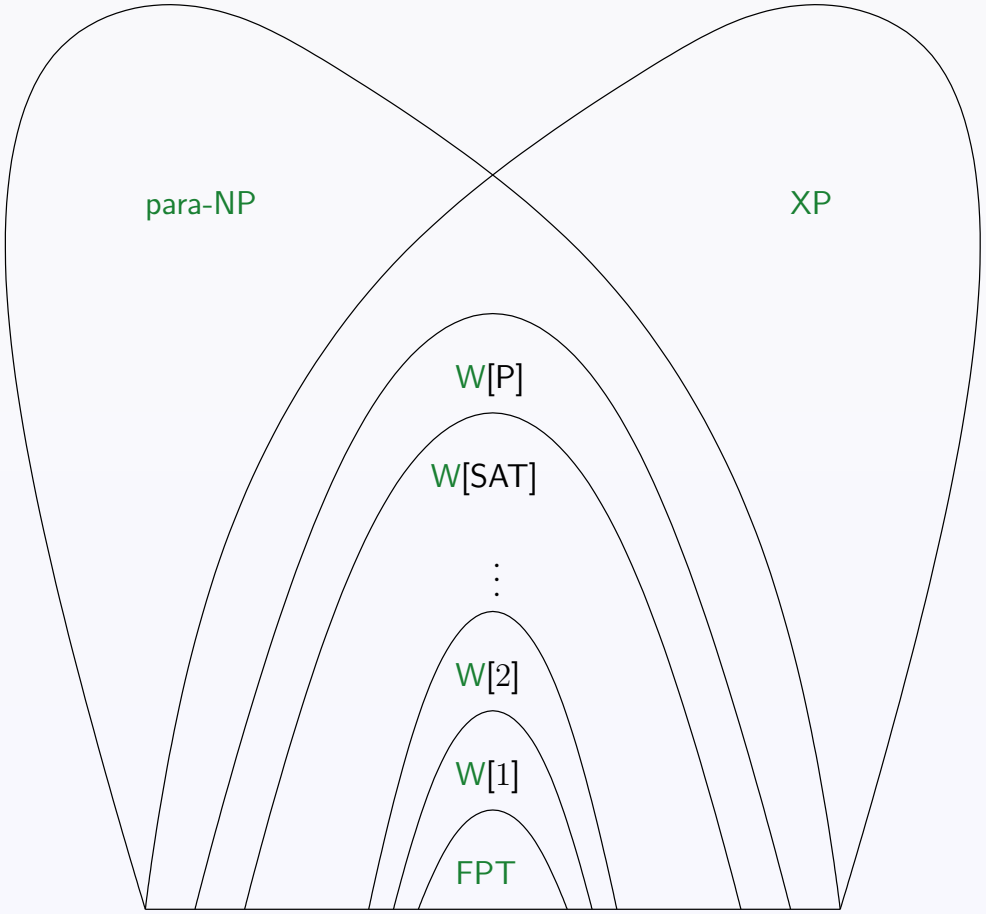

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p-Clique: W[1]-complete

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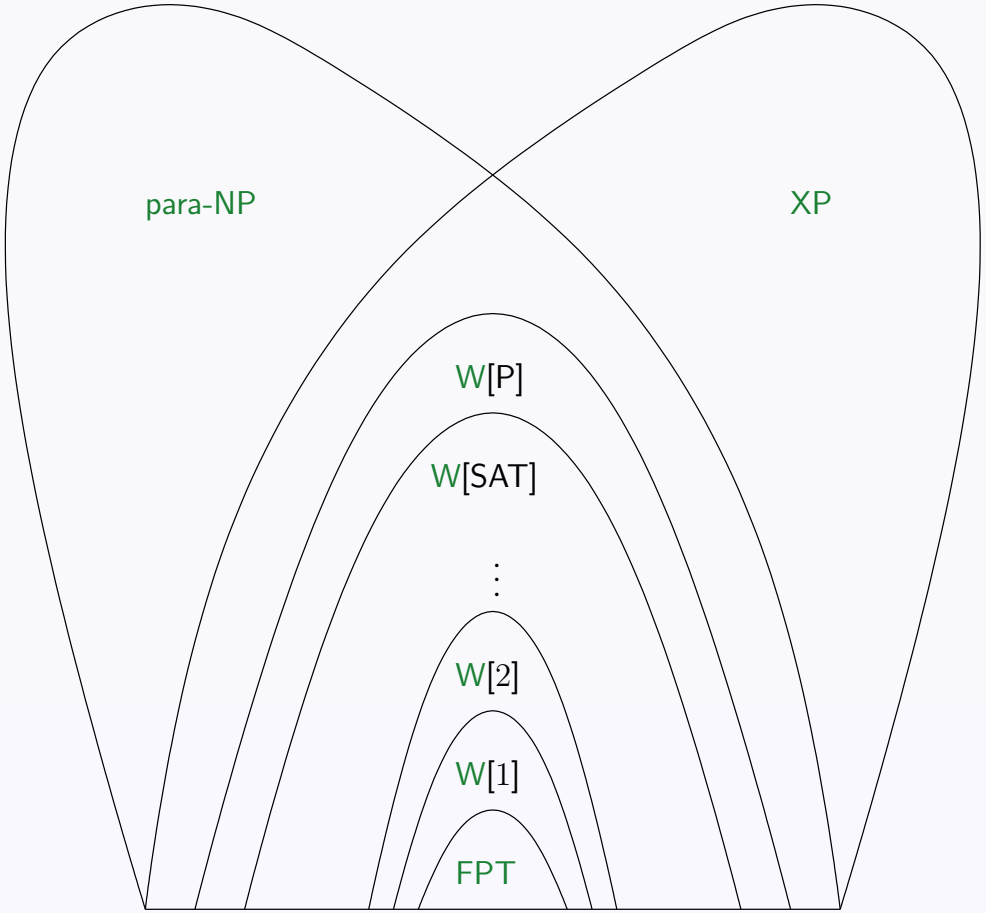

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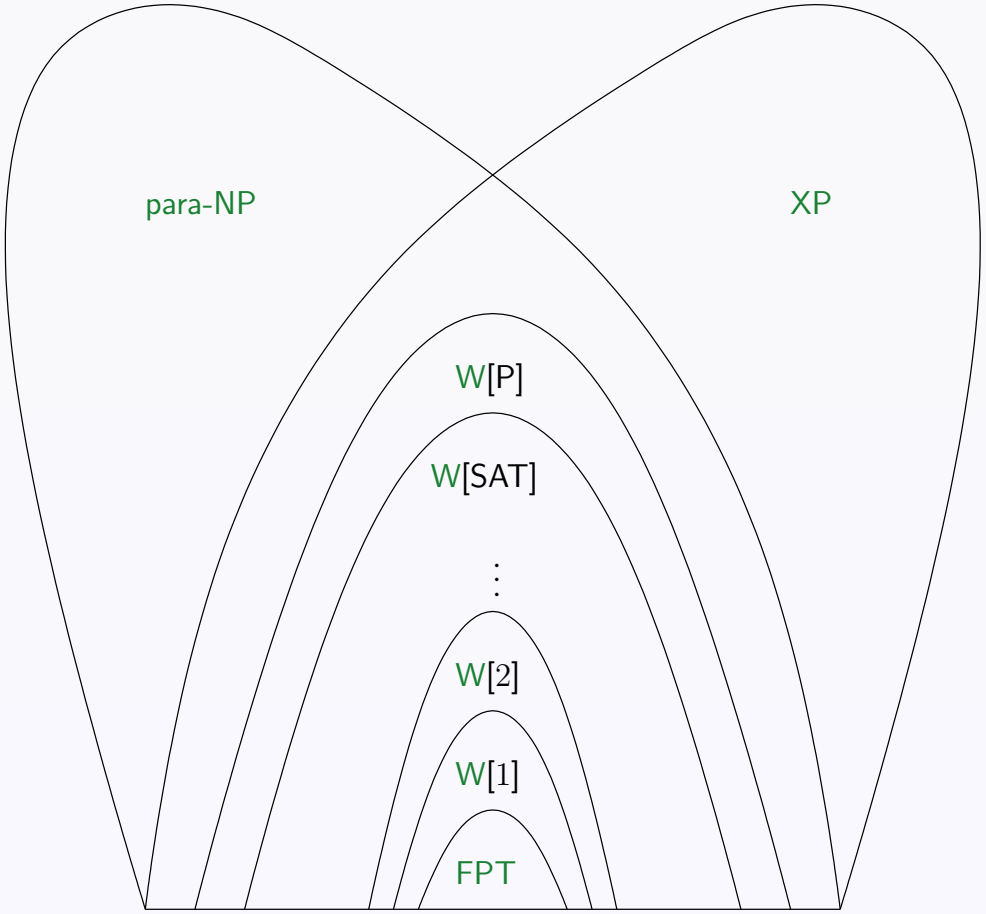

p-Vertex Cover: FPT

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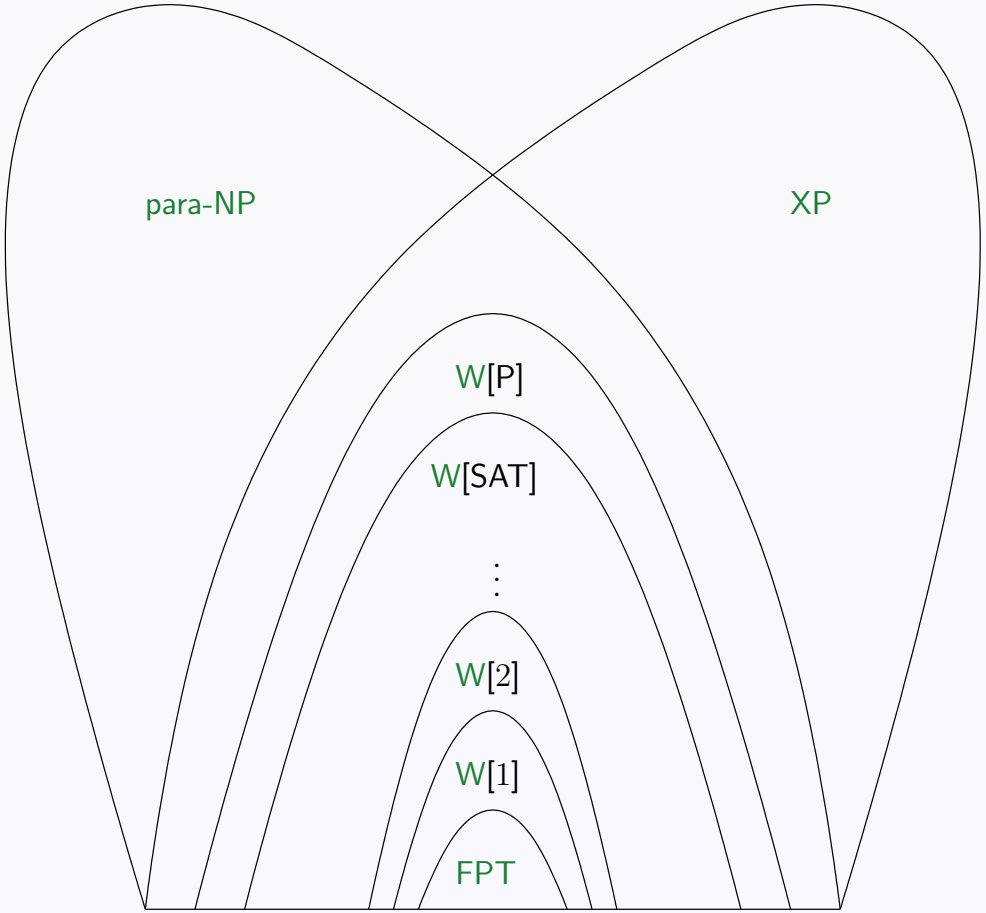

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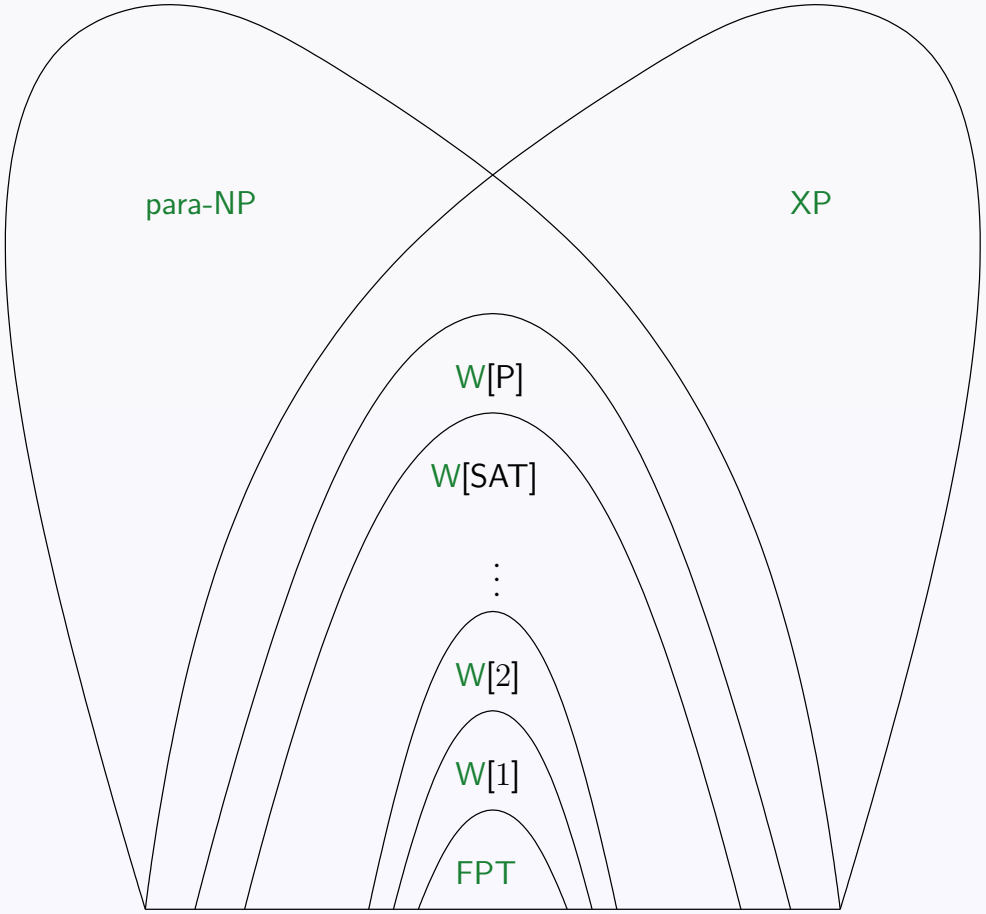

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p-Steiner Tree: W[2]-complete

p-Coloring: para-NP-complete

Part 2, Monday 09/05/2016 - 16:00–17:30 (90')

Tree decompositions

Treewidth

Courcelle's Theorem

Dynamic programming

Tree decompositions

A *tree decomposition* (ou décomposition arborescente) of a graph G is a pair $D = (T, \mathcal{X})$ such that T is a tree and $\mathcal{X} = \{X_t \mid t \in V(T)\}$ is a collection of subsets of G. such that:

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A *tree decomposition* (ou décomposition arborescente) of a graph G is a pair $D = (T, \mathcal{X})$ such that T is a tree and $\mathcal{X} = \{X_t \mid t \in V(T)\}$ is a collection of subsets of G. such that:

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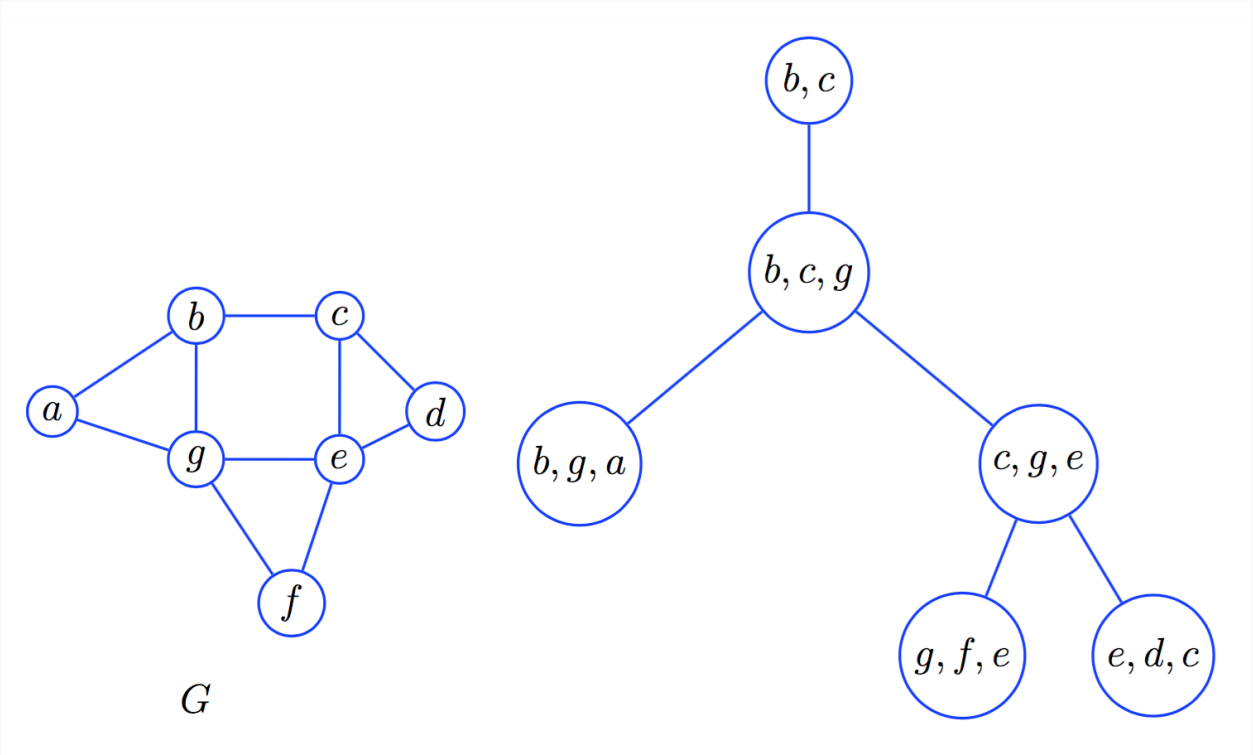
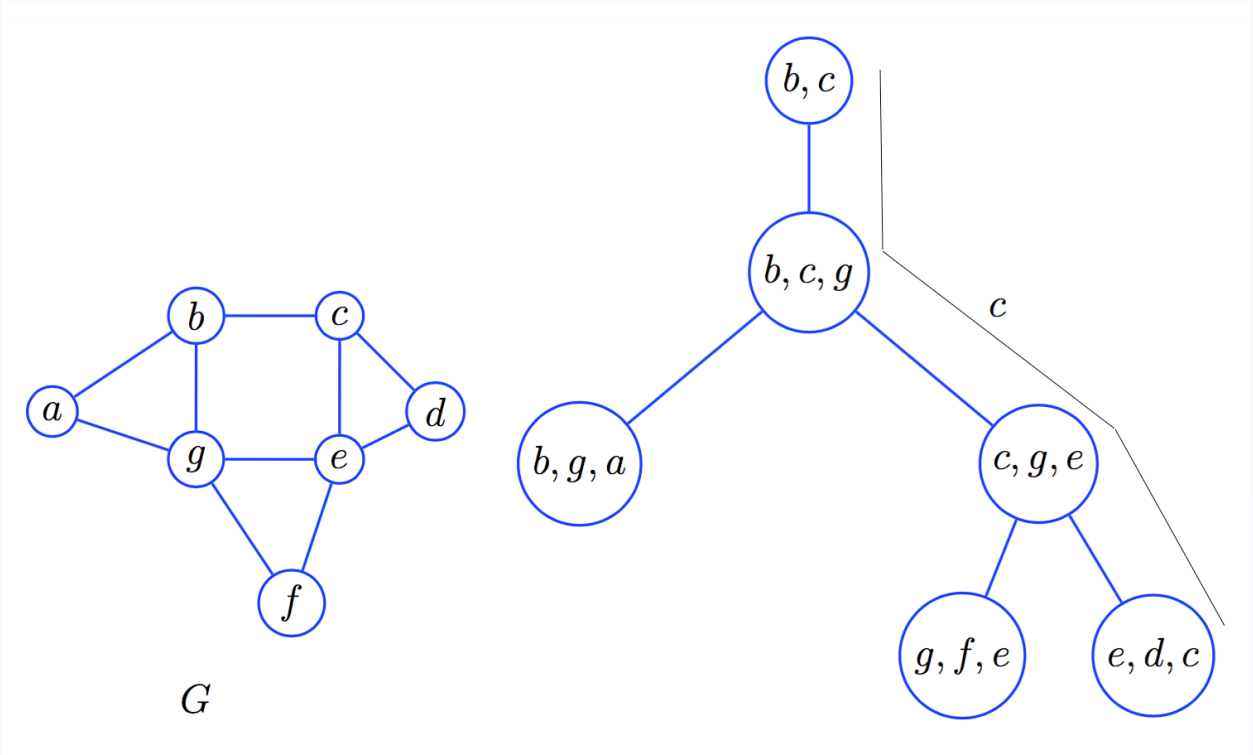
- The *tree-width* (ou largeur arborescente ou largeur d'arbre) of a graph G (**tw**(G)) is the minimum width over all tree decompositions of G



Each vertex of G has a continuous "trace" in the tree of the tree decomposition



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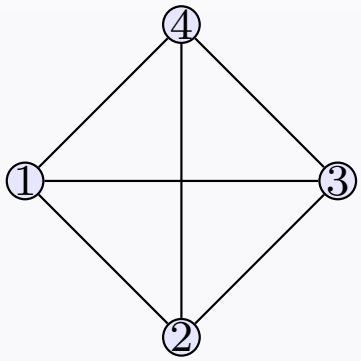
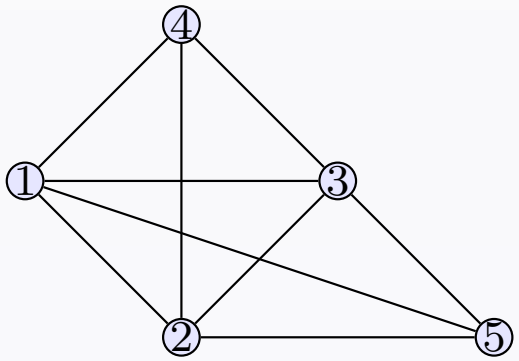
- The treewidth of a graph G is defined as follows

$$\mathbf{tw}(G) = \min\{k \mid G \text{ is a subgraph of some } k\text{-tree}\}$$



A 3-tree



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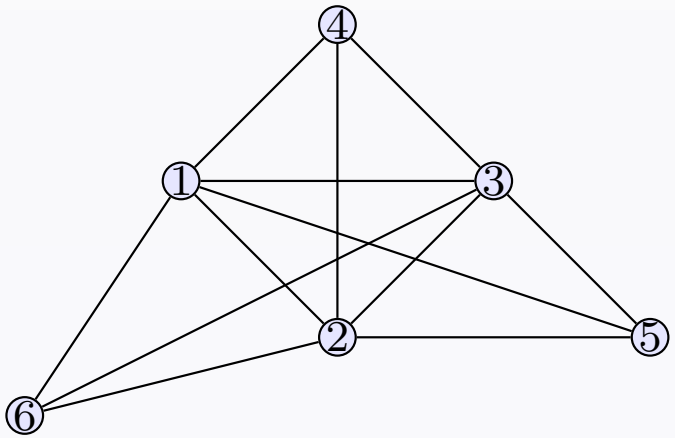

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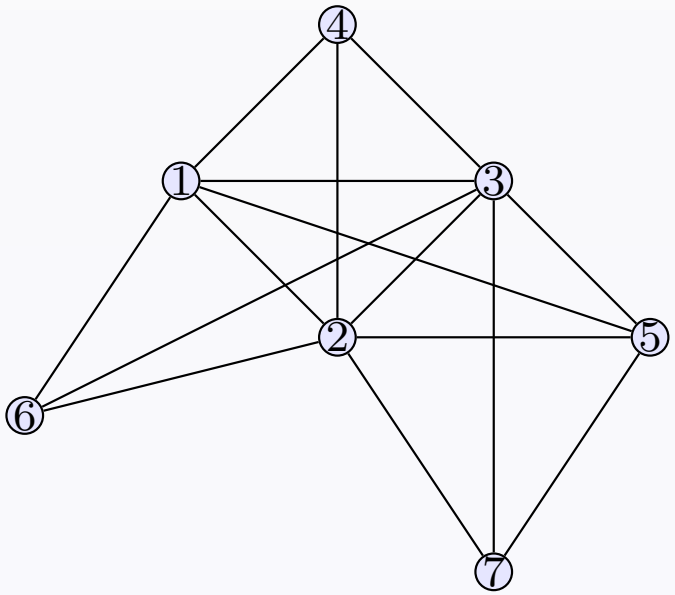

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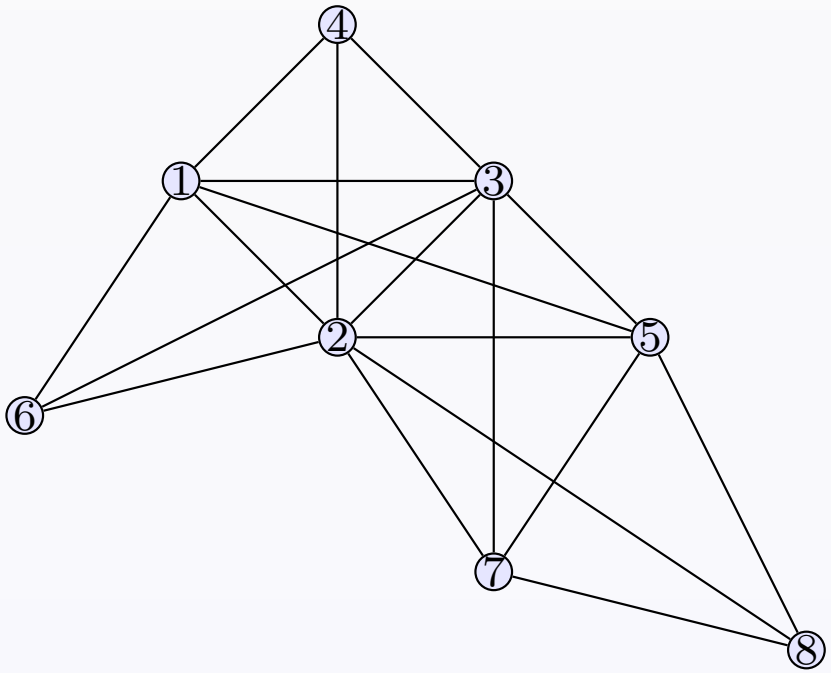

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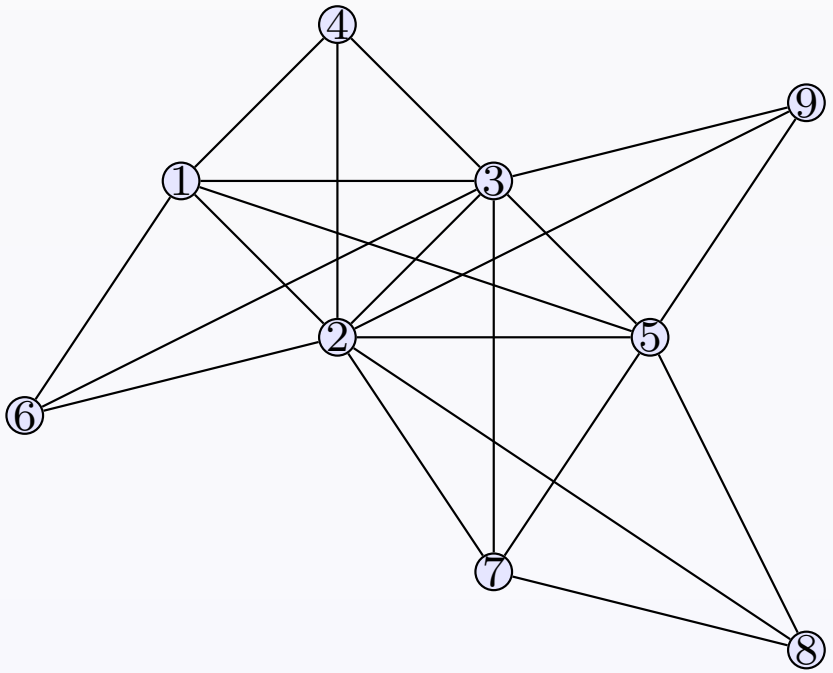

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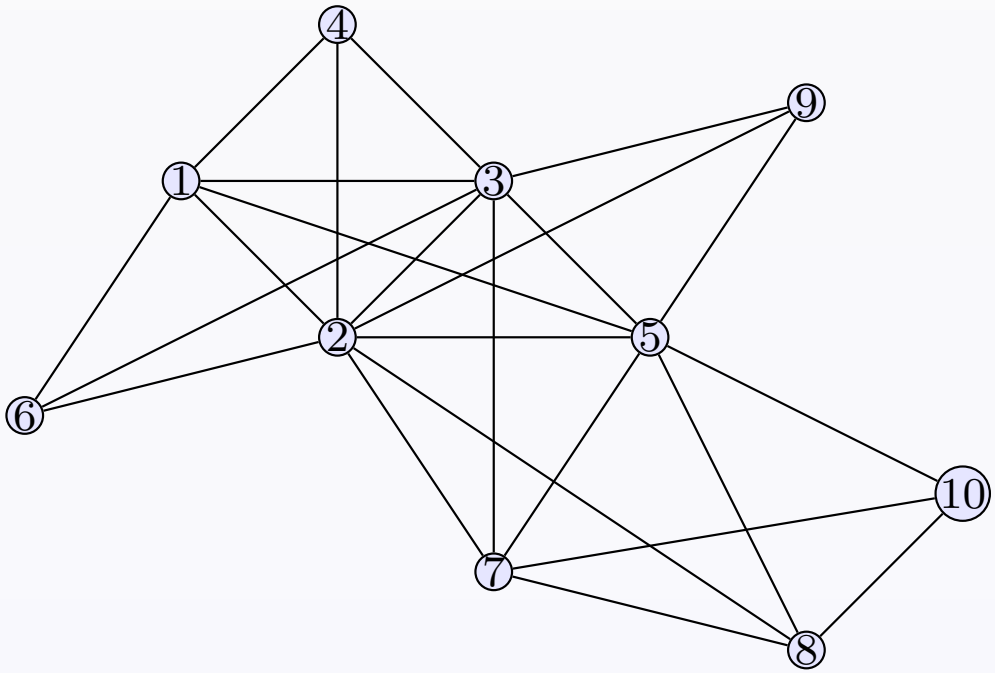

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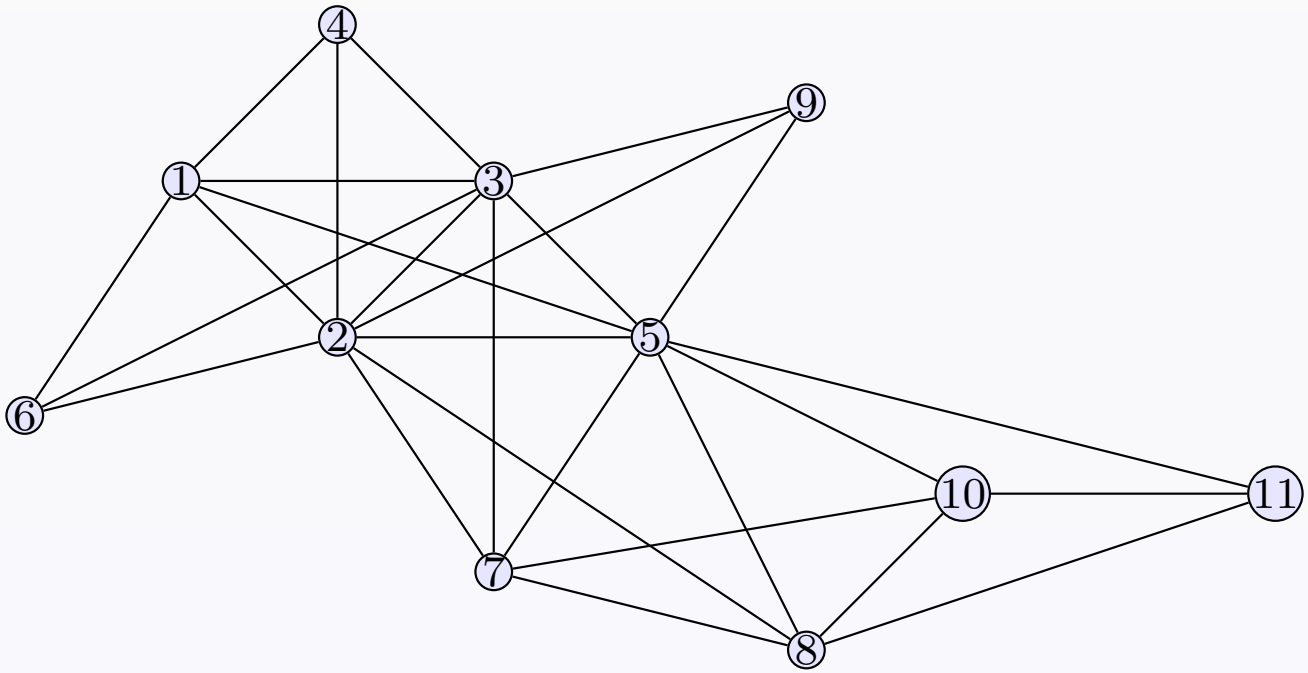

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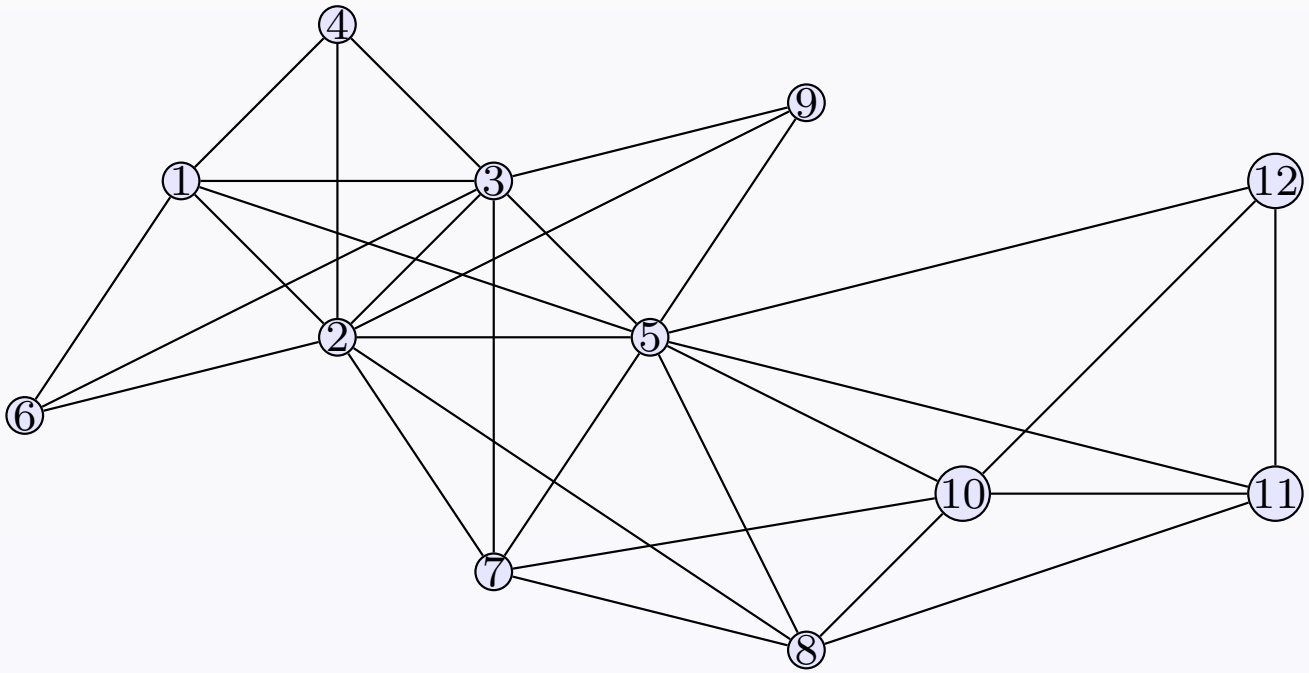

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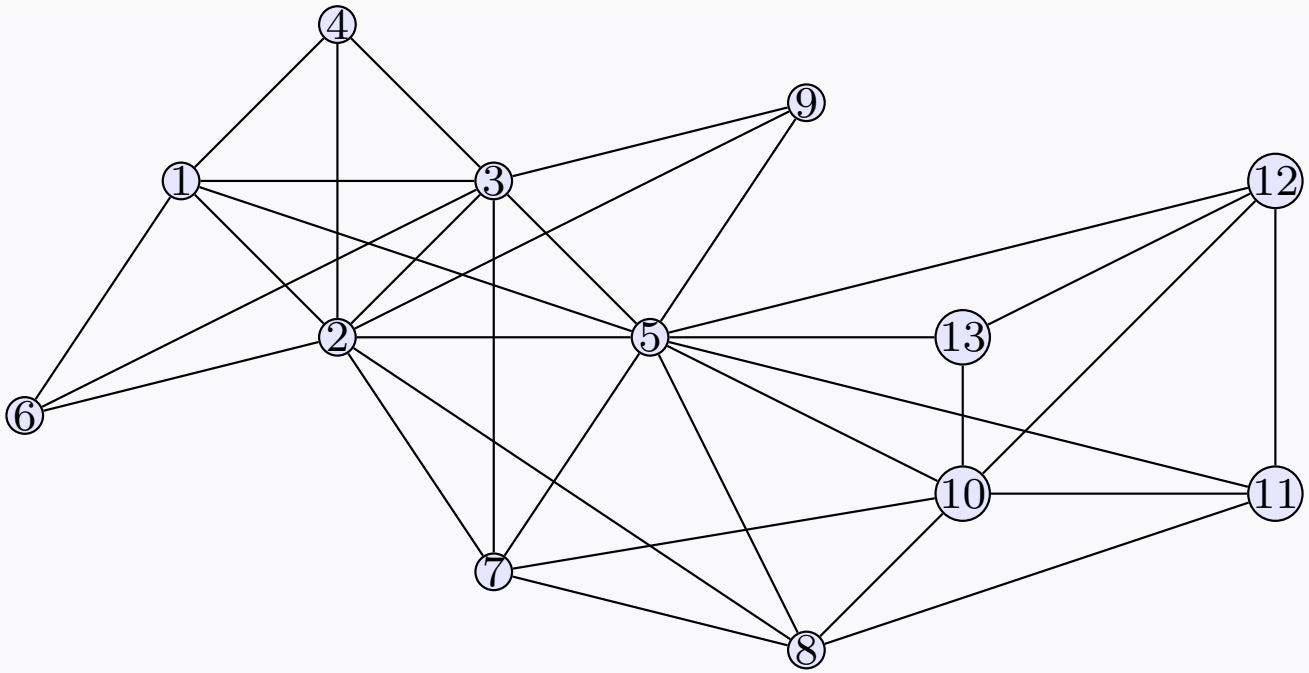

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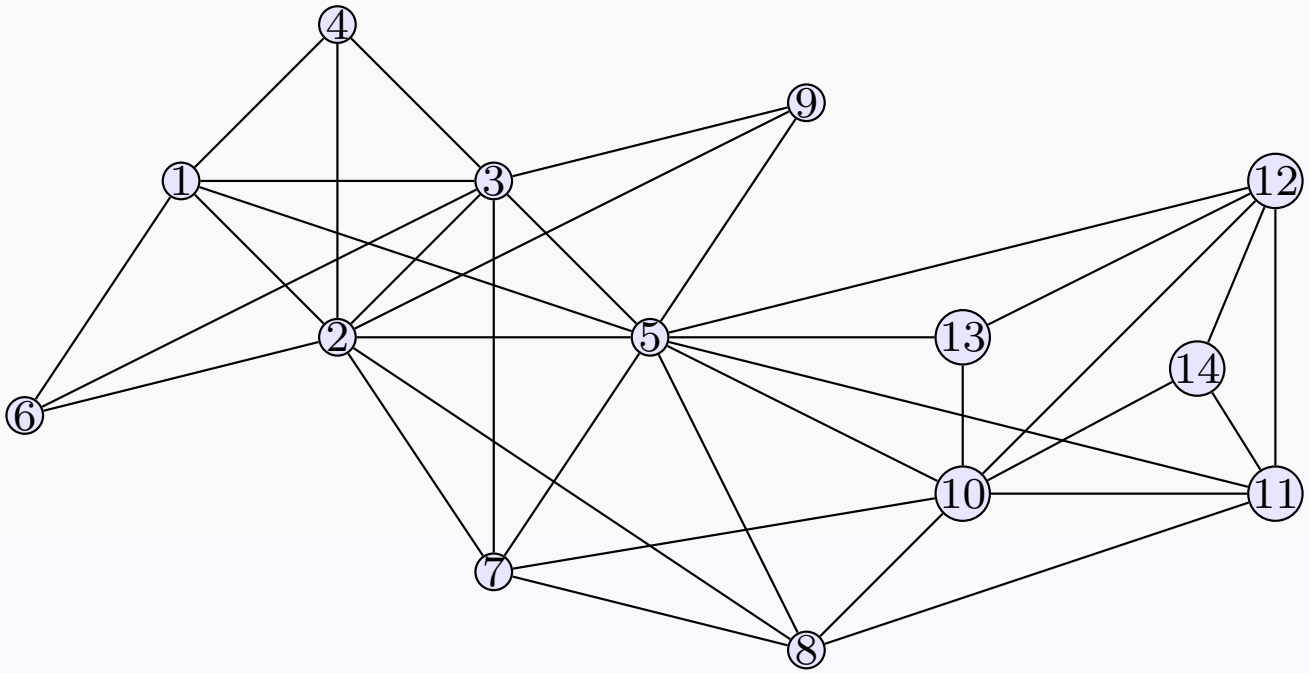

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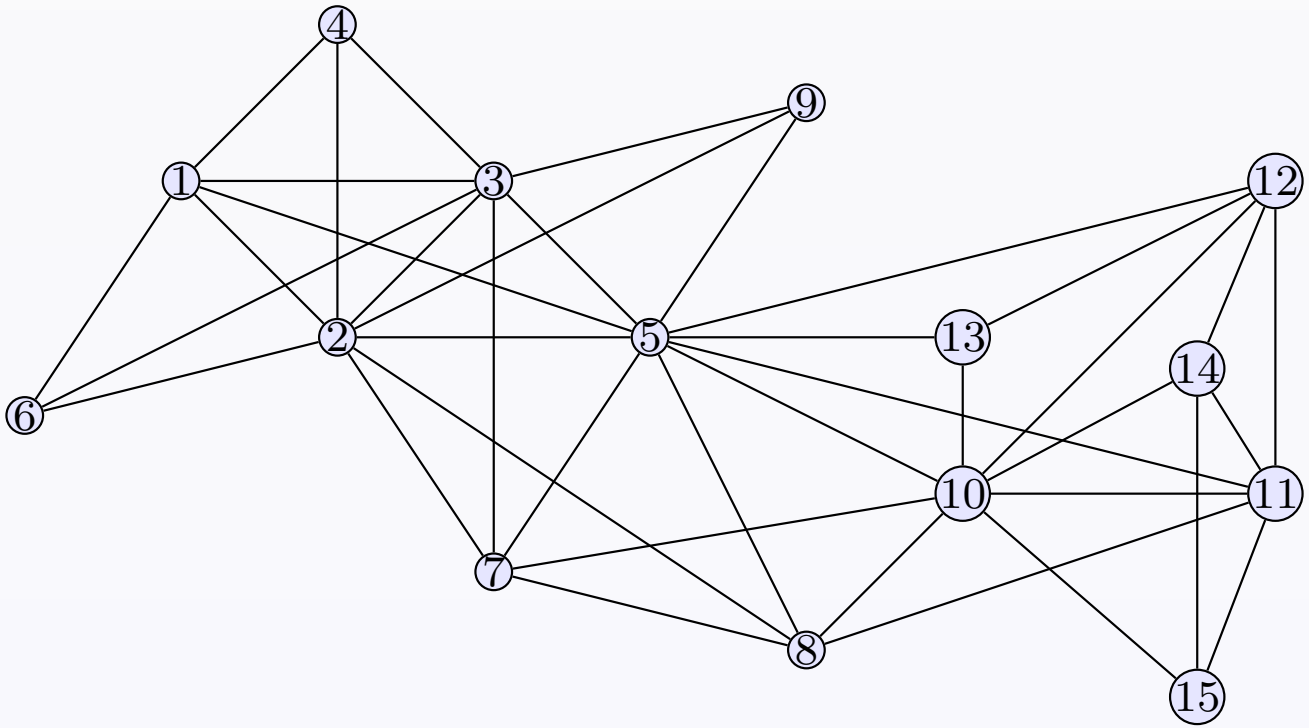

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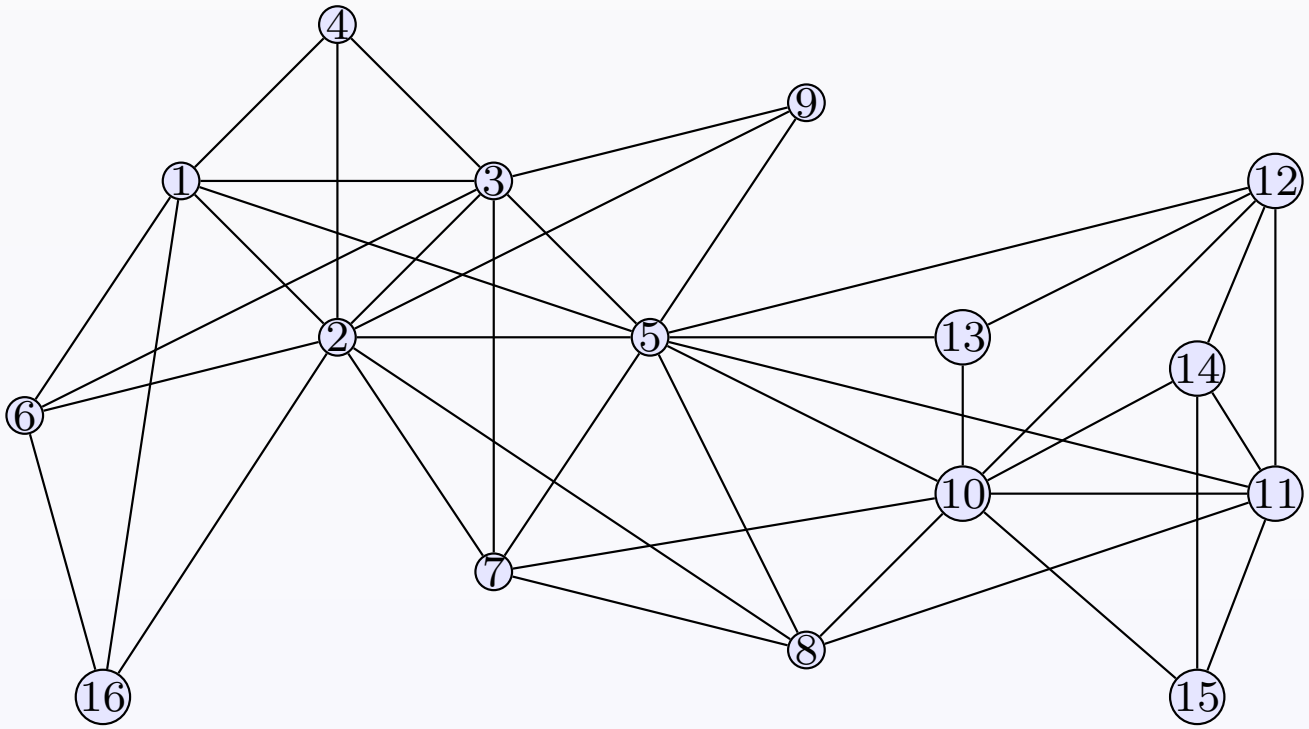

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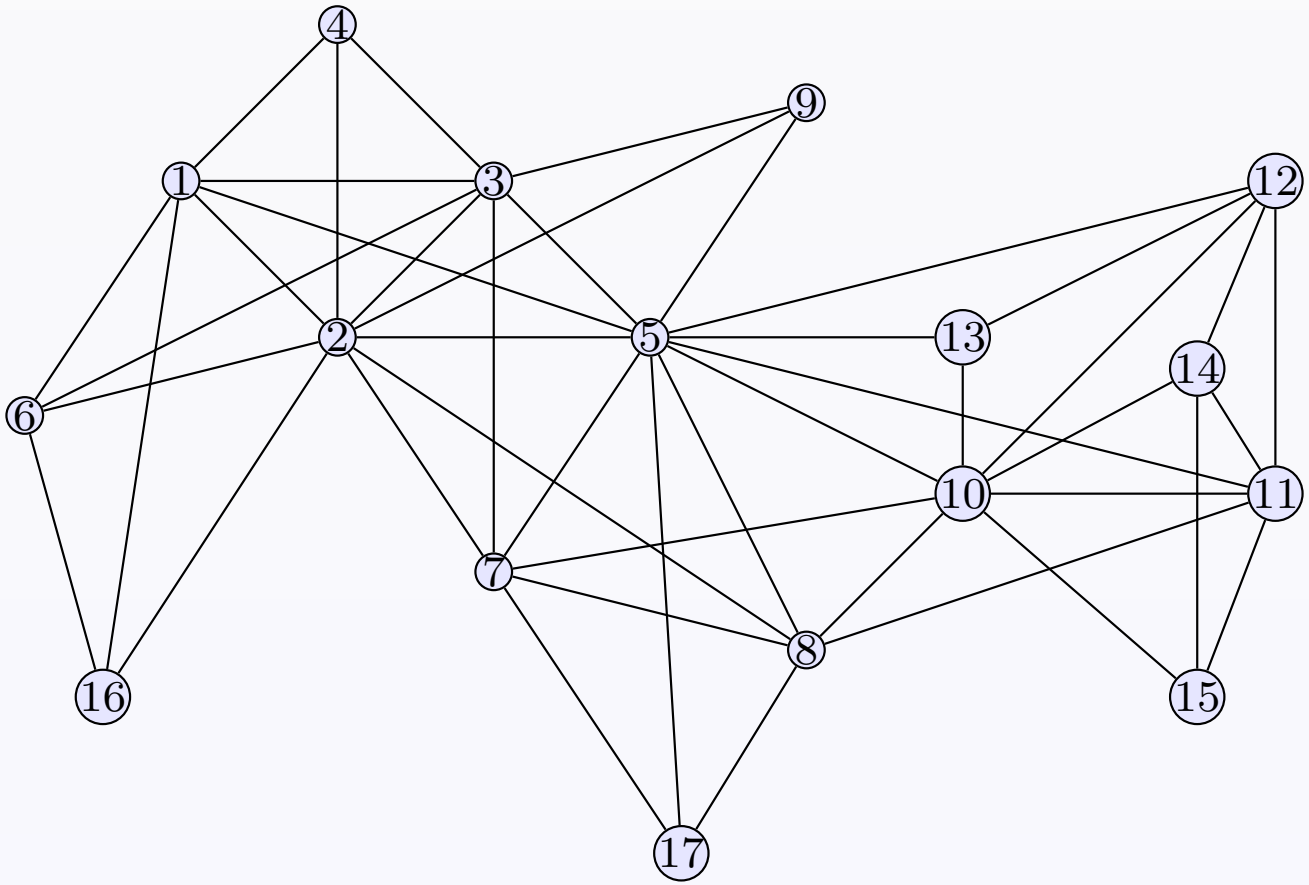

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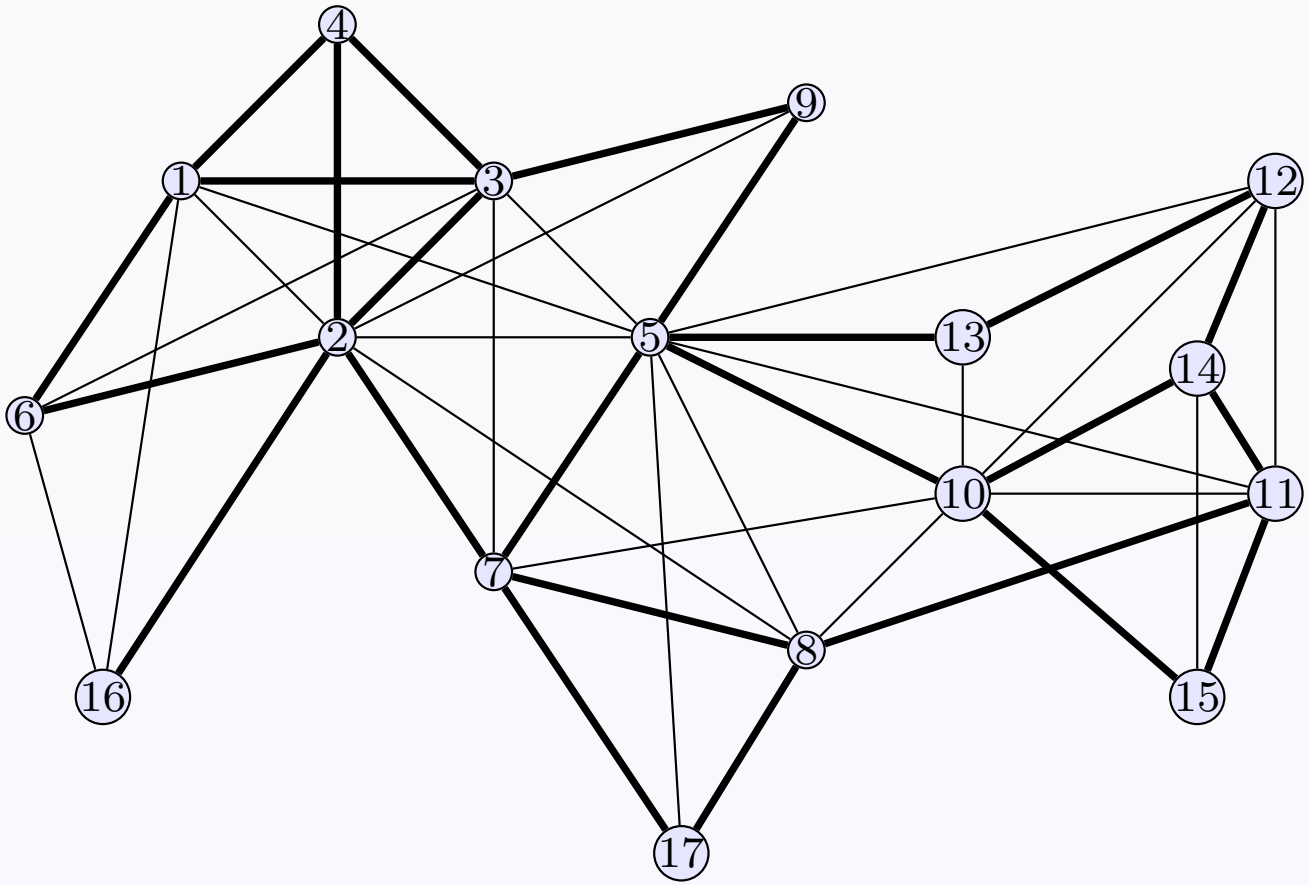

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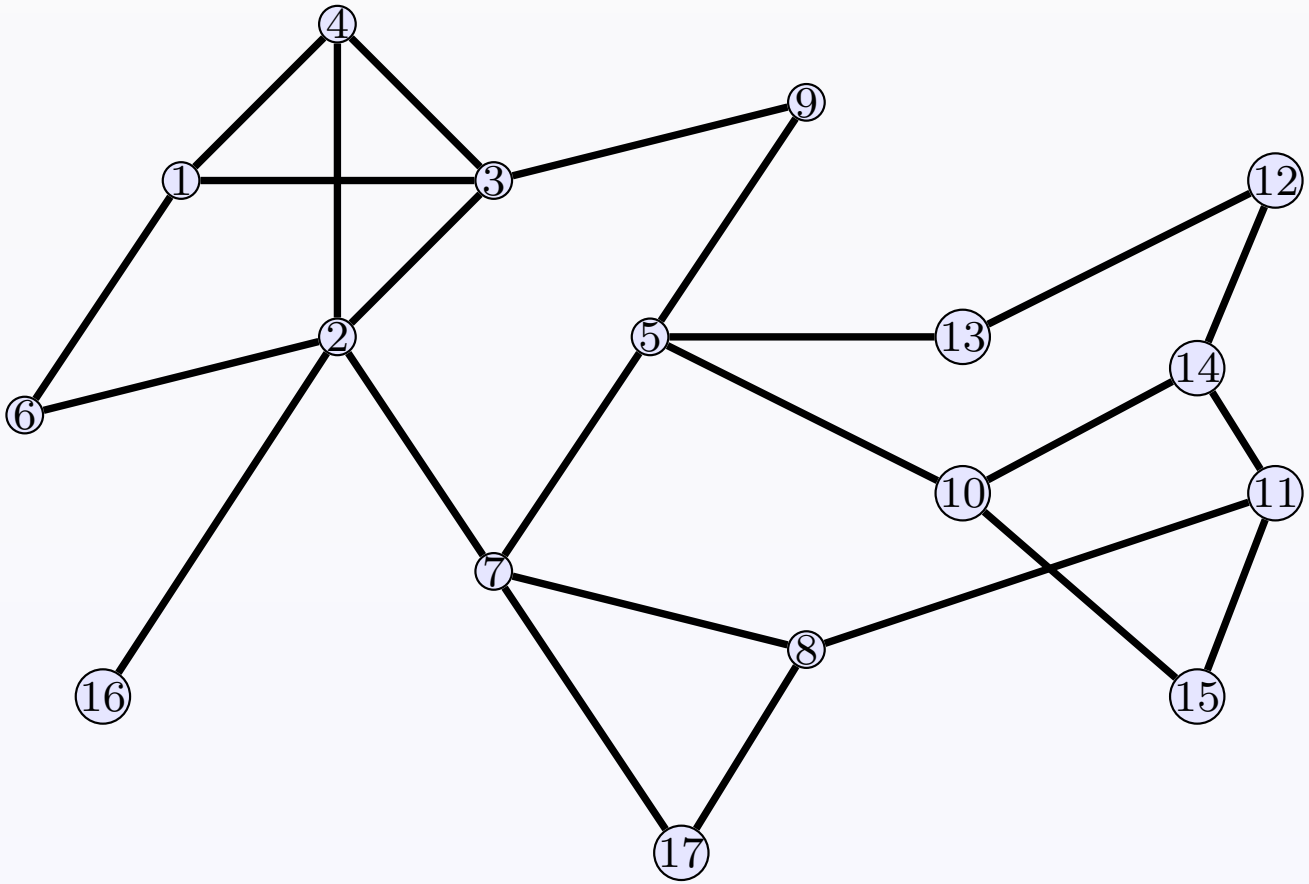
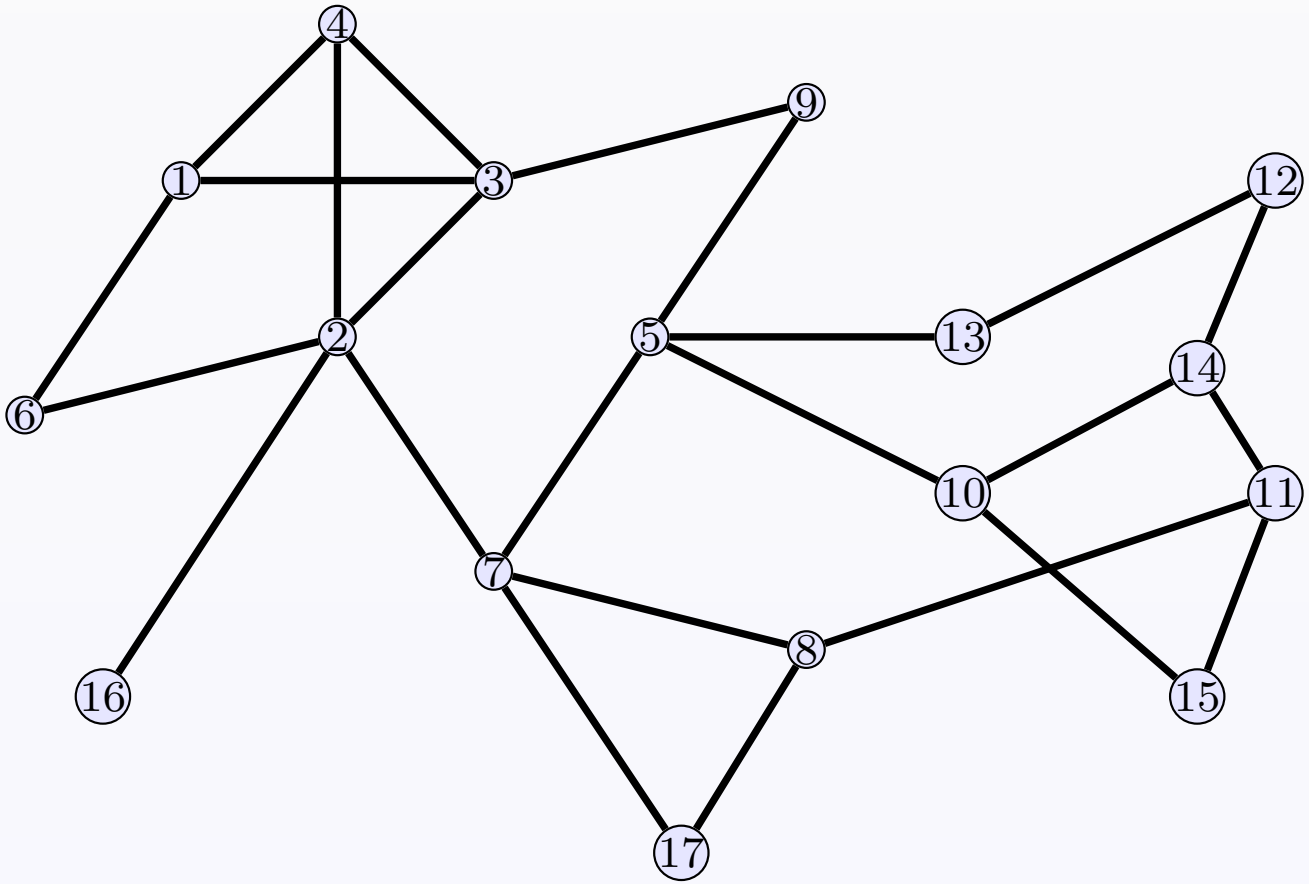
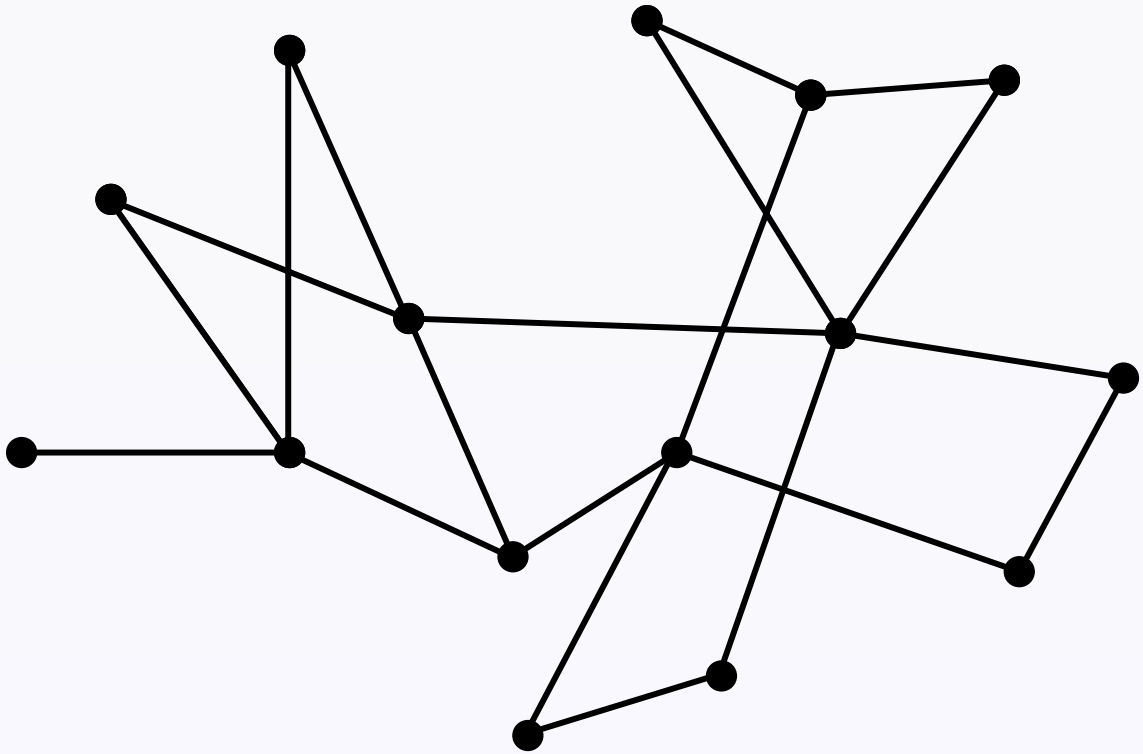
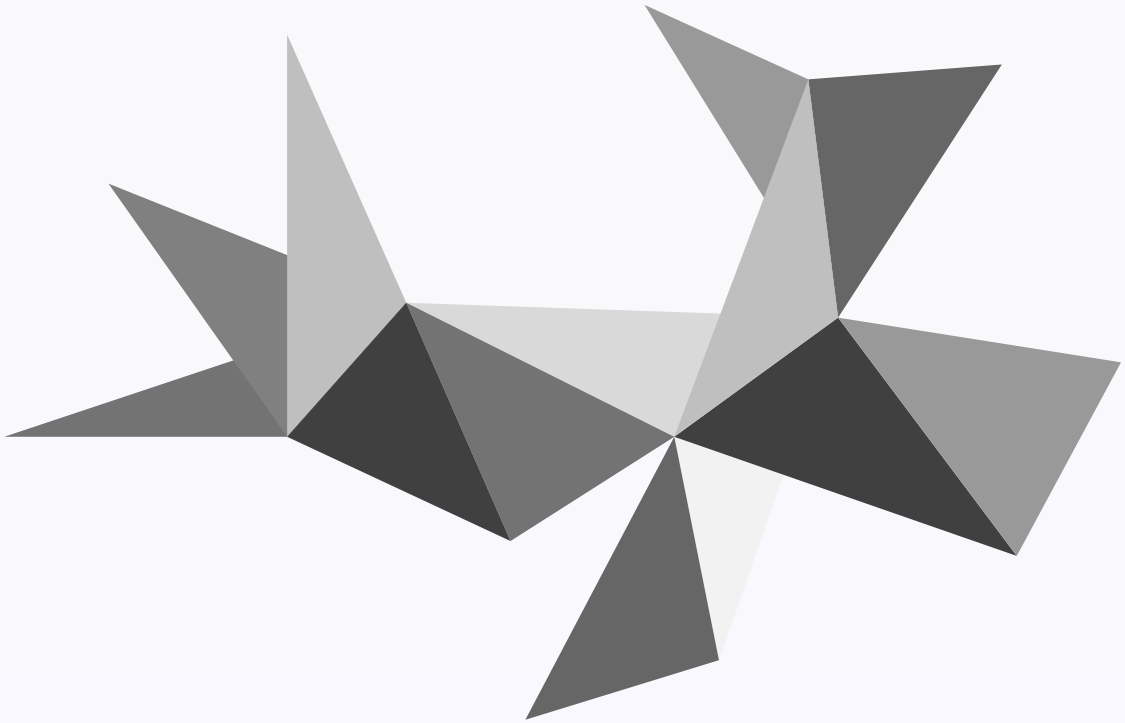


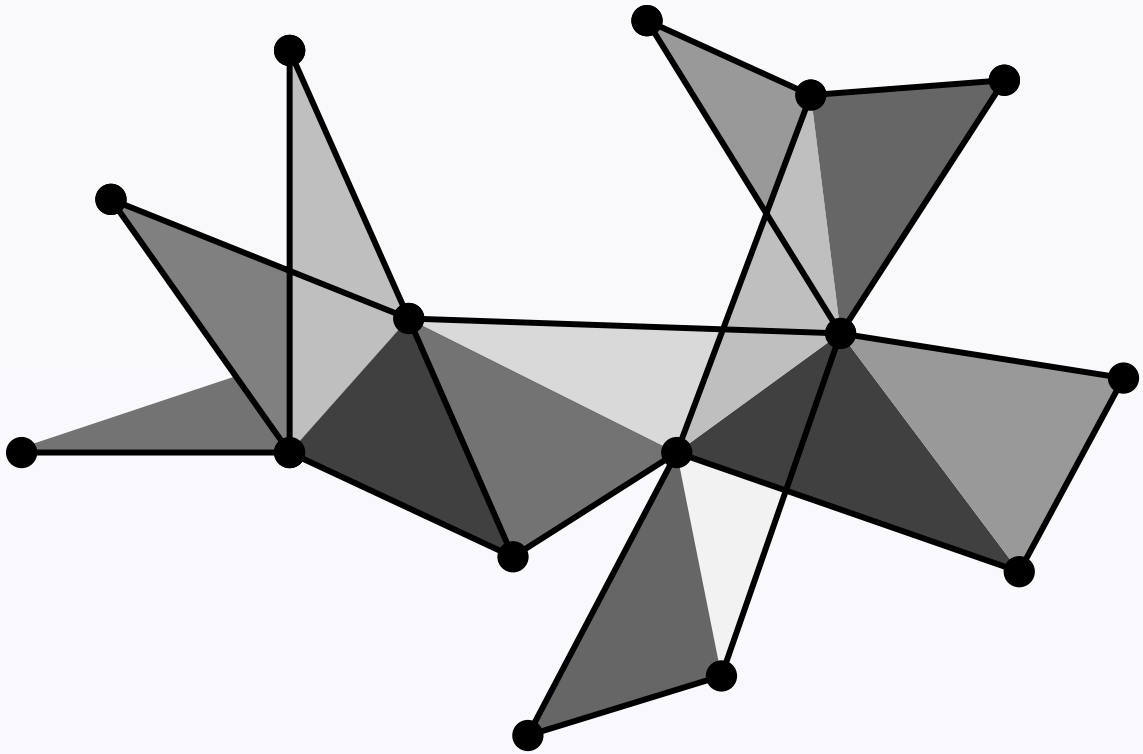
A subgraph of a 3-tree

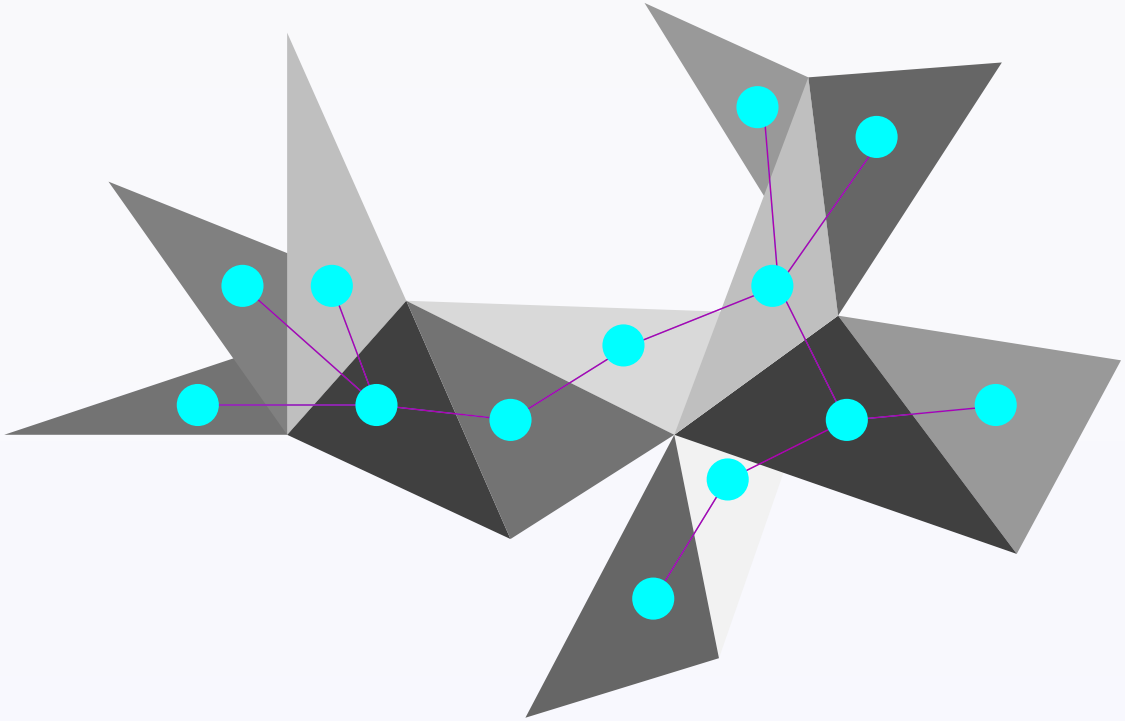


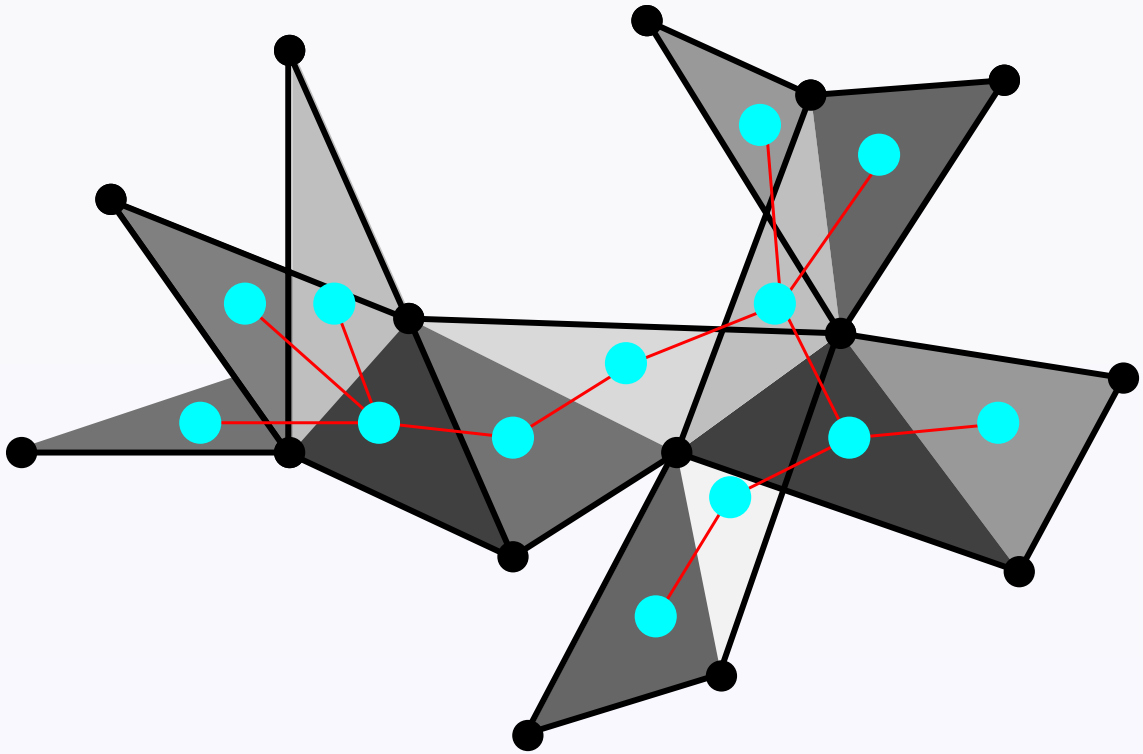
A subgraph of a 3-tree: a graph with treewidth at most 3











Facts about treewidth

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- Many NP-hard problems on graphs become polynomially solvable when their instances are restricted to graphs with constant treewidth.

Parameterizing treewidth

p-Treewidth

Instance: A graph G and an integer $k \geq 0$.

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p-Treewidth is in FPT by an $2^{O(k^3)} \cdot O(n)$ algorithm of Bodlaender

[SIAM J. Comp., 1996]

Monadic Second Order Logic

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Universe: the vertex set V of the graph $G = (V, E)$

An MSO formula can be build using:

Variables: vertices $x, y, z, \ldots$ and sets of vertices $X, Y, Z, \ldots$

Atomic Formulae: $x = y$, $x \in X$, $\{x, y\} \in E$ $\quad$ $(E(x, y))$

Formulae: $\neg x$, $x \vee y$, $x \wedge y$, $x \rightarrow y$, $x \leftrightarrow y$, $\exists x \phi$, $\forall x \phi$, $\exists X \phi$, $\forall X \phi$,

Examples of properties expressible in MSO

3-Colorability:

$$\exists R\ \exists G\ \exists B\ [\forall x\ [(x \in R \vee x \in G \vee x \in B) \wedge$$

$$\neg(x \in R \wedge x \in G) \wedge \neg(x \in B \wedge x \in G) \wedge \neg(x \in R \wedge x \in B)]]$$

$$\wedge \neg[\exists x\ \exists y\ (\{x, y\} \in E \wedge$$

$$((x \in R \wedge y \in R) \vee (x \in G \wedge y \in G) \vee (x \in B \wedge y \in B)))]$$

Examples of properties expressible in MSO

Having an clique of size $\geq k$:

$\exists x_1 \ \exists x_2 \cdots \exists x_k \bigwedge_{1 \leq i < j \leq k} \{x_i, x_j\} \in E$

Examples of properties expressible in MSO

Having an independent set of size k:

$\exists x_1 \ \exists x_2 \cdots \exists x_k \bigwedge_{1 \leq i < j \leq k} (\neg \{x_i, x_j\} \in E) \wedge \neg (x_i \neq x_j)$

Examples of properties expressible in MSO

Having a vertex cover of size k:

$$\exists x_1\ \exists x_2 \cdots \exists x_k\ (\forall x\ \forall y\ \{x, y\} \in E \rightarrow (\bigvee\nolimits_{1 \leq i \leq k} (x = x_i \vee y = x_i)))$$

Examples of properties expressible in MSO

Having a dominating set of size k:

$\exists x_1 \ \exists x_2 \cdots \exists x_k \ \forall y \ \bigvee_{1 \leq i \leq k} (\{x_i, y\} \in E \vee y = x_i)$

Courcelle's theorem

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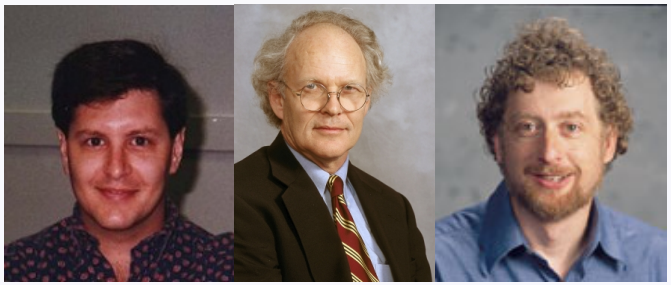

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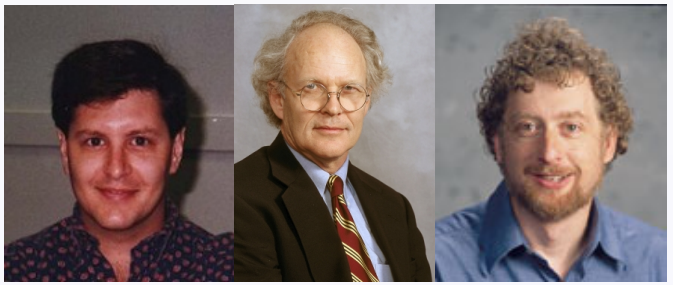

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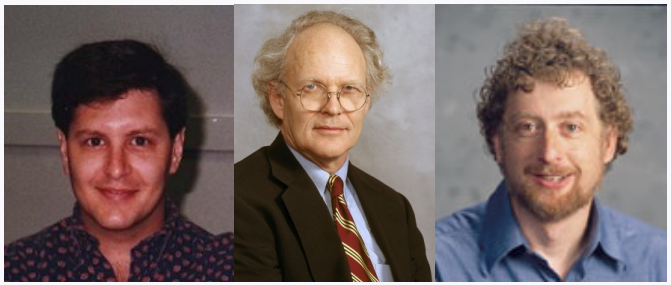

In other words:

If $\Pi \subseteq \mathcal{G}_{\text{all}}$ is a MSO-expressible set, then $(\Pi, \mathbf{tw}) \in$ FPT

or

Every MSO-expressible problem of graphs is fixed parameter tractable when parameterized by the treewidth of its input graph

Inputs of small treewidth can be seen as tree-string: inputs of a tree-automaton generated by the MSO formula expressing $\mathcal{G}$.

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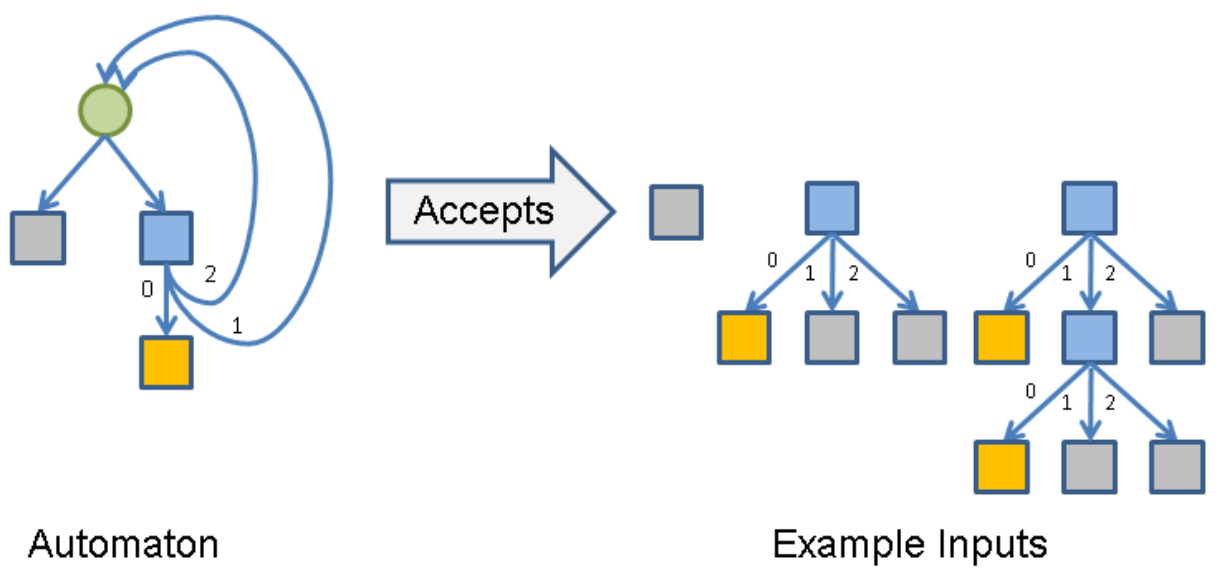

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Drawback of Courcelle's Theorem: the contribution of the formula and the treewidth in the running time is immense.

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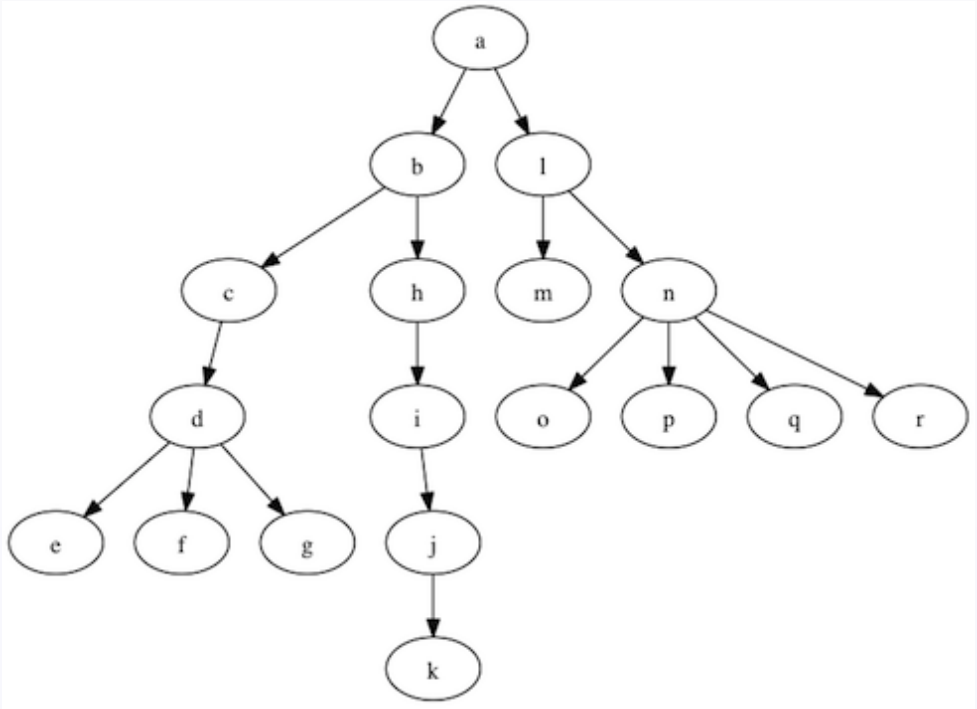

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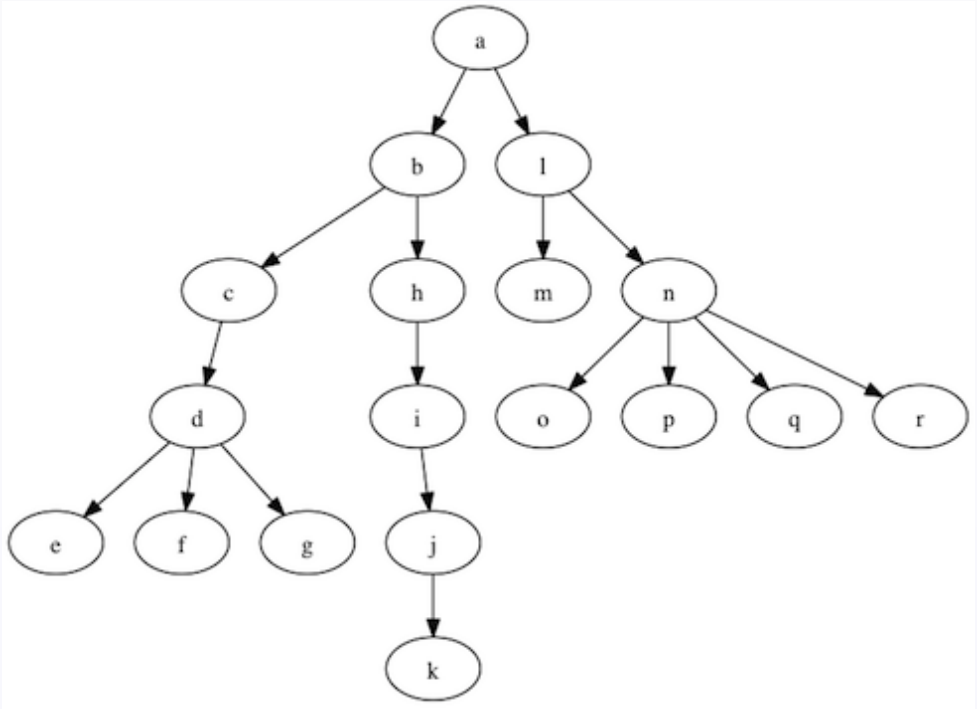
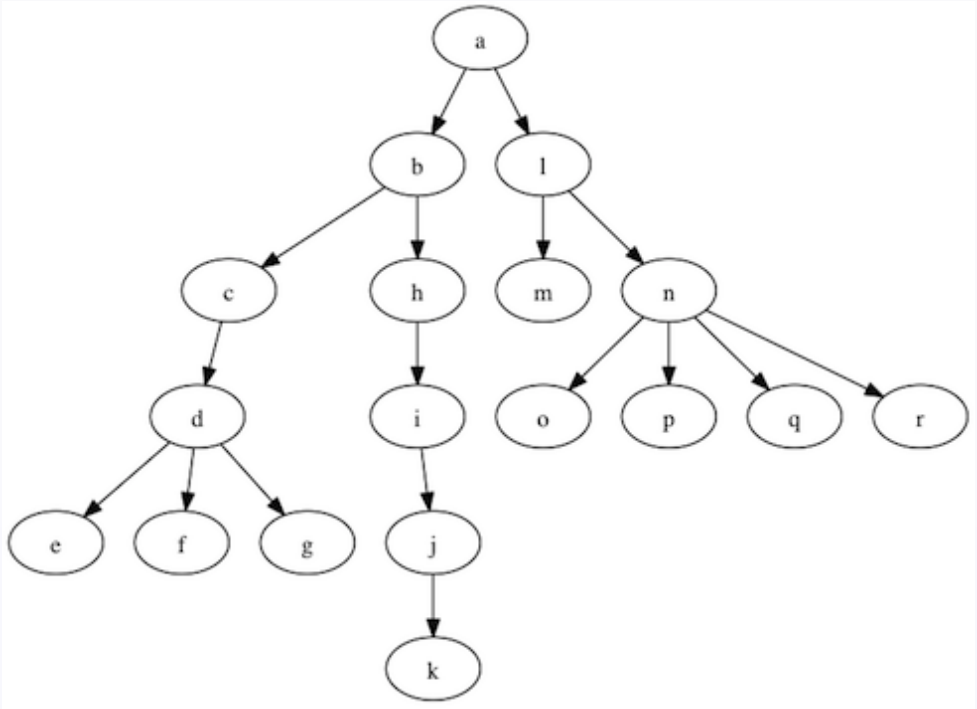


Treewidth is a measure of the possibility of recursively cutting the graph in smaller pieces and process them separately:

In topological terms: treewidth helps us treat the input graph a mono-dimensional entity!



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In an algorithmic terms: Divide and Conquer!

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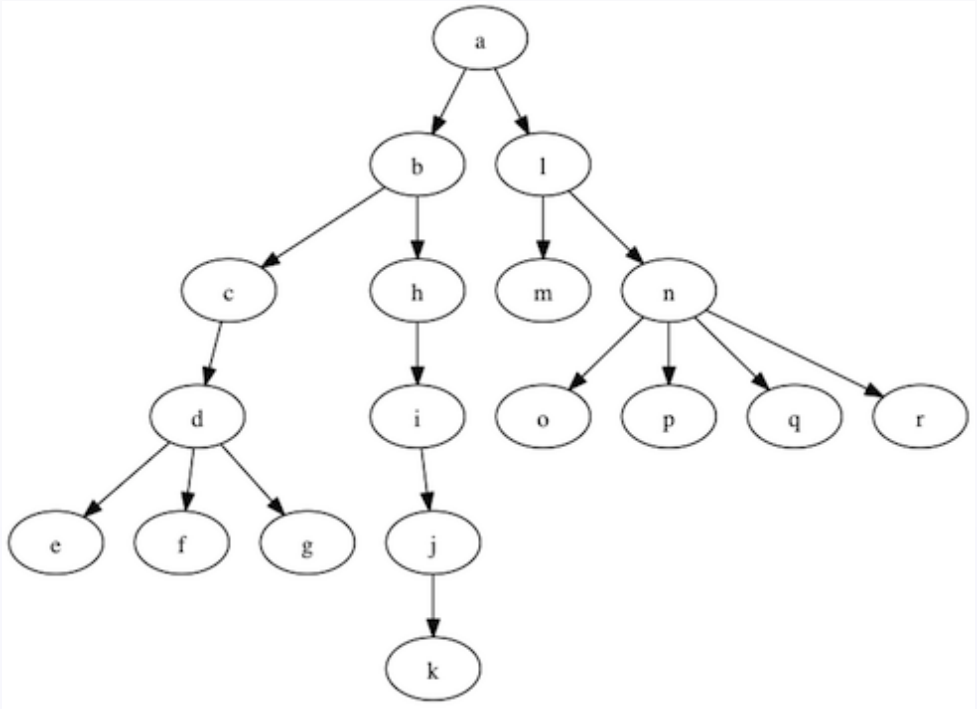

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Which in our case is: Dynamic Programming

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- if t has one child t' then
  - either $X_t = X_{t'} \cup \{v\}$

    (we call X_t *insert* node and v is the *insert vertex*)
  - or $X_{t'} = X_t \cup \{v\}$

    (we call X_t *forget* node and v is the *forget vertex*)

If $(T, \mathcal{X})$ is a nice tree decomposition rooted on r, then

for any $t \in V(T)$, $G_t = G[\bigcup_{t' \text{ is } t \text{ or a descendant of } t \text{ in } T} X_{t'}]$

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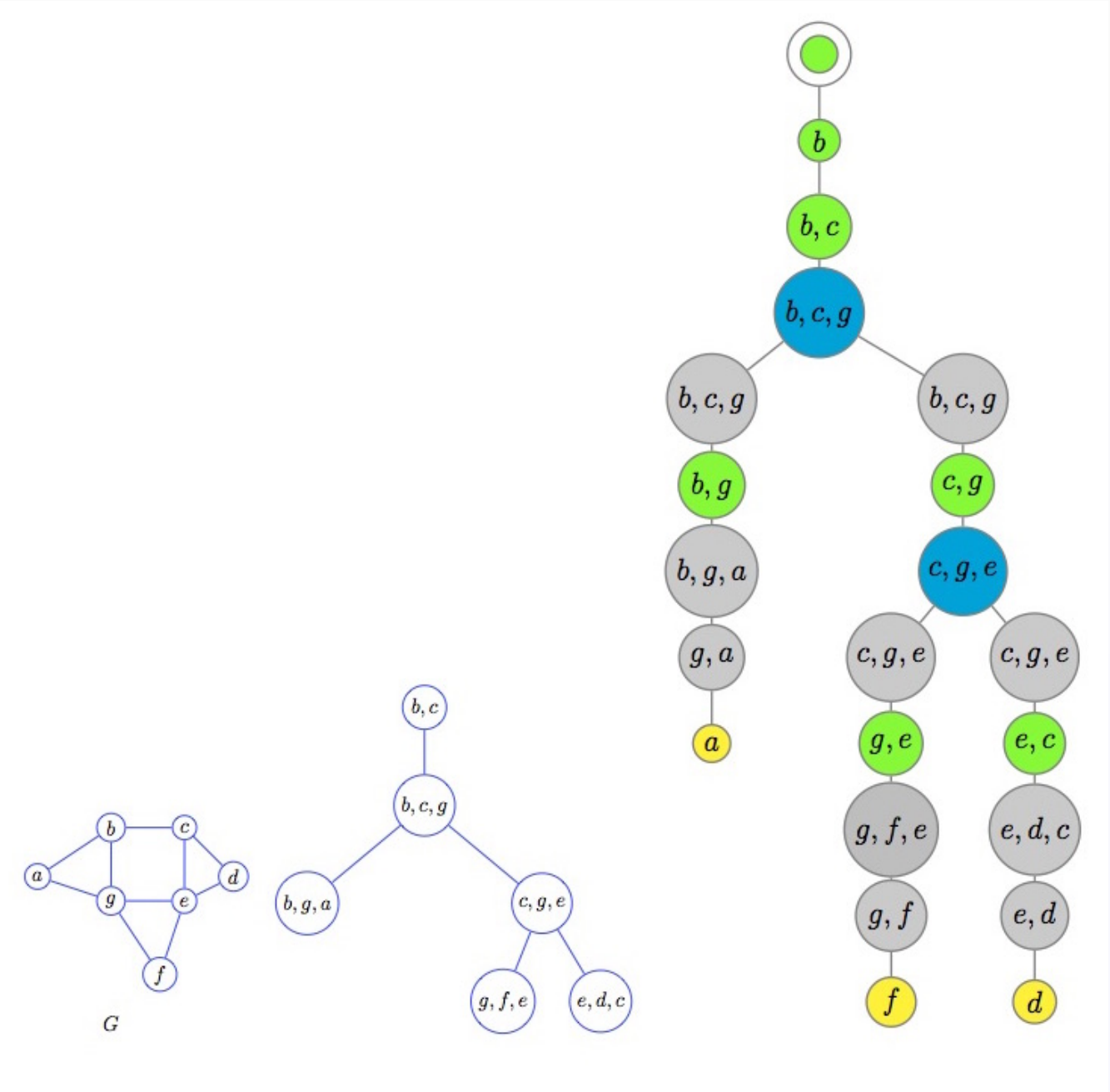
Lemma: *There exists an $O(n)$-step algorithm that transforms any tree decomposition with n nodes to a nice tree decomposition of $\leq 4n$ nodes of the same width.*



A graph G, a tree decomposition, and a nice tree decomposition

How to do dynamic programming for graphs of small treewidth

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5. Provide a way to compute the table of a join node, given the tables of its children.

Parameterizing 3-Coloring by treewidth

tw-3-Vertex Coloring

Instance: A graph G.

Parameter: $k = \mathbf{tw}(G)$

Question: $\exists \chi : V(G) \to \{1, 2, 3\} : \forall \{v, u\} \in E(G)\ \chi(v) \neq \chi(u)$?

For any $\chi : S \to I$ and $R \subseteq S$, we define $\chi[R] = \{(v, \chi(v)) \in \chi \mid v \in R\}$

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1st step: Definition of the tables:

For any $t \in V(T)$ and any 3-coloring $\phi : X_t \to \{1, 2, 3\}$, we define

$$B_t(\phi) = [\exists \chi : V(G_t) \to \{1, 2, 3\} \text{ such that } \chi[X_t] = \phi]$$

(the table of t contains an array of $3^{|X_t|}$ bits)

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$G = G_r$ is 3-colourable iff $B_r(\varnothing) = 1$

2nd step: tables for leaf nodes:

Let X_l be an leaf node

we have

$$B_l(\varnothing) = 1$$

3rd step: tables for insert nodes:

Let X_t be an insert node

let t' be the child of t and v be the insert vertex.

For any $\phi : X_t \to \{1, 2, 3\}$, we have

$$B_t(\phi) = B_{t'}(\phi - (v, \phi(v))) \bigwedge_{u \in N_{G_t}(v)} [\phi(v) \neq \phi(u)]$$

4nd step: tables for forget nodes:

Let X_t be a forget node

let t' be the child of t and v be the forget vertex.

For any $\phi : X_t \to \{1, 2, 3\}$, we have

$$B_t(\phi) = \bigvee_{i \in \{1,2,3\}} B_{t'}(\phi \cup \{v, i\})$$

5th step: tables for join nodes:

Let X_t be an join node

let t_1, t_2 be the children of t

For any $\phi : X_t \to \{1, 2, 3\}$, we have

$$B_t(\phi) = B_{t_1}(\phi) \wedge B_{t_2}(\phi)$$

Conclusion:

Given a tree decomposition of G,

the following tw-3-Vertex-Coloring problem is in $2^{O(k)}$-FTP:

(we gave an $O(3^k \cdot k \cdot n)$ dynamic programming algorithm)

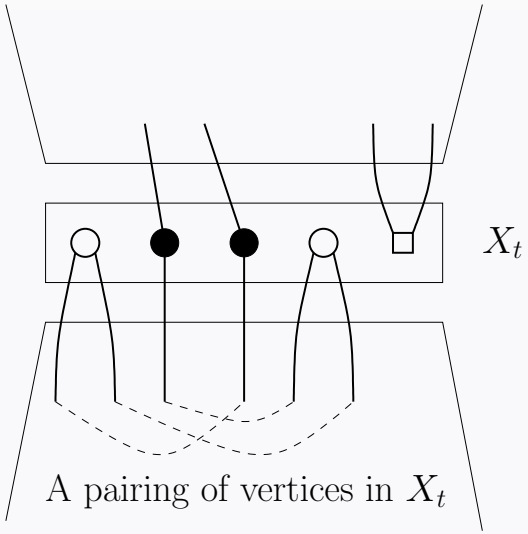
Parameterizing Hamiltonian Cycle by treewidth:

tw-Hamiltonian Cycle

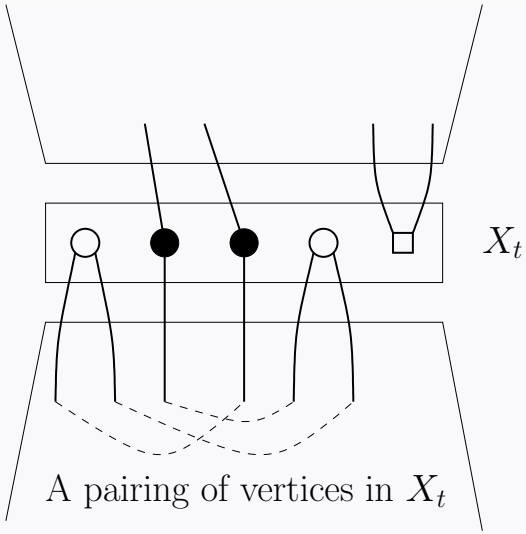
Instance: A graph G.

Parameter: $k = \mathbf{tw}(G)$

Question: does G contain a spanning cycle?

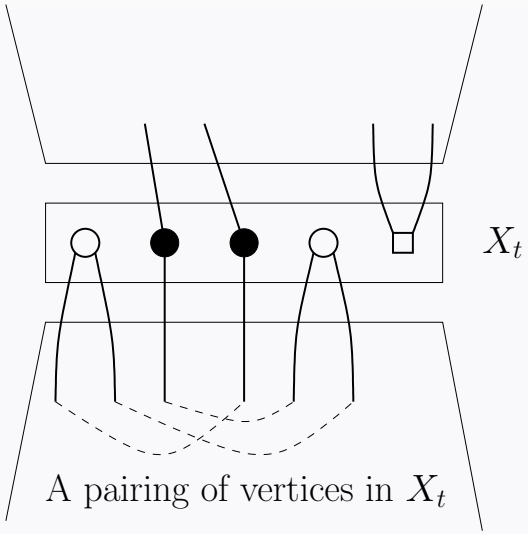


A pairing of vertices in X_t



A pairing of vertices in X_t

A *pairing* of X_t is a graph H (with loops) s.t. $V(G) = X_i$ and $\forall x \in X_i$ $\mathbf{deg}_H(x) \leq 2$



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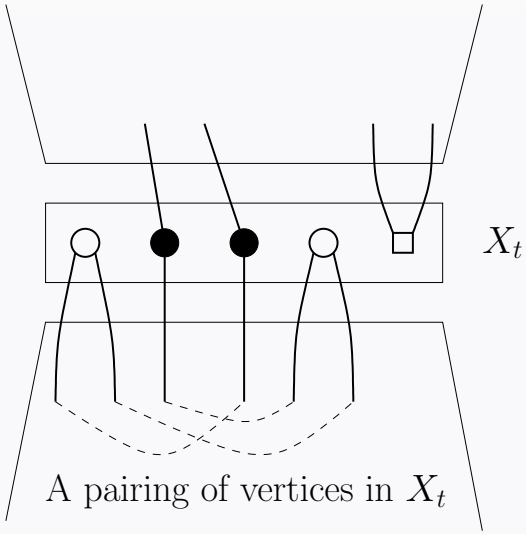

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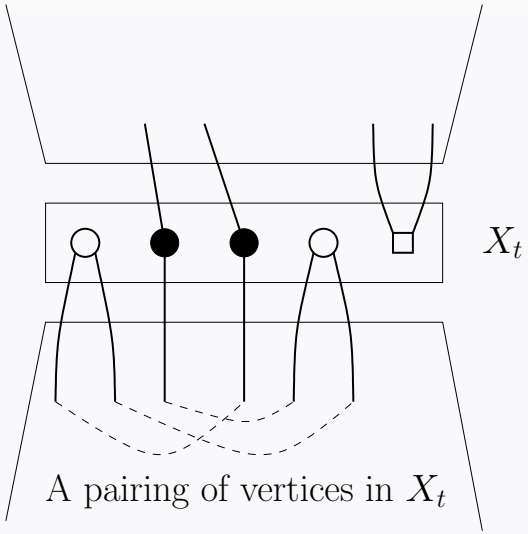

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- The restriction of a cycle to G_t is a collection $\mathcal{P}$ of internally disjoint paths in G_t with ends in X_i.
- Each $\mathcal{P}$ corresponds to some pairing $H_{\mathcal{P}}$ of X_t
- For any set S, let $\texttt{pairs}(S)$ be the set of all pairings of S

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let H_w be just the vertex w looped.

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$\forall H \in \texttt{pairs}(X_i)$,

$$B_t(H) = [H \text{ is the pairing of some } t\text{-path collection } \mathcal{P}]$$

$G = G_r$ has a Hamiltonian cycle iff $B_r(H_w) = 1$

2nd step: tables for leaf nodes:

Let X_l be an leaf node (assume that $X_l = \{y\}$)

Notice that $\texttt{pairs}(t) = \{H_0, H_1\}$

where $H_0(H_1)$ is the vertex y looped (unlooped)

$$\forall H \in \texttt{pairs}(t)\ B_l(H) = [|E(H)| = 0]$$

3rd step: tables for insert nodes:

Let X_t be an insert node

let t' be the child of t and v be the insert vertex.

For any $\forall H \in \texttt{pairs}(t)$ we have

$$B_t(H) = [B_{t'}(H - v)] \wedge \ [N_H(v) \subseteq N_{G_t}(v)]$$

4st step: tables for forget nodes:

Let X_t be an forget node

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For any $\forall H \in \texttt{pairs}(t)$ we have

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Our next step is to show that tw-Planar Hamiltonian Cycle $\in 2^{O(k)}$-FPT

Part 3, Tuesday 10/05/2016 - 16:00–17:00 (60')

Branch decompositions

Sphere cut decompositions

Dynamic programming on planar graphs

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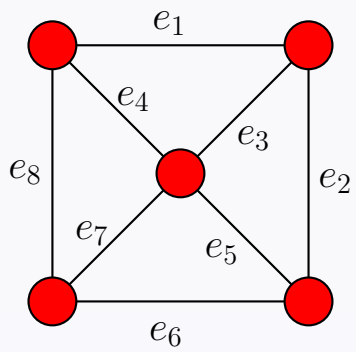
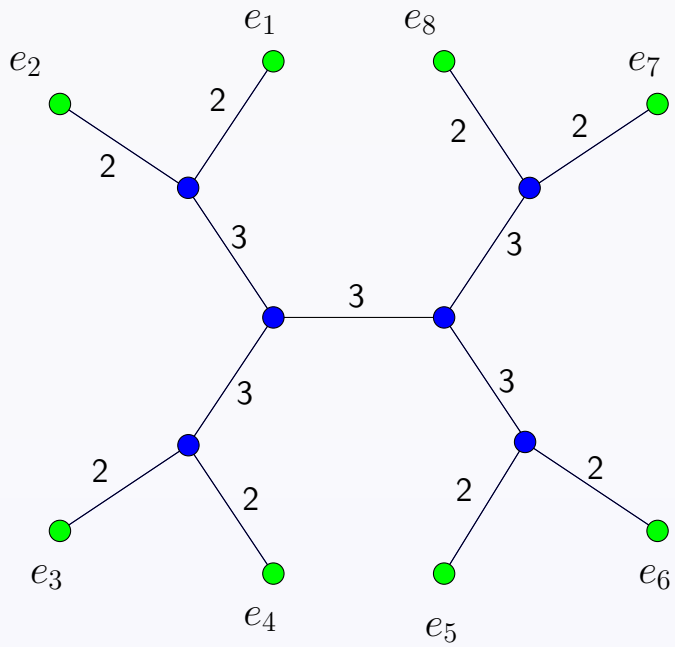
if T_1 is one of the connected components of $T - e$ then we set

$E_e = \tau^{-1}(\text{leaves of } T_1)$ and $\mathbf{mid}(e) = \partial E_e$.

A graph G and a branch decomposition of it.





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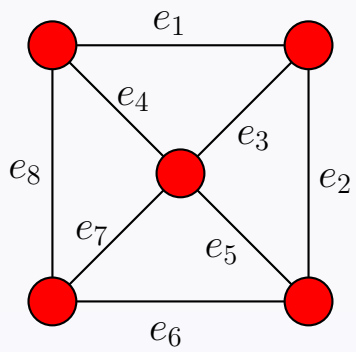

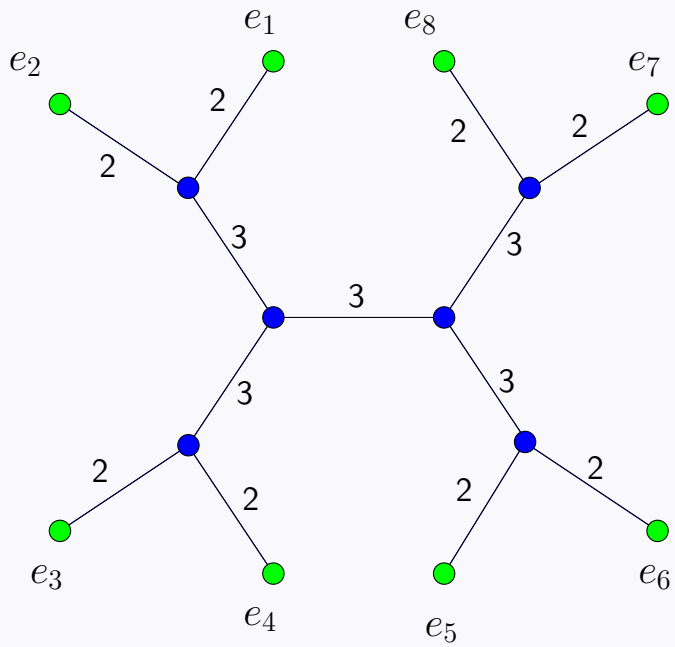

The *width* of a branch decomposition (T, τ) is $\max\{|\mathbf{mid}(e)| \mid e \in E(T)\}$

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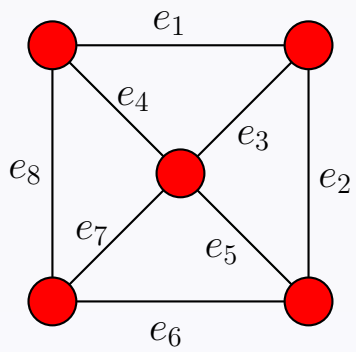

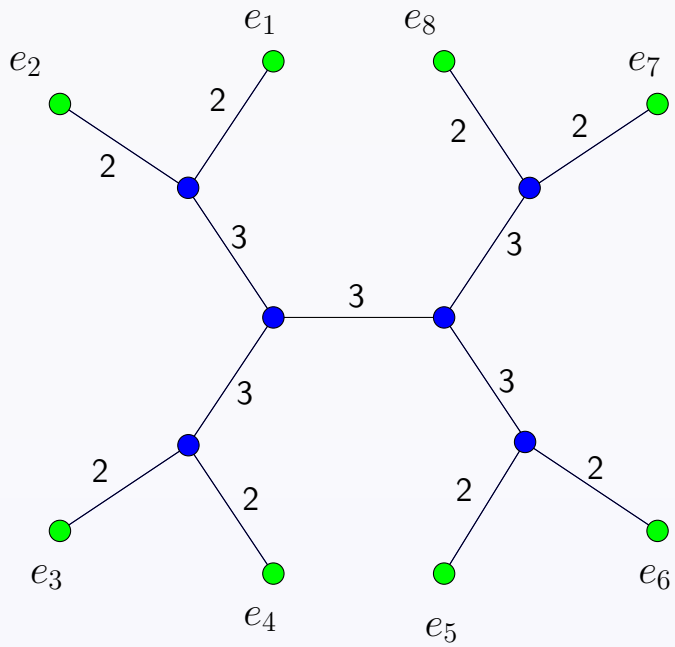

The *width* of a branch decomposition (T, τ) is $\max\{|\mathbf{mid}(e)| \mid e \in E(T)\}$

The *branchwidth*, $\mathbf{bw}(G)$, of a graph G is then minimum width a branch decomposition of G may have.

Combinatorics of branchwidth

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Theorem: [Robertson and Seymour, GM-10] *If G is not acyclic, then*

$\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(\mathbf{G})$

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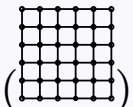
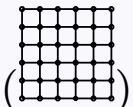
Theorem: [Robertson and Seymour, GM-10] *If G is not acyclic, then*

$\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(\mathbf{G})$

If T is a tree, then $0 \leq \mathbf{bw}(G) \leq 2$.

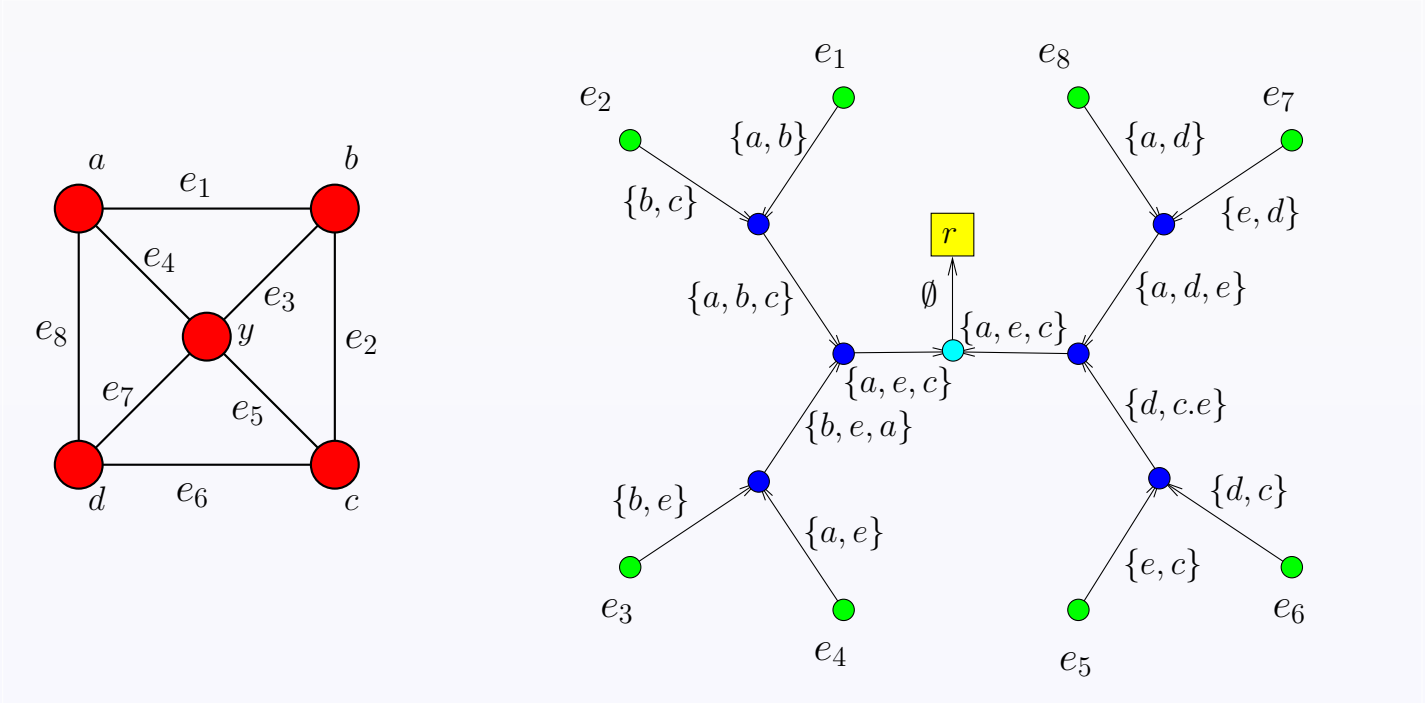
$\mathbf{tw}($  $) = \mathbf{bw}($  $) = 6$

$\mathbf{bw}(K_6) = 4 < \mathbf{tw}(K_6) = 5$

Dynamic programming for graphs of small branchwidth

Given a branch decomposition (T, τ), (of small width)

1. Root T to some vertex r without preimage



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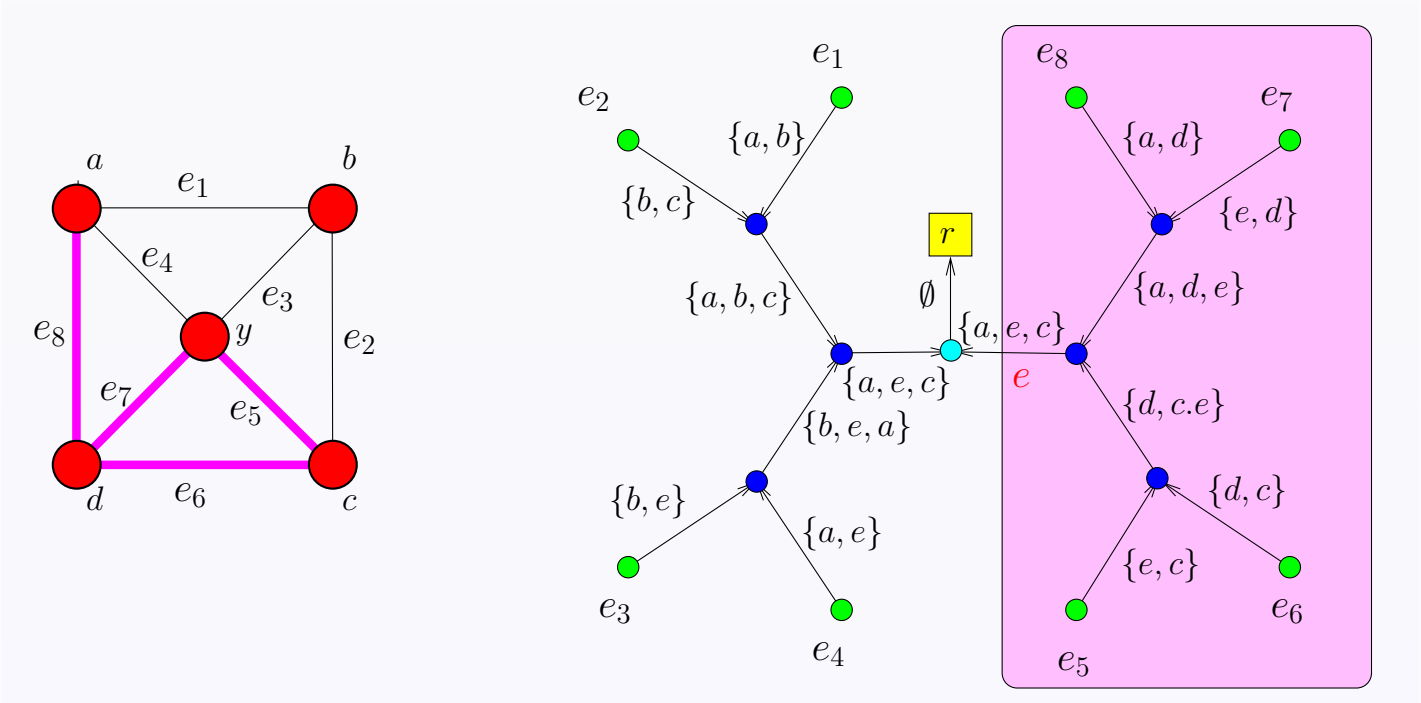

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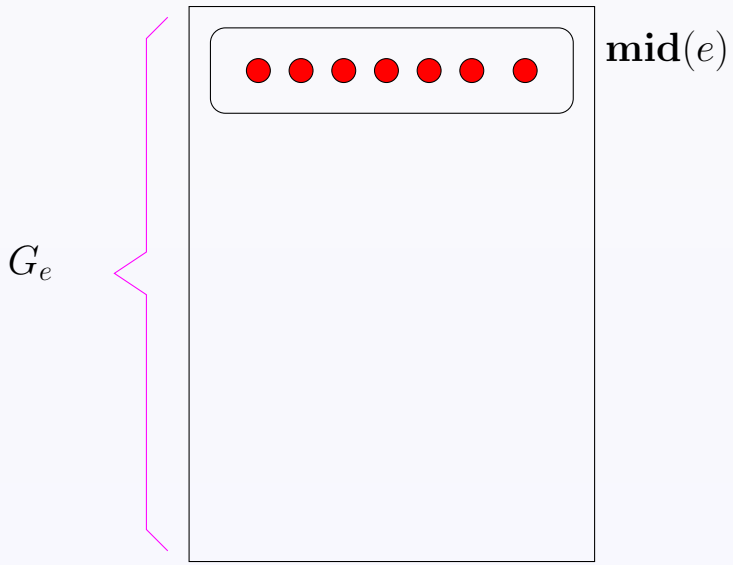
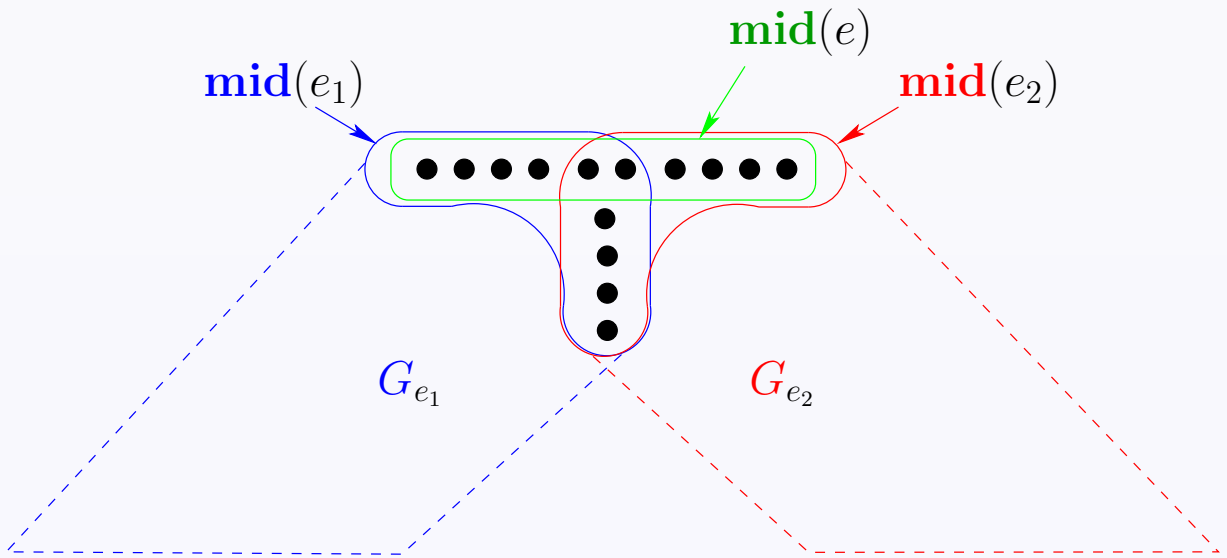
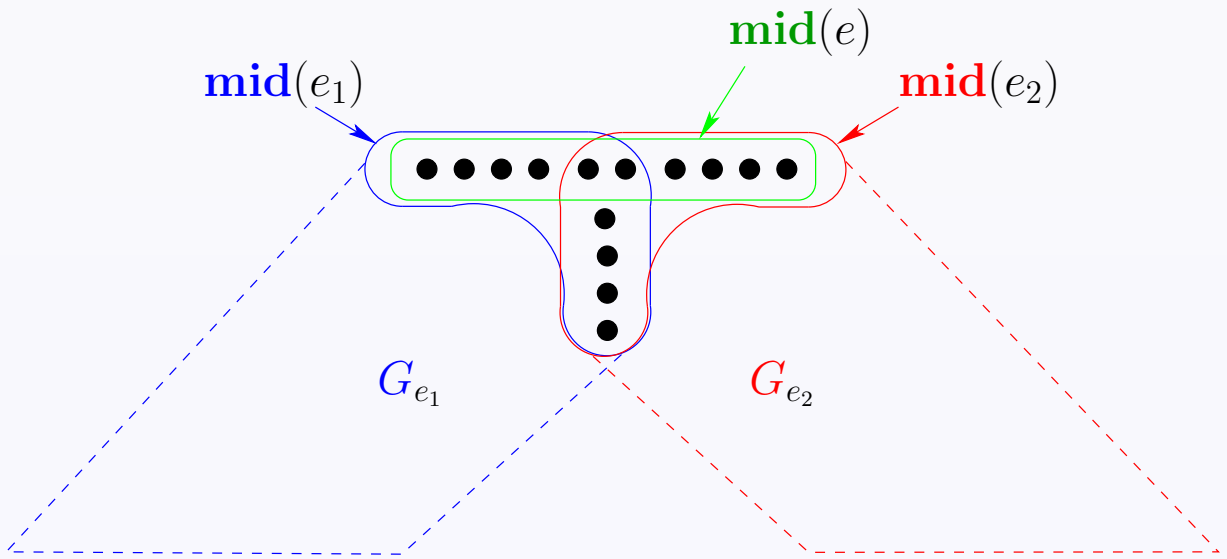
2. Define, for each $e \in E(T)$, a table encoding the information of a partial solution for G_e as restricted to $\mathbf{mid}(e)$. The values of this table for the root node should provide a global answer.



3. Define the values of this table for the leaf nodes



3. Define the values of this table for the leaf nodes
4. Provide the way to compute the table of an edge using the tables of its children edge.



An example: Vertex Cover

Let G be a graph and $X, X' \subseteq V(G)$ where $X \cap X' = \emptyset$.

We say that $\mathbf{vc}(G, X, X') \leq k$ if G contains a vertex cover S where $|S| \leq k$ and $X \subseteq S \subseteq V(G) \setminus X'$.

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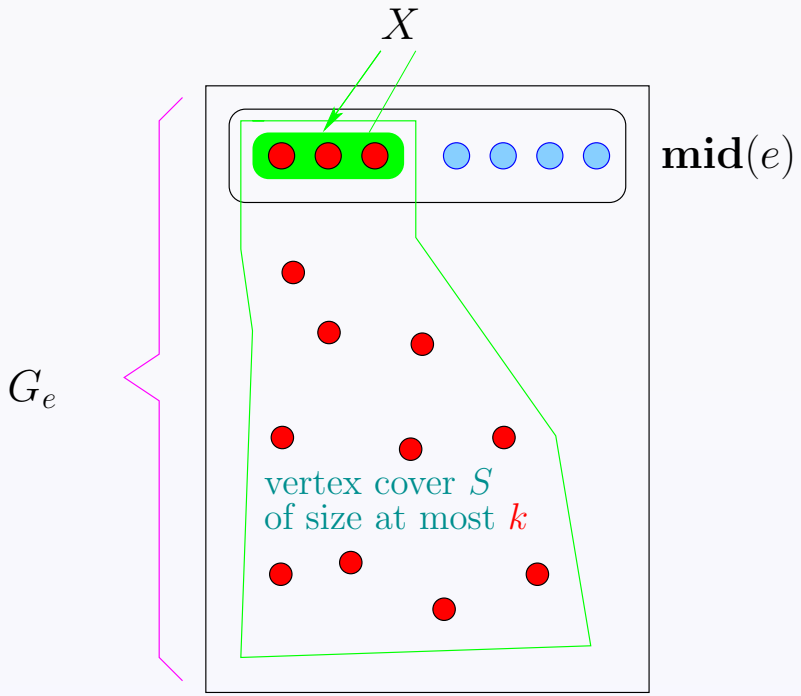

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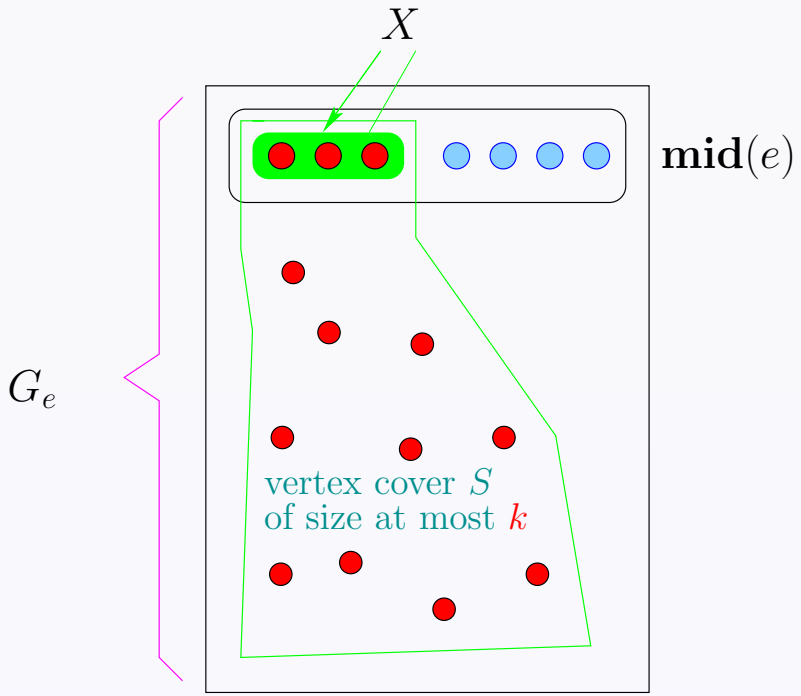

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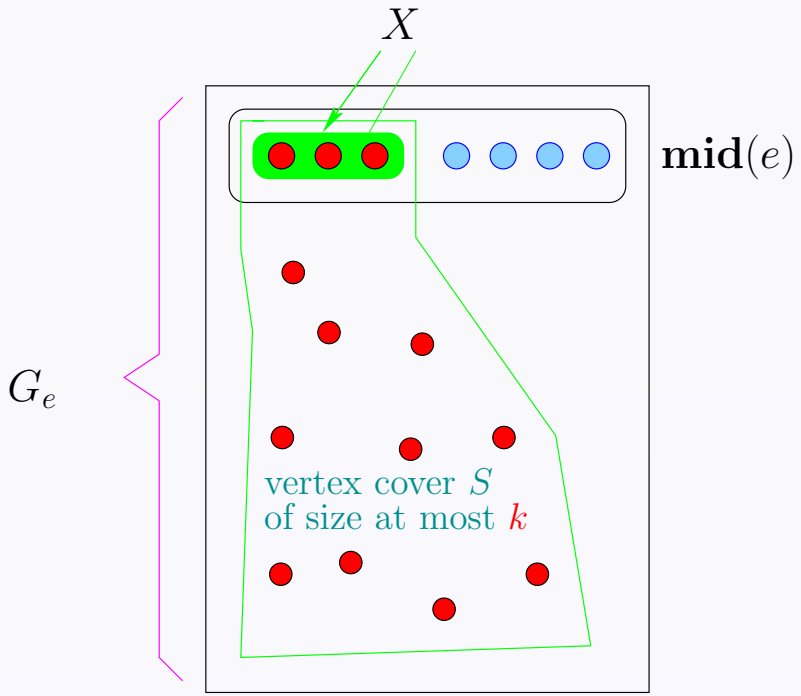

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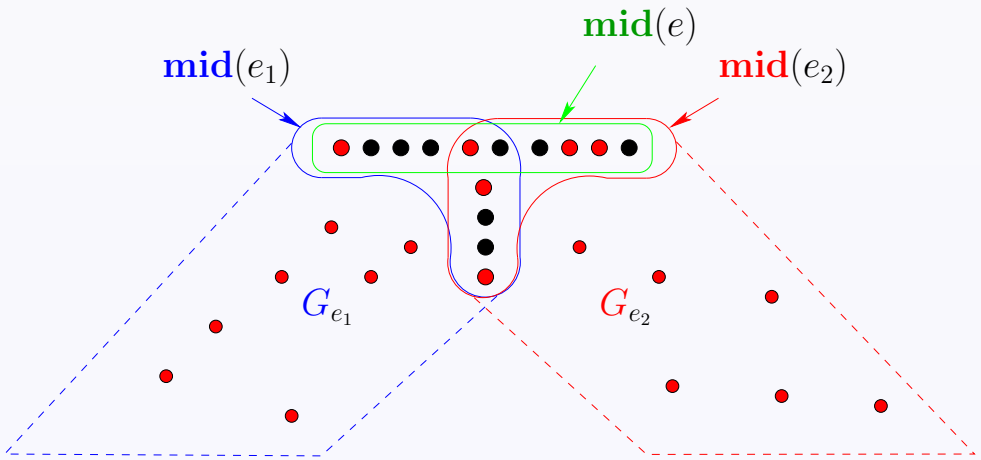

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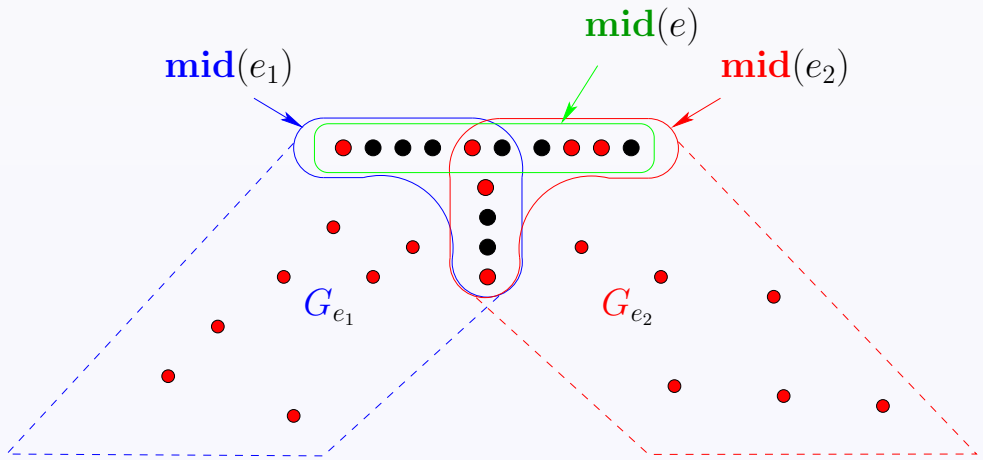

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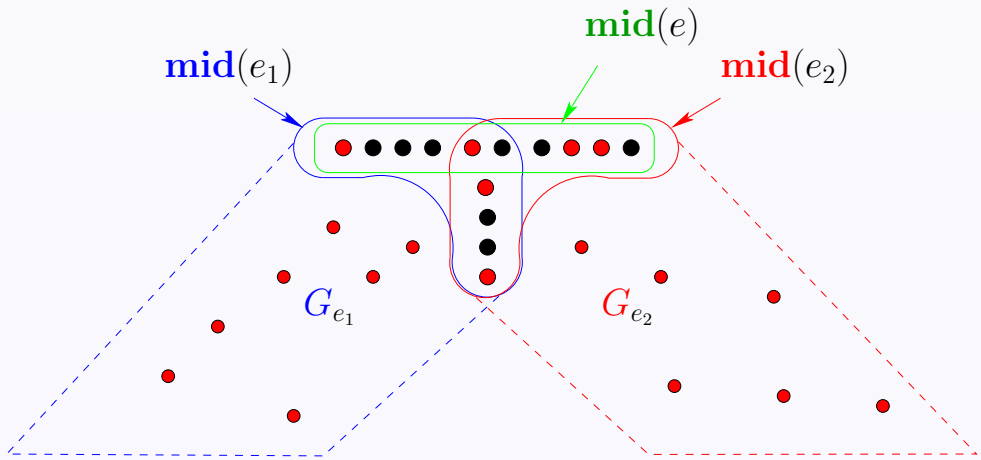

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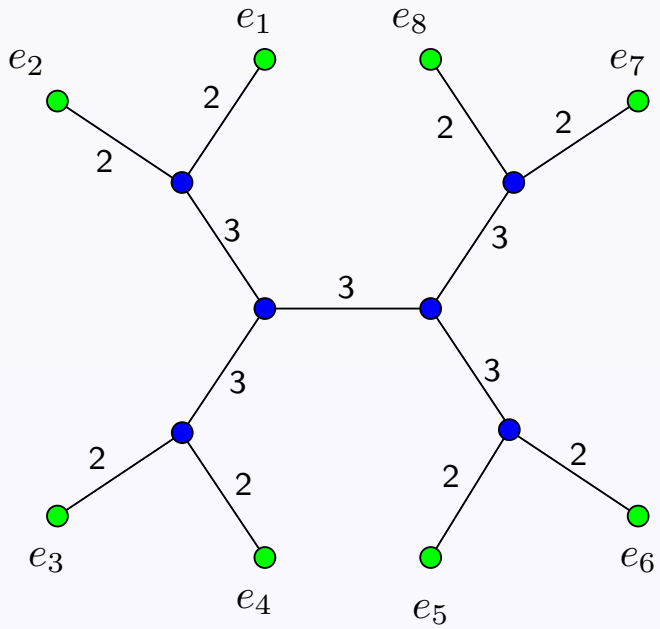

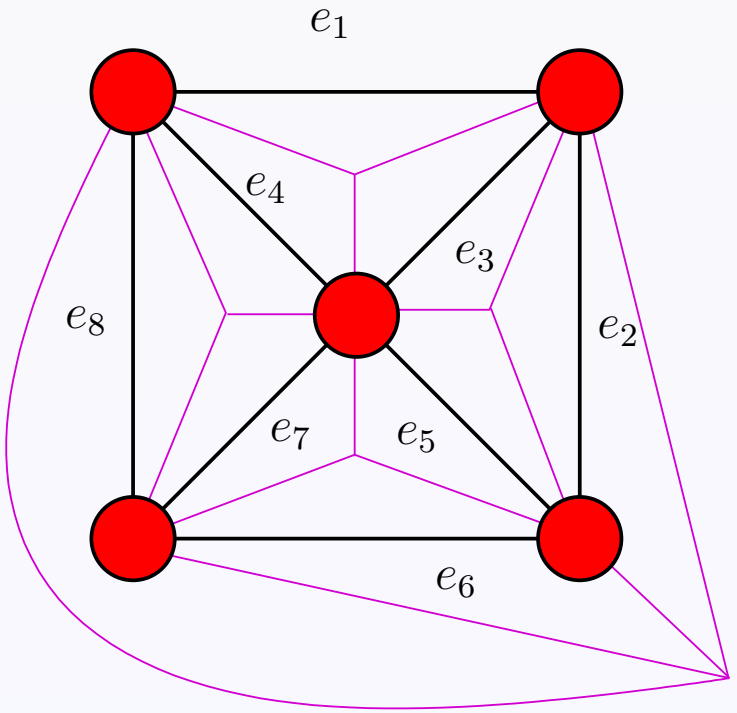

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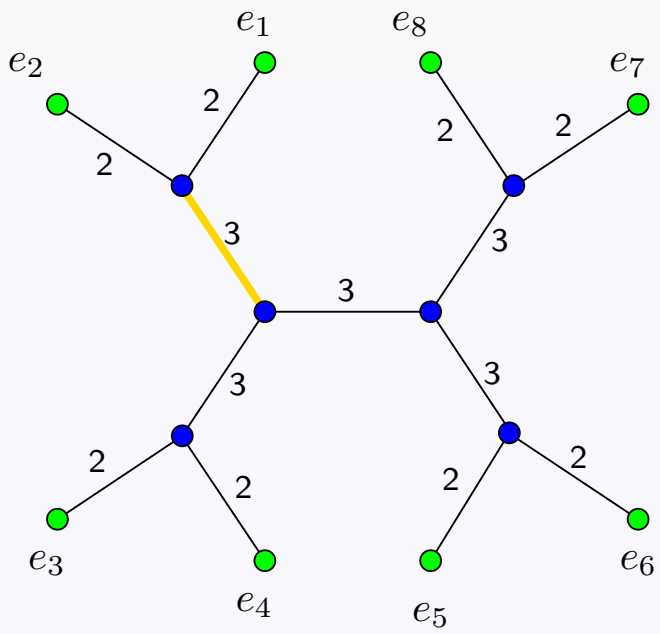

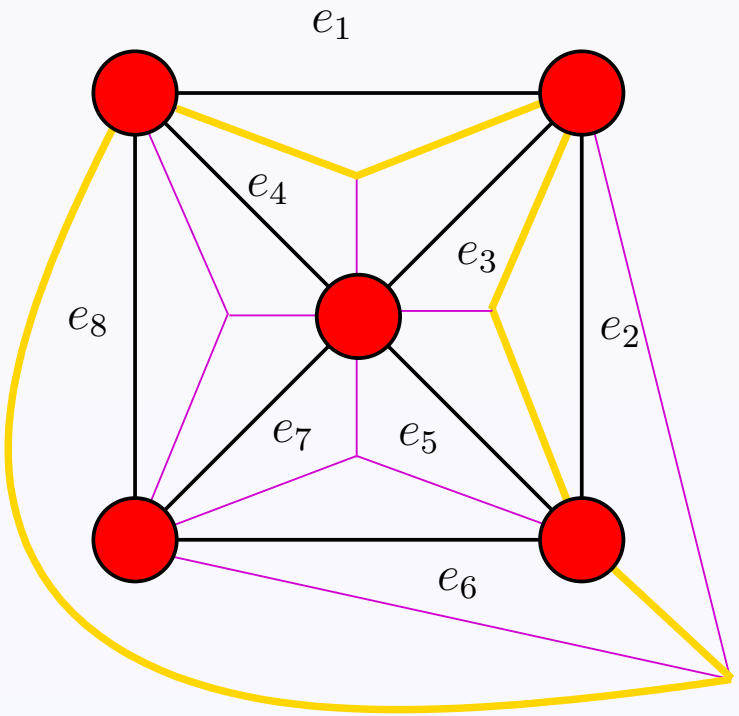

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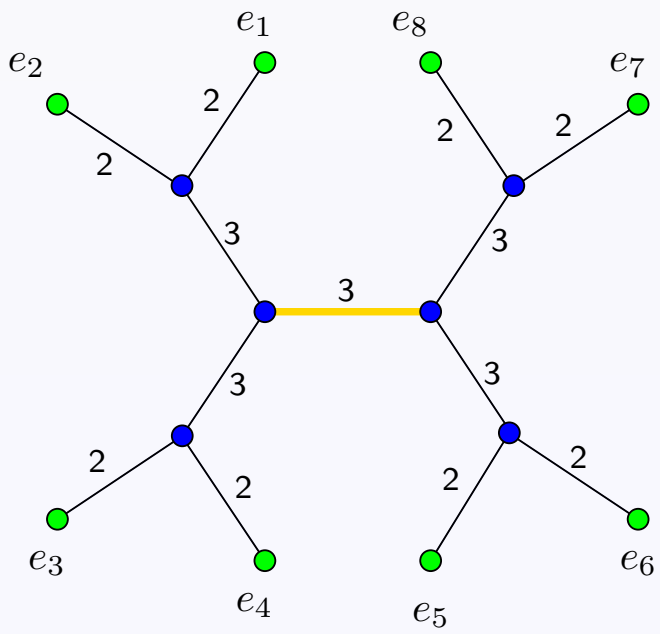

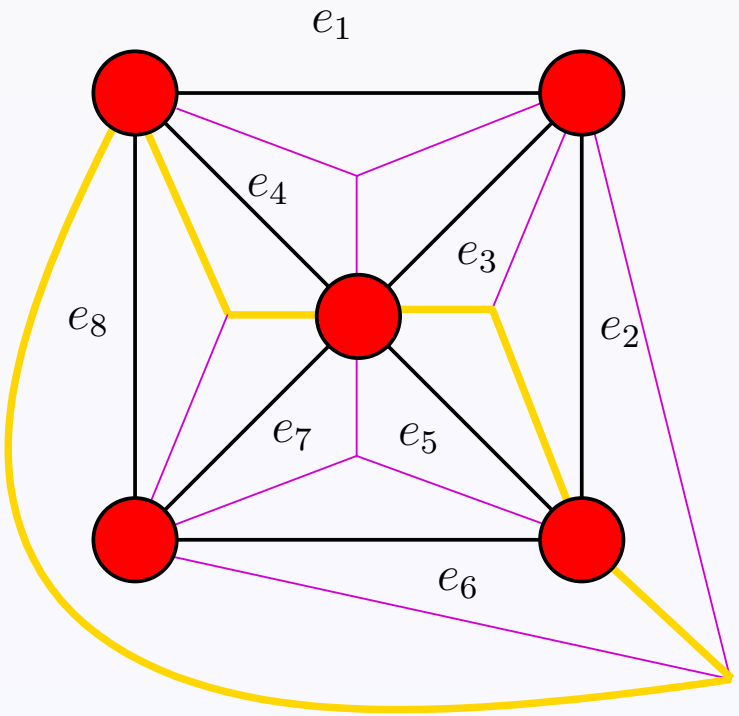

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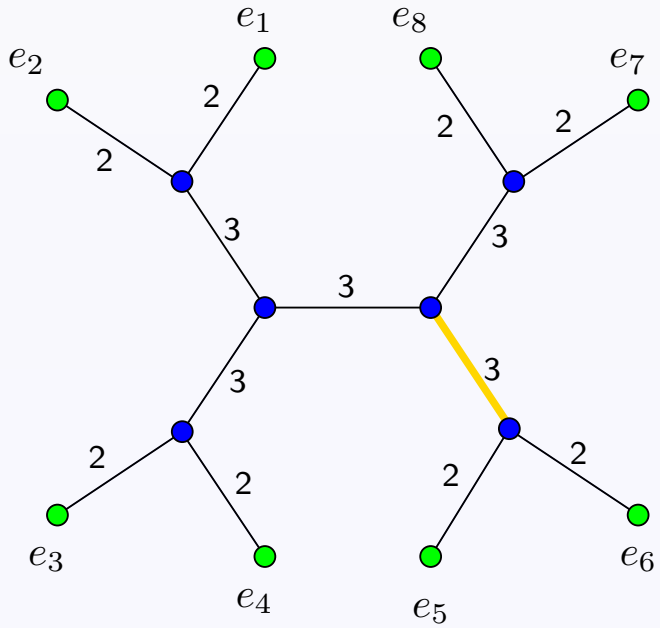

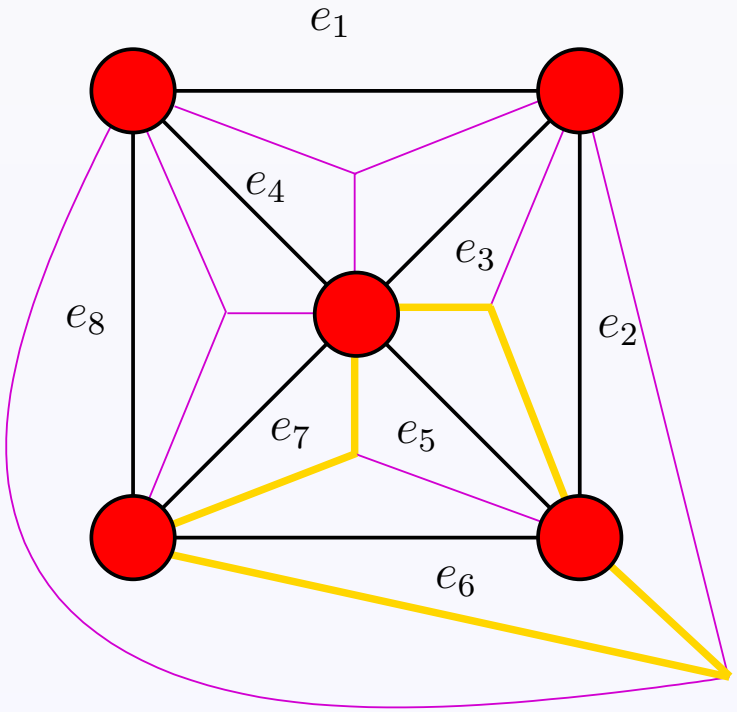

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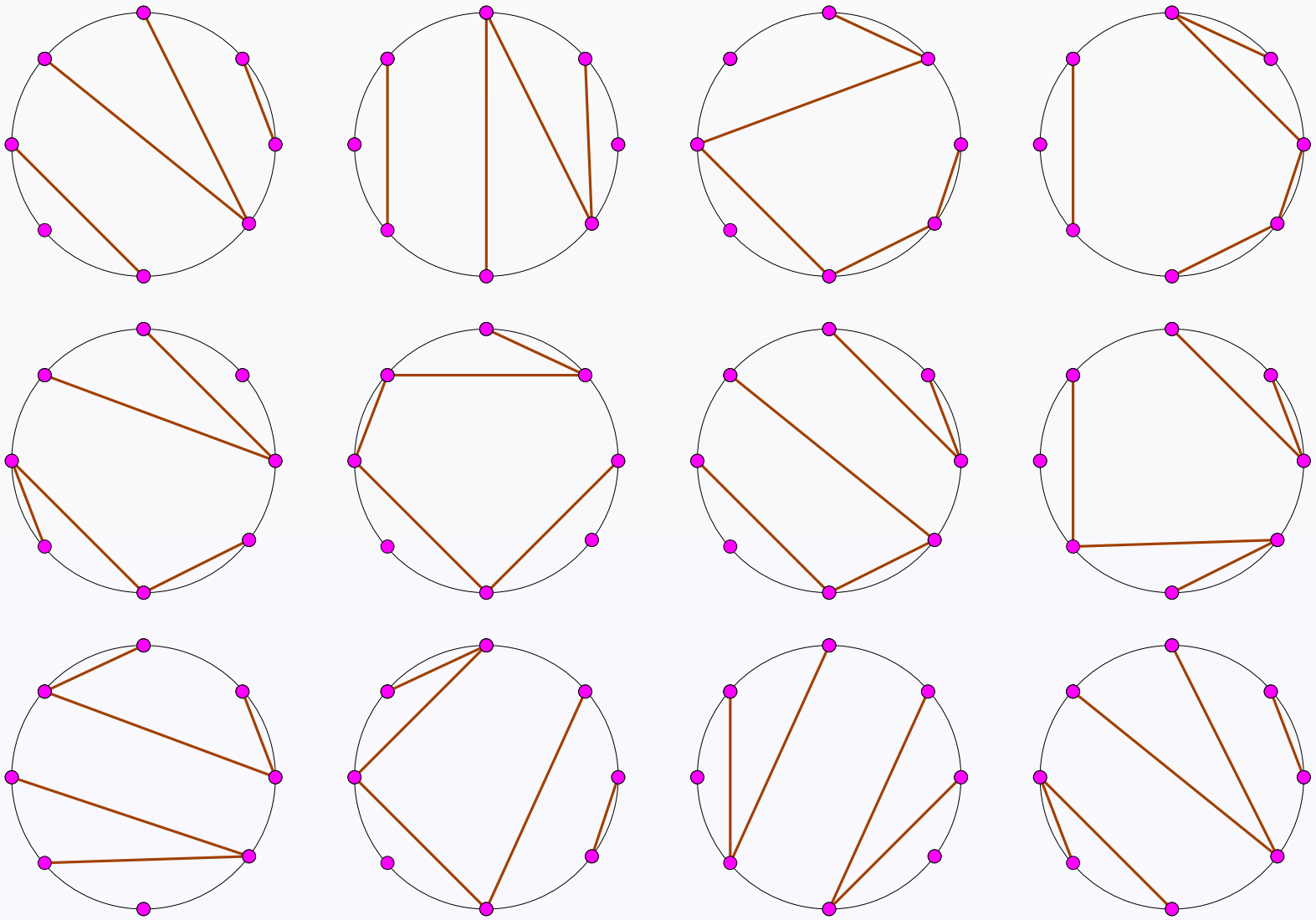
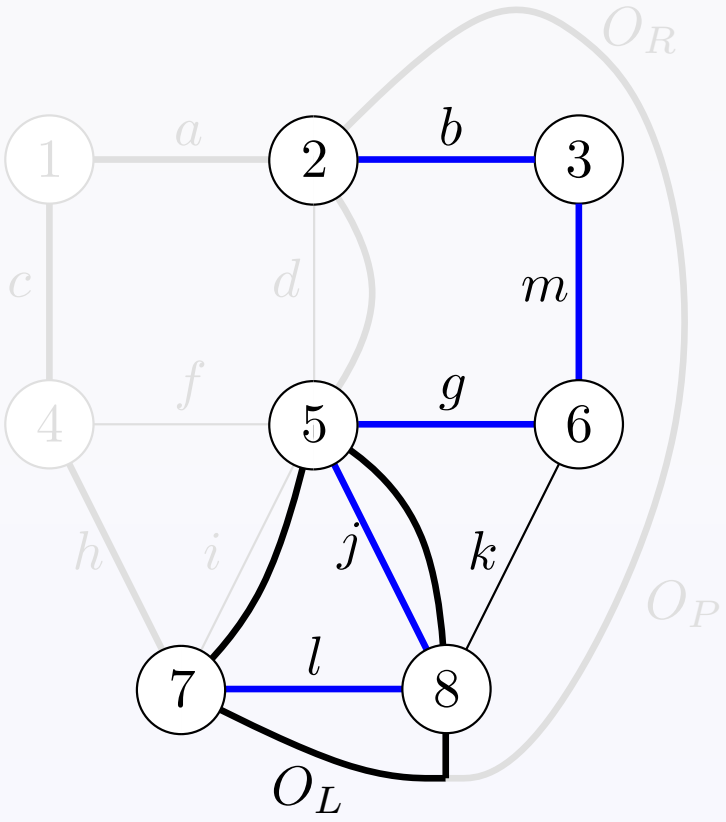
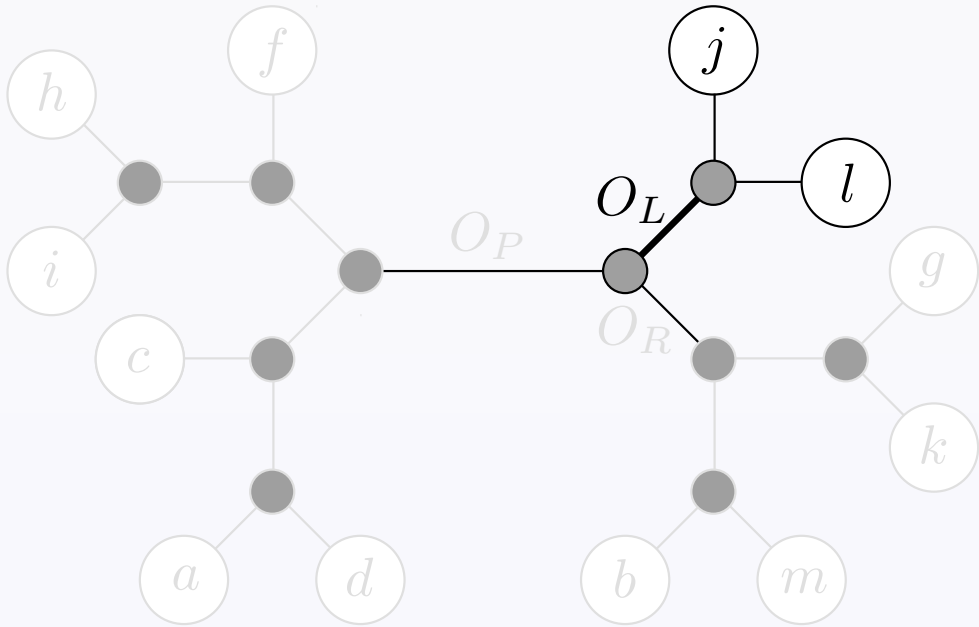
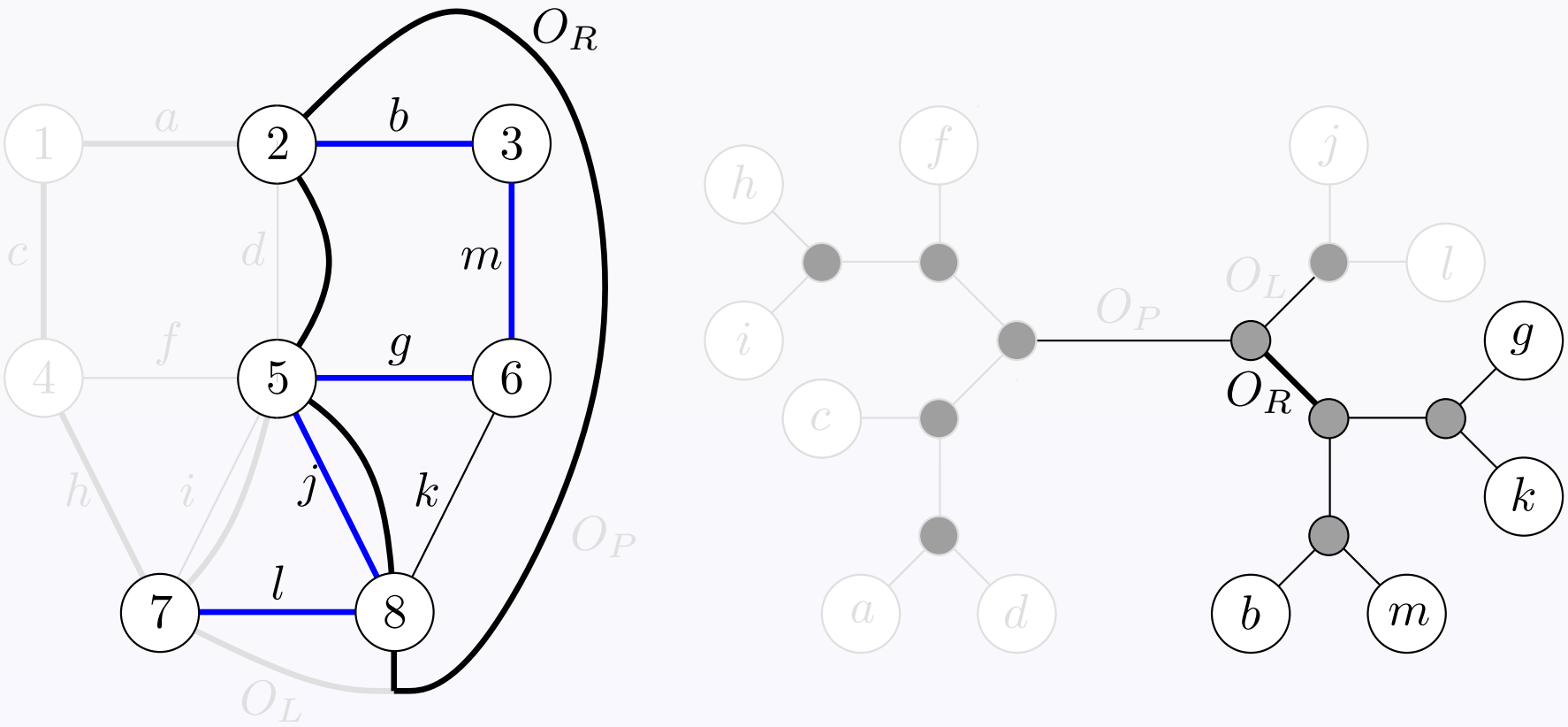


Non crossing pairings





The two nooses O_L and O_R of the two children for a nose O_P for the parent.



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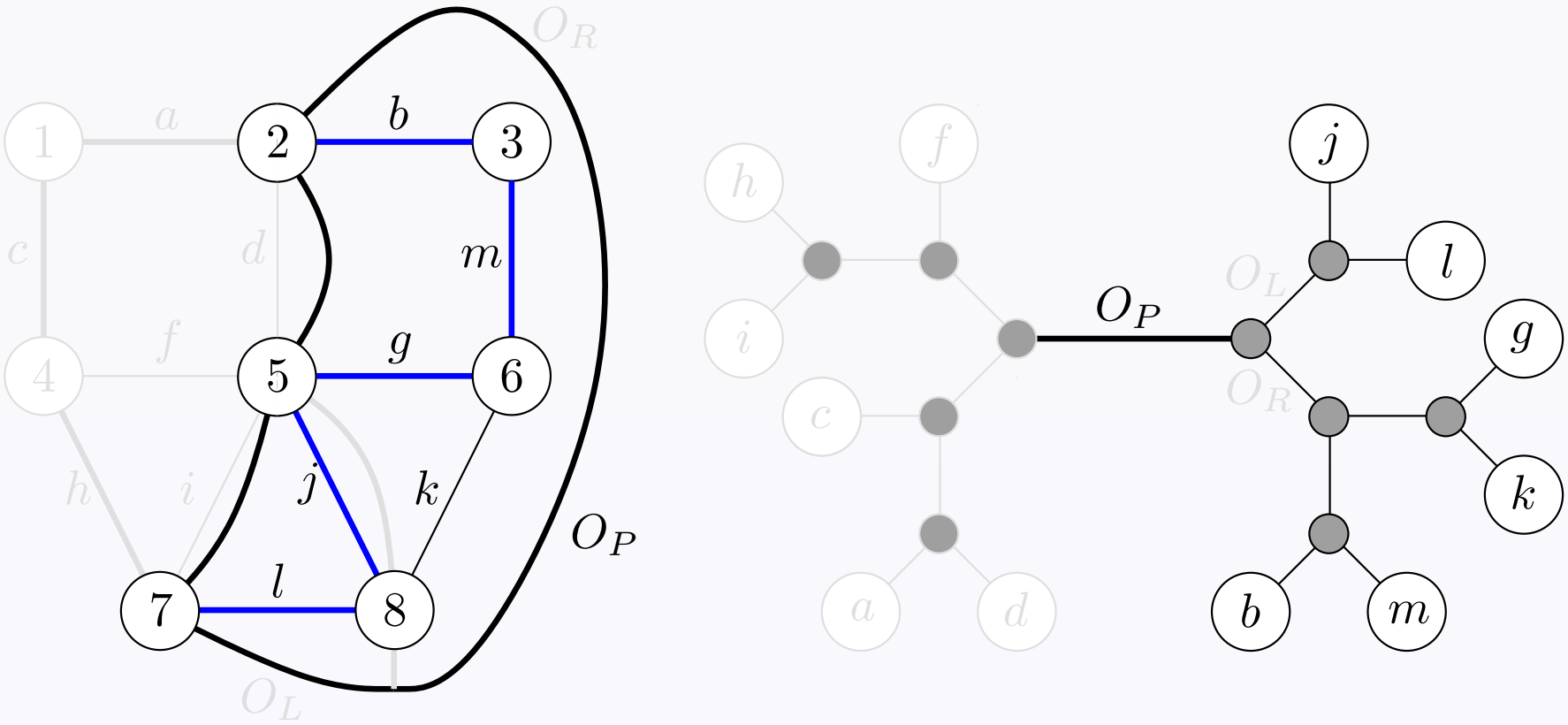

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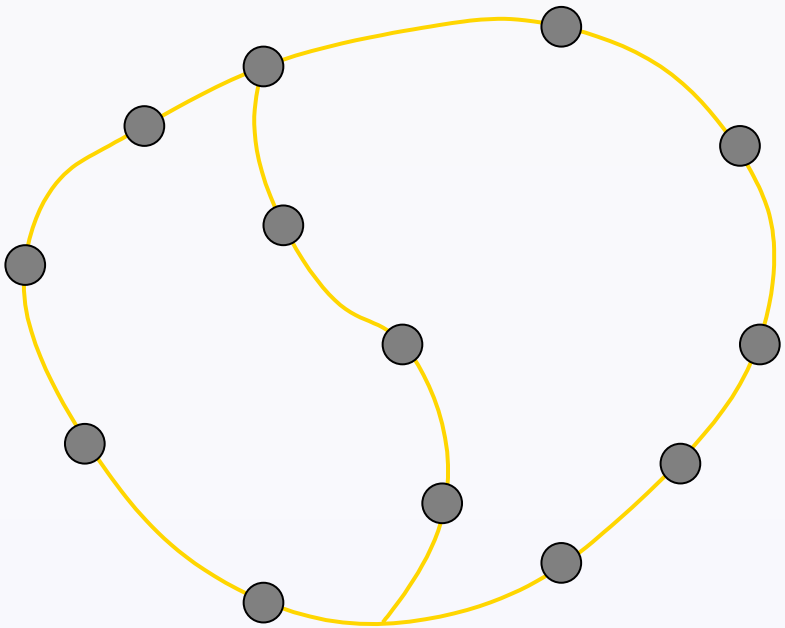

O_P

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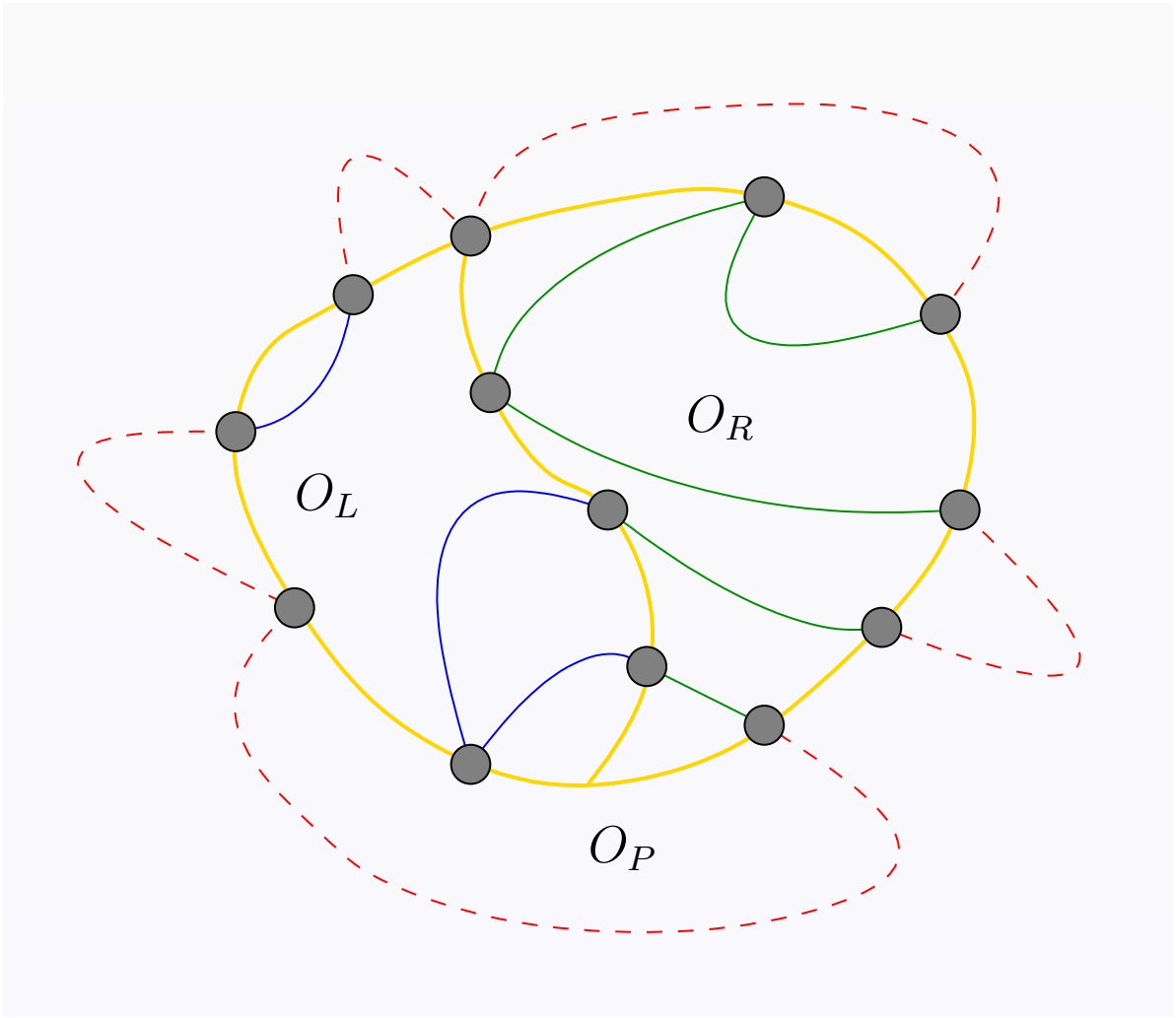

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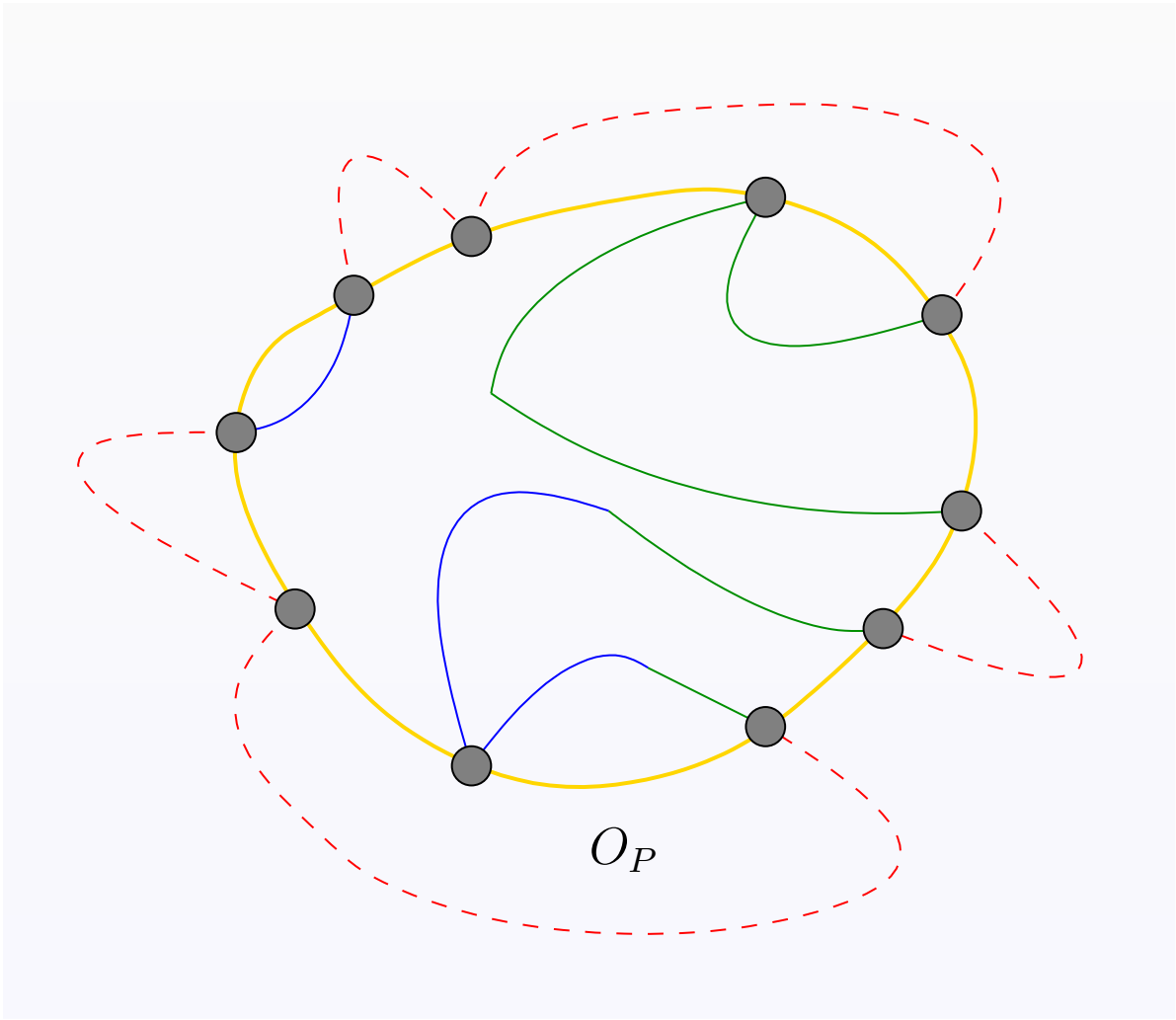

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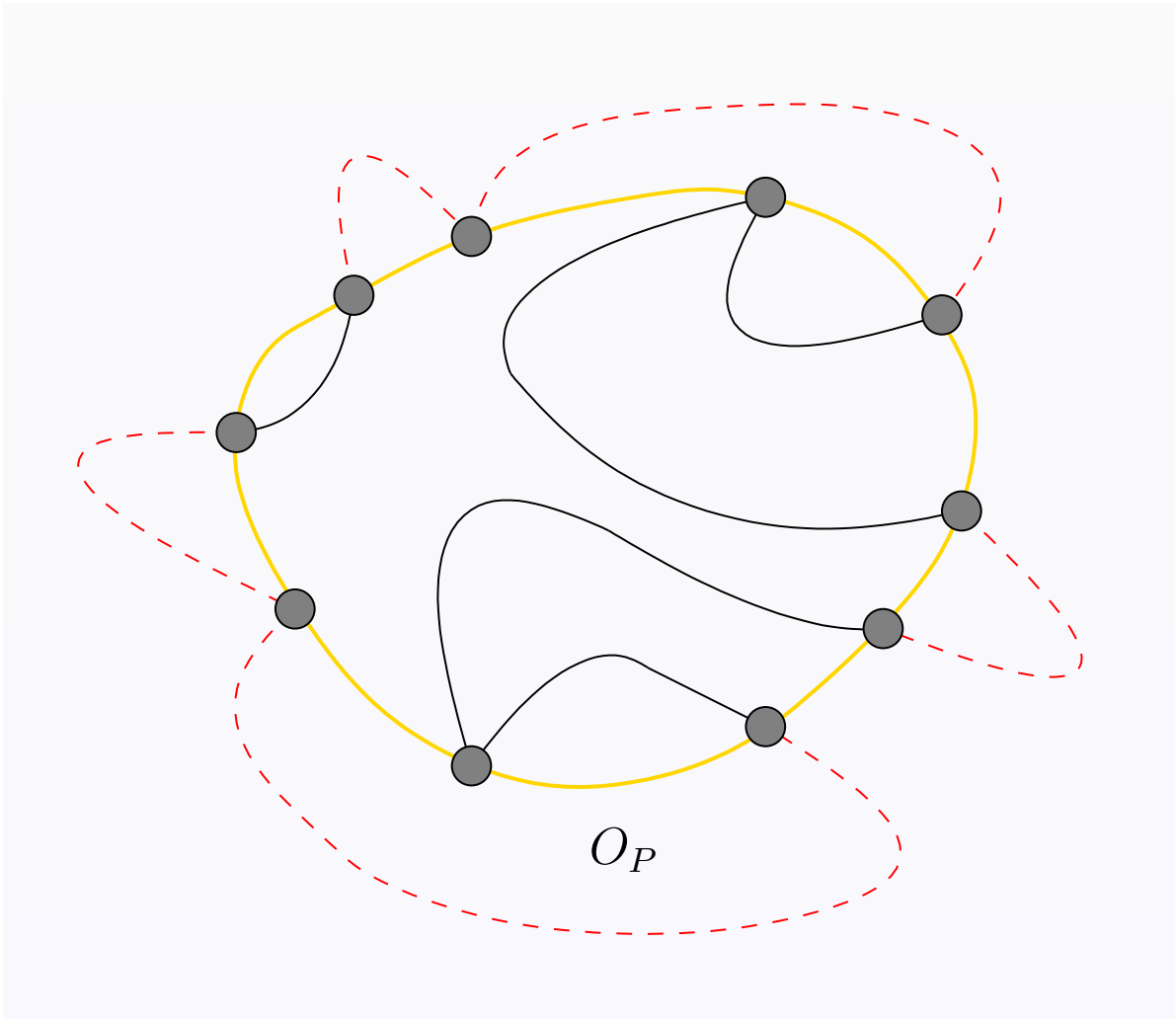

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Therefore: dynamic programming for Hamiltonian Cycle of a planar graph G on a sphere cut decompositions of G with width $\leq k$ takes $2^{O(k)} \cdot O(n)$ steps.

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The idea of using properties of the embedding (for pairings) has been extended for bounded genus graphs in [Dorn, Fomin, and Thilikos. SWAT 2006]

and for H-minor-free graphs in [Dorn, Fomin, and Thilikos. SODA 2008]

Common idea: Planarization

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Extensions/alternatives:

- For H-minor free graphs: [Sau, Rué, Thilikos, COCOON 2012]
- Surface split decompositions: [Bonsma, STACS 2012]
- Brick Decompositions: [Cohen-Addad & de Mesmay, ESA 2015]

Part 4, Thursday 10/05/2016 - 11:00–12:30 (90')

Bidimensionality

Subexponential parameterized algorithms

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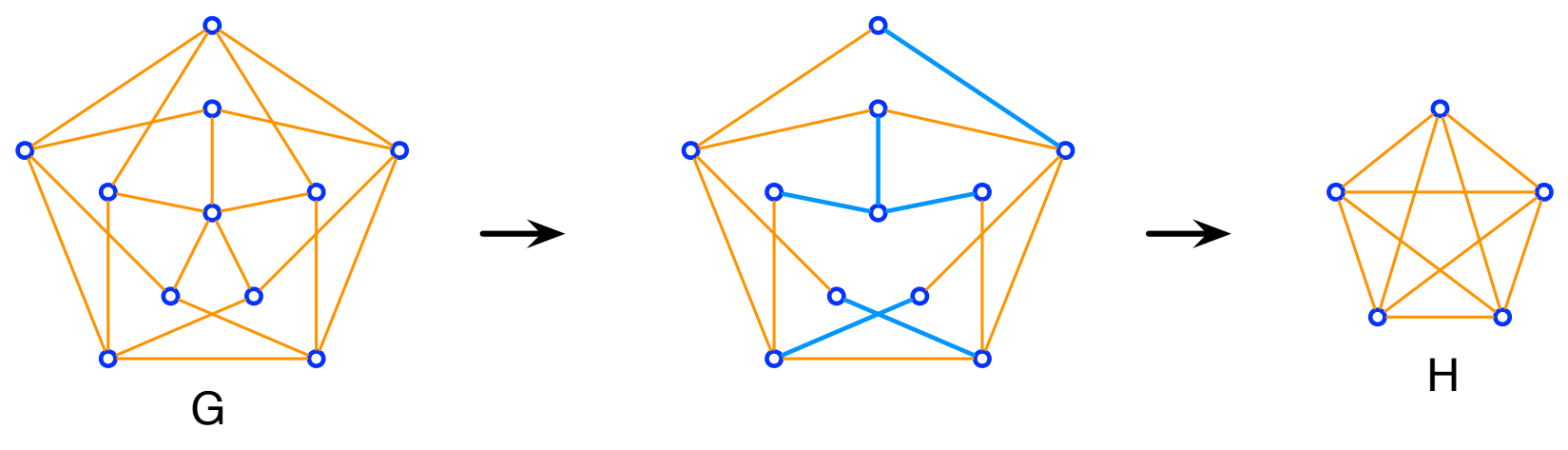

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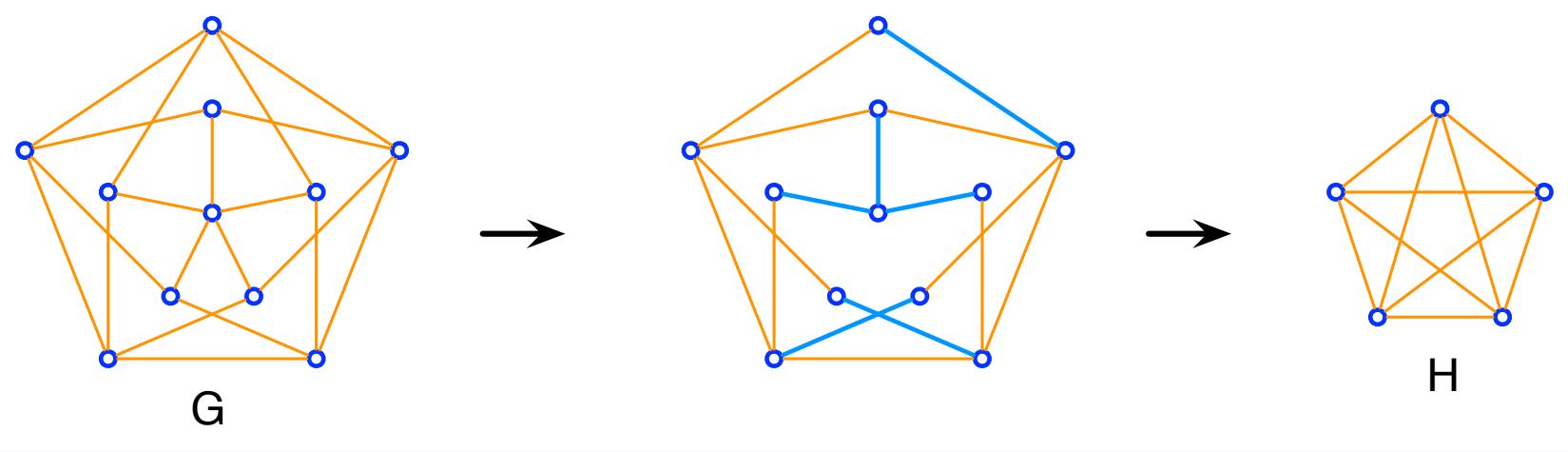

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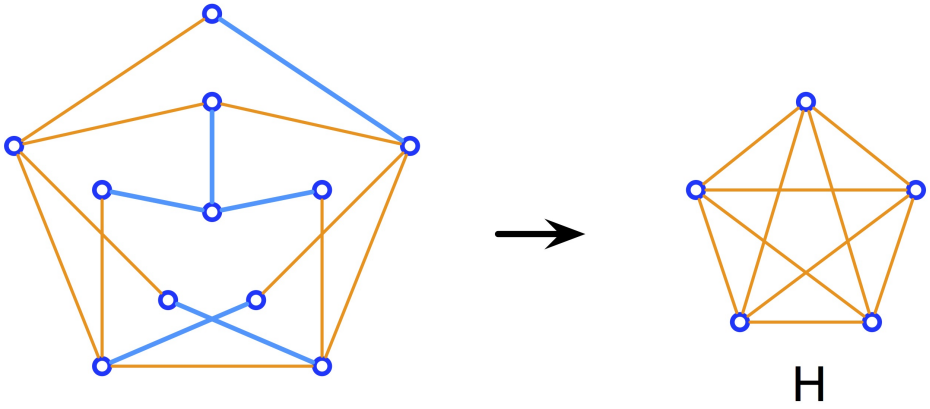

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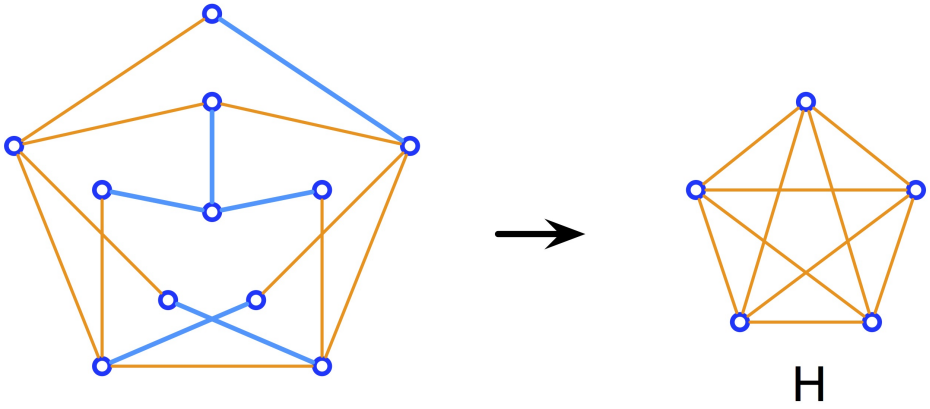

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[S is a scattered: no two distinct vertices of S have a common neighbor.]

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These are YES-instances for minimization problems and

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In other words:

There exists an algorithm that solves p-Checking Value of $\mathbf{p}$ in $f(k) \cdot n^3$ steps.

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- Can we derive results for problems closed under other relations?

Comments on P versus NP: from CS and Maths

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NP-completeness: A Retrospective

Christos H. Papadimitriou*

University of California, Berkeley, USA

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Mathematical Problems for the Next Century[1]

Steve Smale

Department of Mathematics
City University of Hong Kong
Kowloon, Hong Kong

August 7, 1998

"P versus NP – a gift to mathematics from computer science." Steve Smale, 1998

A metaphore on graph minors

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General requirements

We need the following two facts (for suitable functions f and G):

1. [Combinatorial] $\mathbf{p}(G) \leq k \Rightarrow \mathbf{tw}(G) \leq f(k)$

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Proof:

This algorithm first checks whether $\mathbf{tw}(G) \leq f(k)$.

If the answer is negative, then outputs a negative/positive answer (by **1**).

If the answer is positive, then runs DP algorithm (by **2**).

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$\boxed{2^{O(\sqrt{k})} \cdot n^{O(1)}}$ for their planar restrictions $\longrightarrow$ **when** can we match this?

- Here we care about such questions!

Exponential Time Hypothesis (ETH):

There is no $2^{o(n)}$-step algorithm that solves 3-SAT

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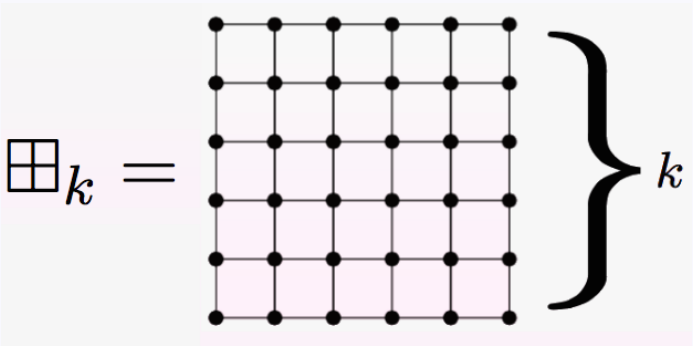
The $(k \times k)$-grid



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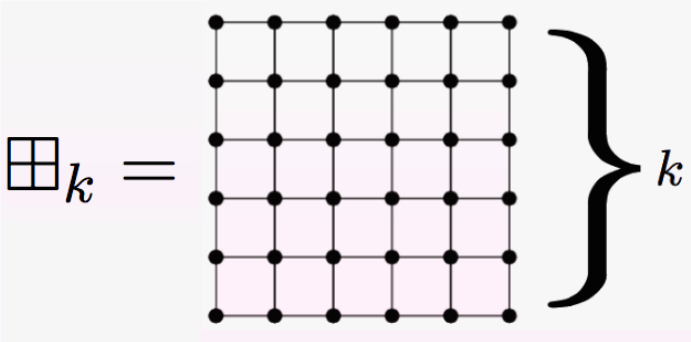

This fact was first proved in GM-V by Robertson and Seymour.

Theorem: [Robertson & Seymour – GM-V]

There is a $\delta : \mathbb{N} \to \mathbb{N}$ *such that* $\forall \alpha \quad \mathbf{tw}(G) > \delta(\alpha) \Rightarrow G \geq_m \boxplus_\alpha$.

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- The best known lower bound is $\delta(k) = \Omega(k^2 \cdot \log k)$

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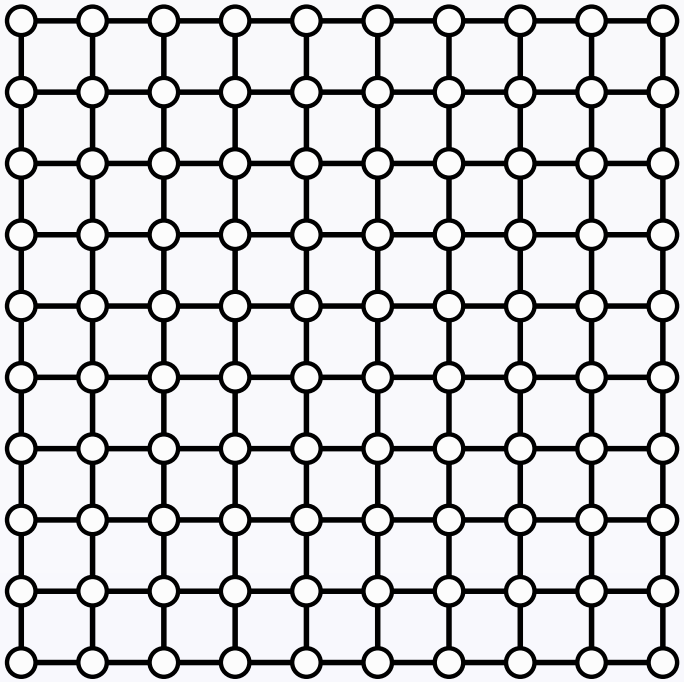
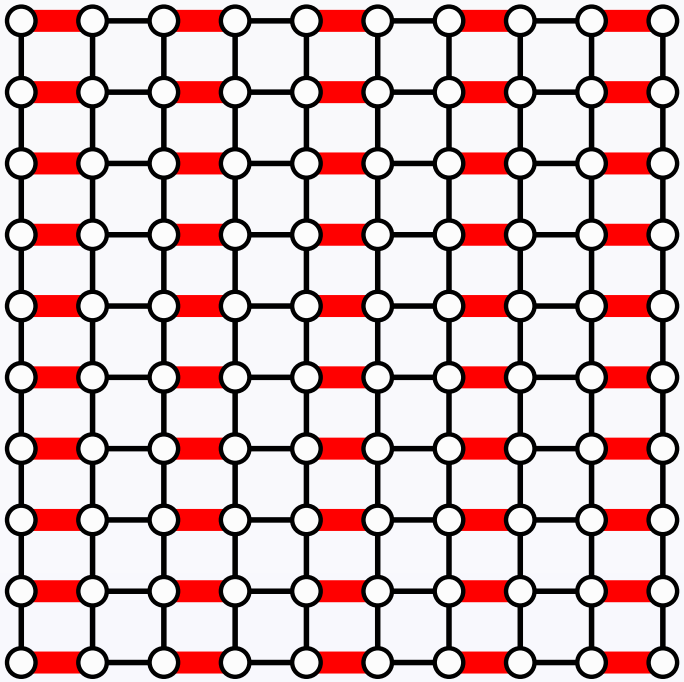
(1) and **(3)** $\Rightarrow k < \mathbf{p}(G)$.





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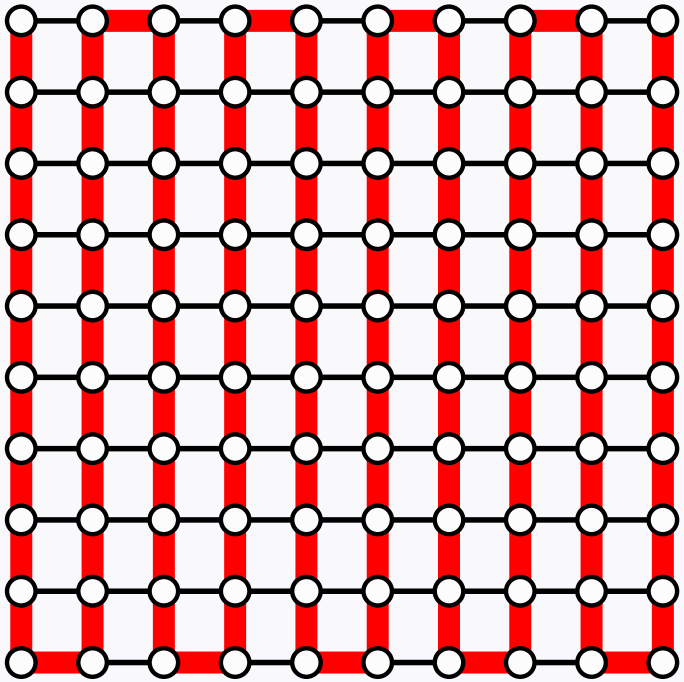

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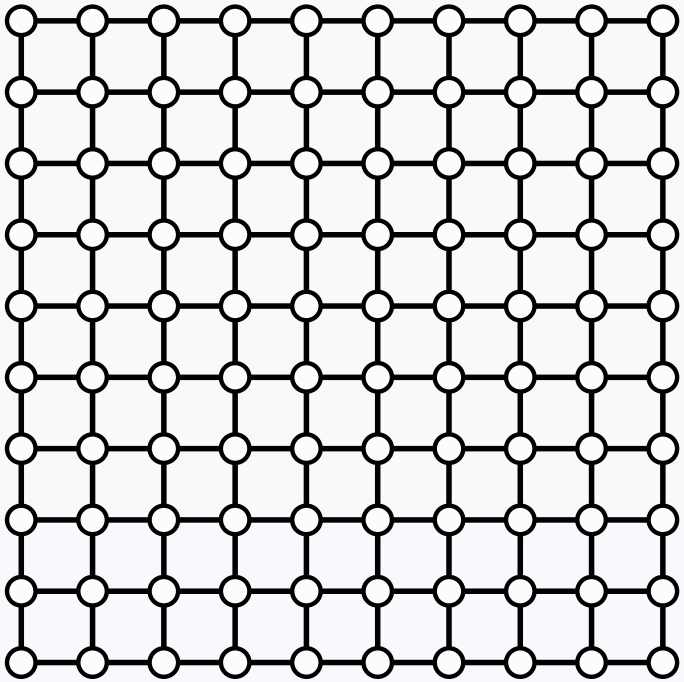

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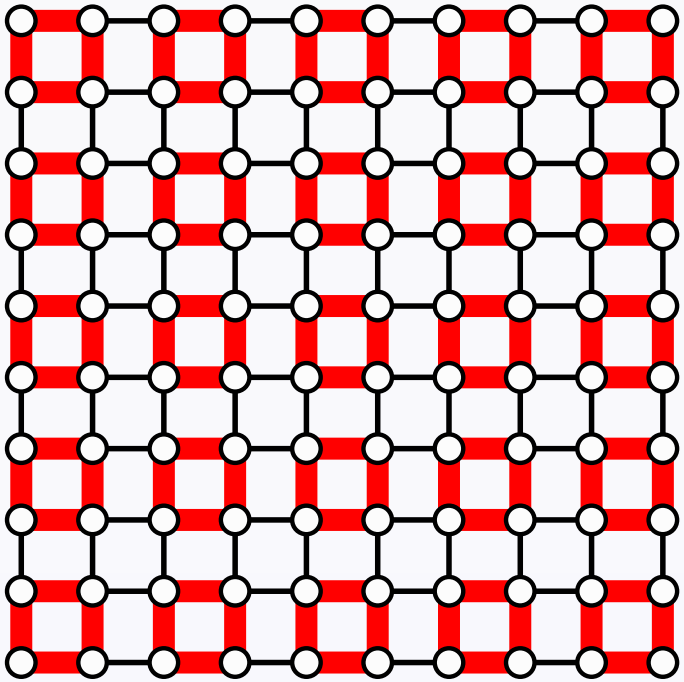

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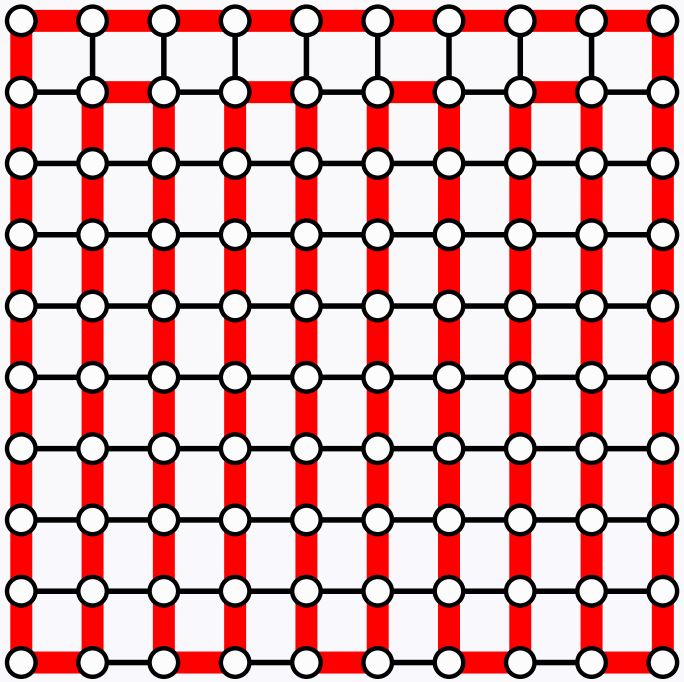

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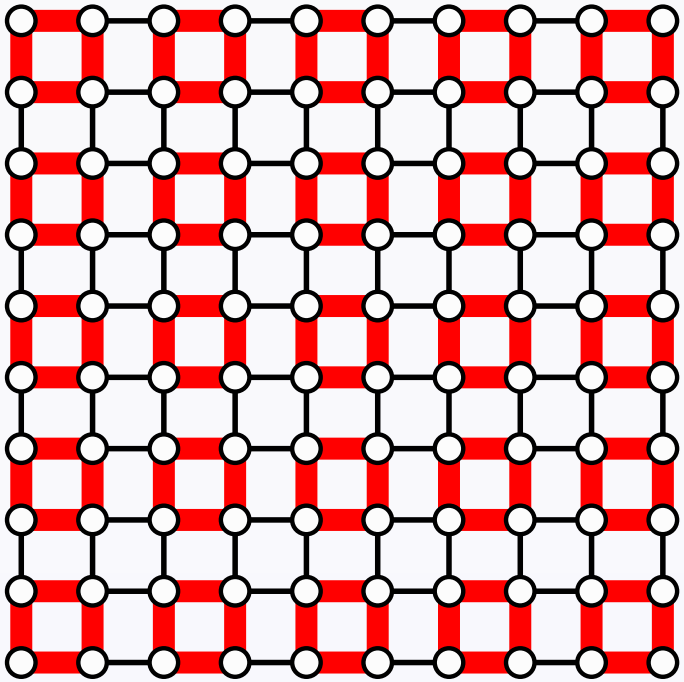

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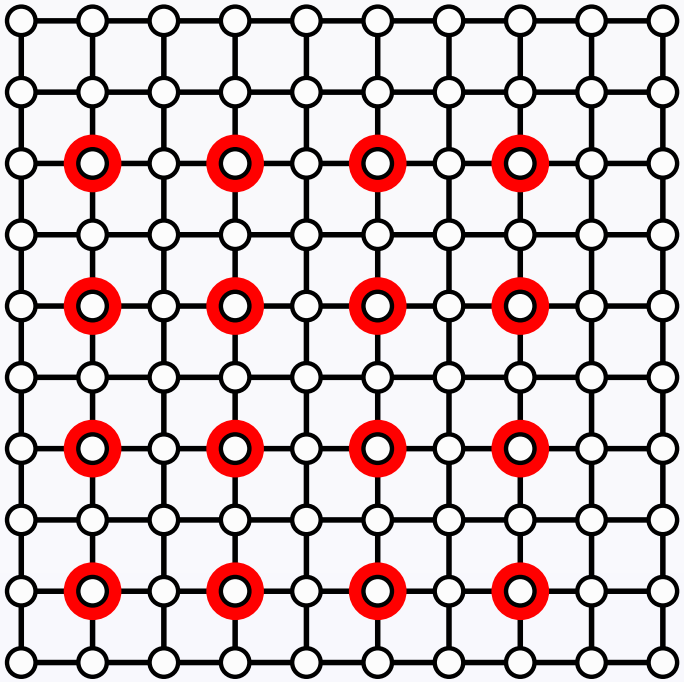

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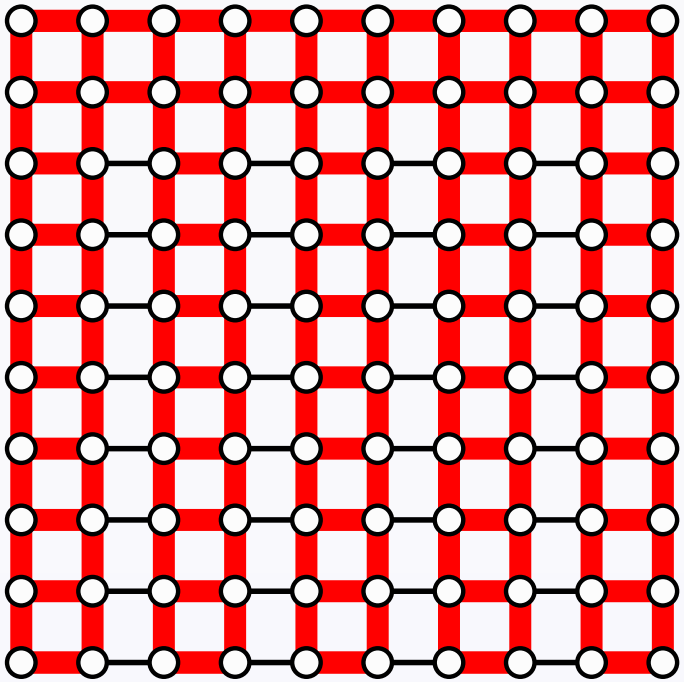

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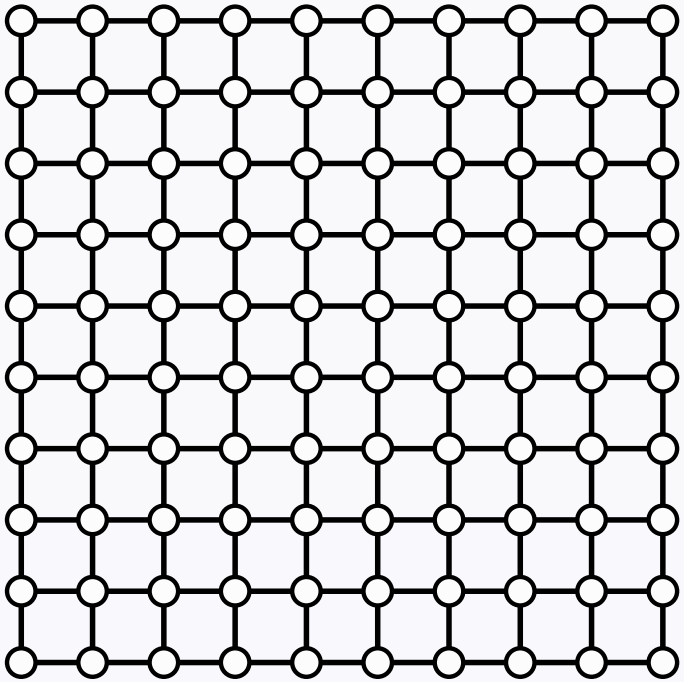

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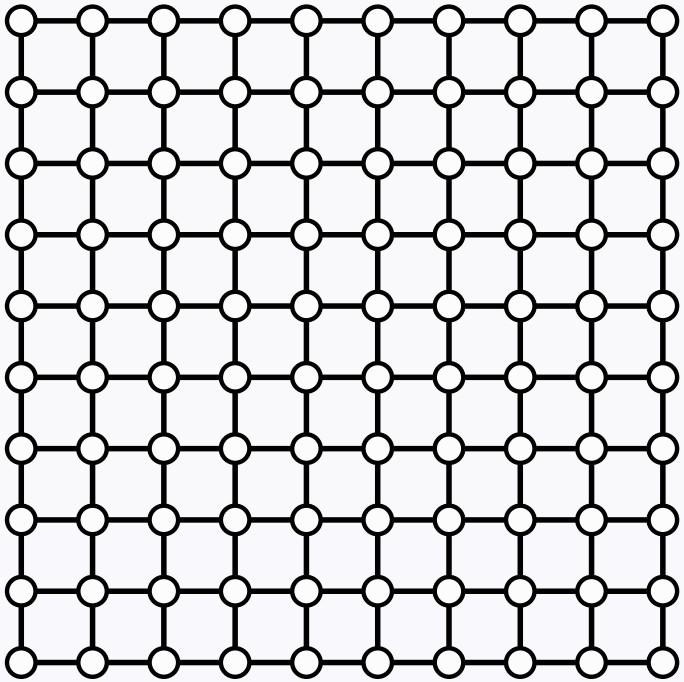

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- Not all minor-closed problems are bidimensional!

such as Treewidth, Pathwidth, Branchwidth, Tree-depth, and Genus

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- $\boxed{\Pi_{\mathbf{p}}}$ is singly exponentially solvable w.r.t. treewidth

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4. For even better (e.g. **subexponential**) parameterized dependency we must restrict our attention to special graph classes.

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Definition: *A graph class* $\mathcal{G}$ *has the* subquadratic grid minor property *(***SQGM***) if there exist* $1 \leq c < 2$ *such that* $\forall k \boxplus_k \not\leq G \Rightarrow \mathbf{tw}(G) = O(k^c)$

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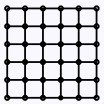
Theorem: [Robertson, Seymour, & Thomas 1994] *If G is planar and $\mathbf{bw}(G) \geq 4k$, then*  $_k \leq_m G$.

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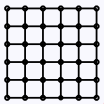
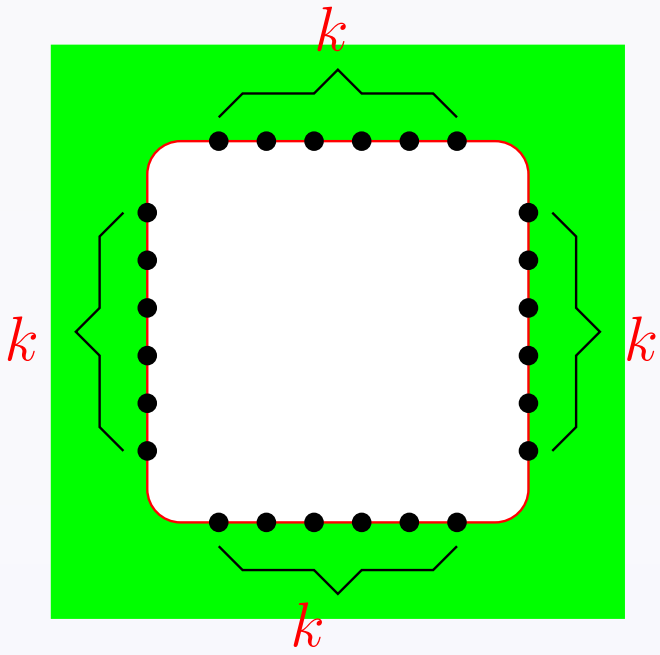
Theorem: [Robertson, Seymour, & Thomas 1994] *If* G *is planar and* $\mathbf{bw}(G) \geq 4k$, *then*  $_k \leq_m G$.

- We sketch the Idea of the proof of the above theorem:


k
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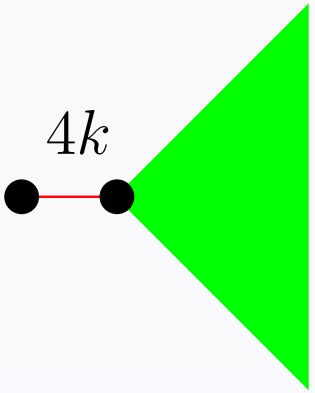
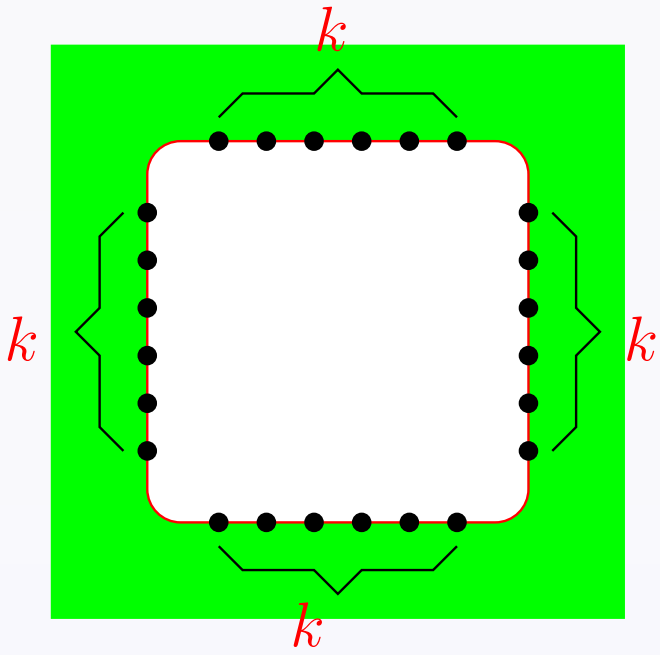
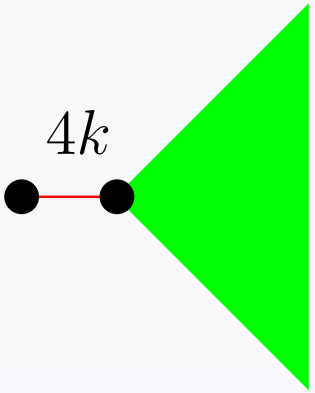
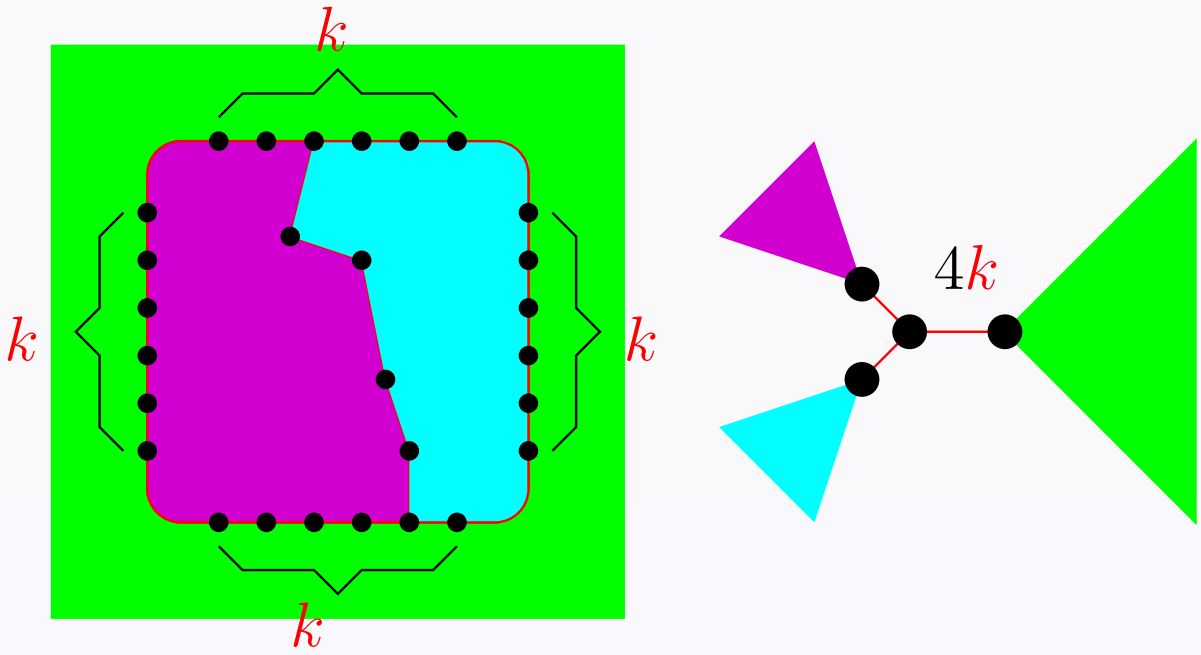
k


4k

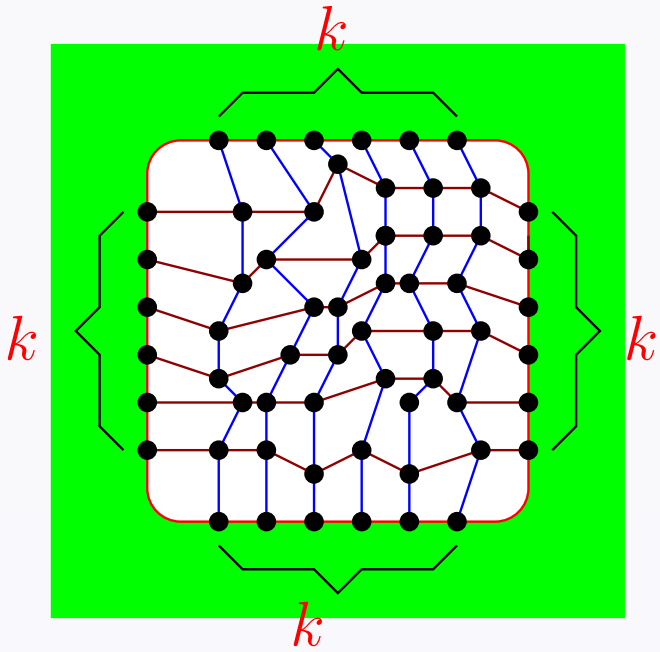


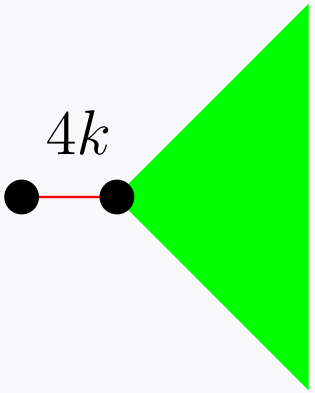


"Suppose" that we constructed a partial branch decomposition of the part of the graphs that is inside a disk.

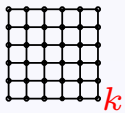


If there is a path from north-south or east-west, partition the disk: one more step further with the construction of a branch decomposition of width $\leq 4k$.





Such a path must exist,

otherwise, from Menger's theorem, the graph contains  as a minor.

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- are there more general graph classes where $1 < c < 2$?

Bidimensionality for contraction-closed problems

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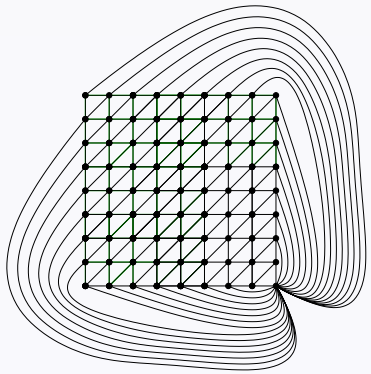

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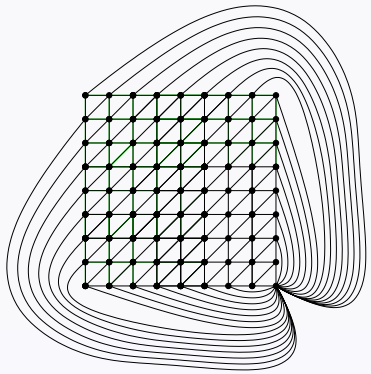

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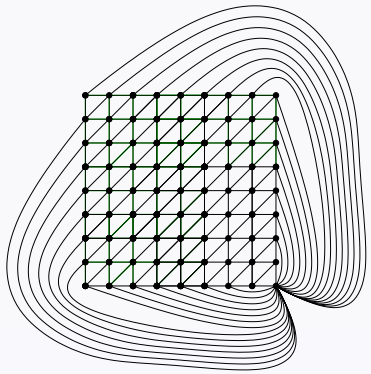

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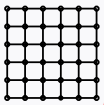
Theorem: [Robertson, Seymour, & Thomas 1994] *If* G *is planar and* $\mathbf{bw}(G) \geq 4k$, *then*  $_k \leq_m G$.

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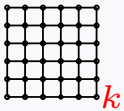
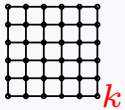
Theorem: [Robertson, Seymour, & Thomas 1994] *If* G *is planar and* $\mathbf{bw}(G) \geq 4k$, *then*  $_k \leq_m G$.

Proof: if we do not apply edge removals while obtaining  $_k$ from G we end up to a partially triangulated grid that can be further be contracted to the uniformly triangulated grid Γ_k.

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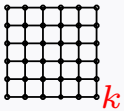
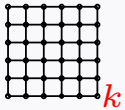
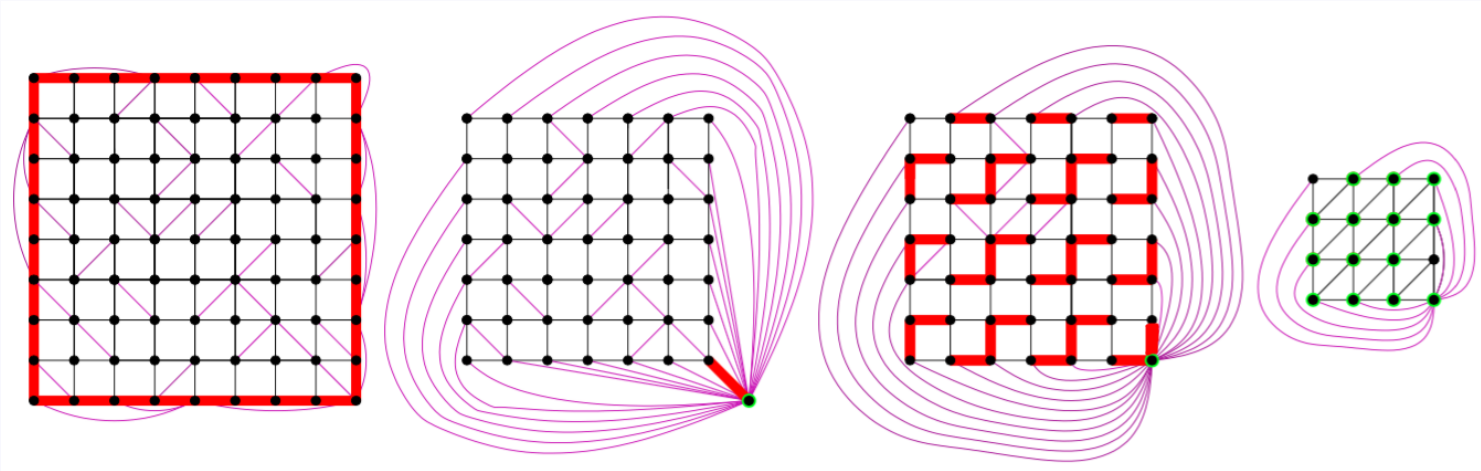
Theorem: [Robertson, Seymour, & Thomas 1994] *If* G *is planar and* $\mathbf{bw}(G) \geq 4k$, *then*  $_k \leq_m G$.

Proof: if we do not apply edge removals while obtaining  $_k$ from G we end up to a partially triangulated grid that can be further be contracted to the uniformly triangulated grid Γ_k.



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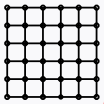
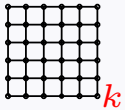
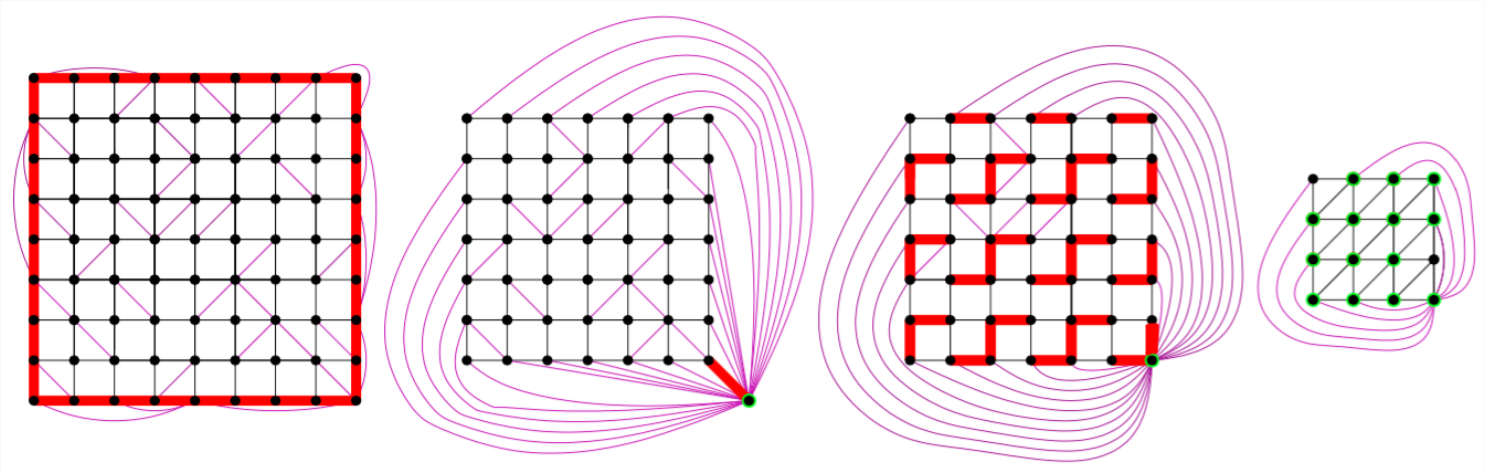
Theorem: [Robertson, Seymour, & Thomas 1994] *If* G *is planar and* $\mathbf{bw}(G) \geq 4k$, *then*  $_k \leq_m G$.

Proof: if we do not apply edge removals while obtaining  $_k$ from G we end up to a partially triangulated grid that can be further be contracted to the uniformly triangulated grid Γ_k.



Therefore $\Gamma_k \not\leq_c G \Rightarrow \mathbf{tw}(G) = O(k^c)$, thus **SQGC** holds for $c = 1$.

Bidimensionality race:

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When **SQGC** property holds?

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Bounded Genus graphs: [Demaine, Hajiaghayi, Thilikos, SIDMA 2006]

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Families of 2D-geometric graphs [Baste, Thilikos, in preparation]

Bidimensionality race:

SQGC: $\forall k\ \Gamma_k \not\leq_c G \Rightarrow \mathbf{tw}(G) = O(k^c)$ for some $c < 2$.

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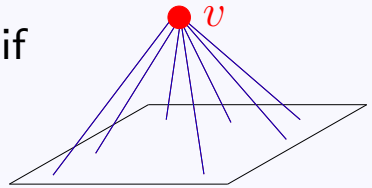
A graph H is an *apex graph* if



planar graph

$\exists v \in V(H)$: $H - v$ is planar

Bidimensionality and subexponential algorithms.

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Then Π *can be solved in* $2^{o(k)} \cdot n^{O(1)}$ *steps*

Some bidimensional problems

(Connected) Vertex Cover, (Connected) Dominating Set, (Connected) Feedback Vertex Set, Induced Matching, Longest Cycle, (Connected) (Induced) Cycle Packing, (Connected) Cycle Domination, d-Scattered Set, Longest Path, (Induced) Path Packing, (Connected) r-Center, (Connectd) Diamond Hitting Set, Minimum Maximal Matching, Face Cover, Unweighted TSP Tour, Max Bounded Degree Connected Subgraph

- The previous theorem can become an algorithmic meta-theorem as

  ii. is singly exponentially solvable w.r.t. treewidth

is implied by expressibility in Existential Counting Modal Logic

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[Bodlaender, Cygan, Kratsch, Nederlof, ICALP 2013] $\star\star$

[Fomin, Lokshtanov, Saurabh, SODA 2014] $\star\star$

Part 5, Friday 10/05/2016 - 09:00–10:30 (90')

Bidimensionality and Kernelization

Irrelevant vertex technique

Bidimensionality and Kernelization

Kernelization

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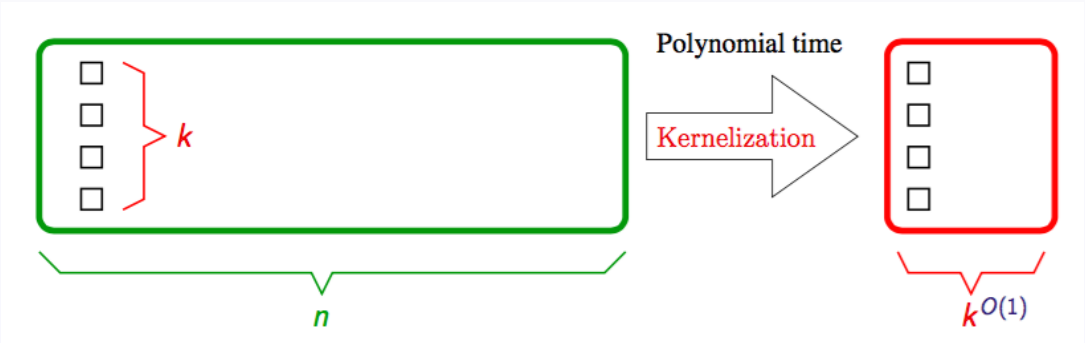

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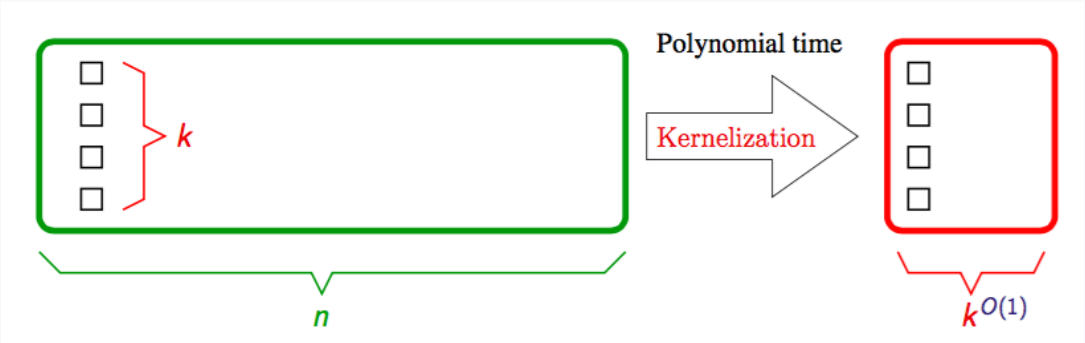

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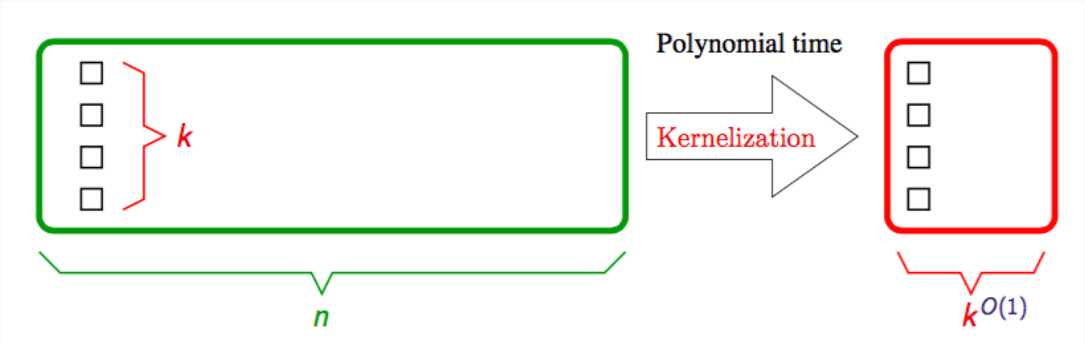

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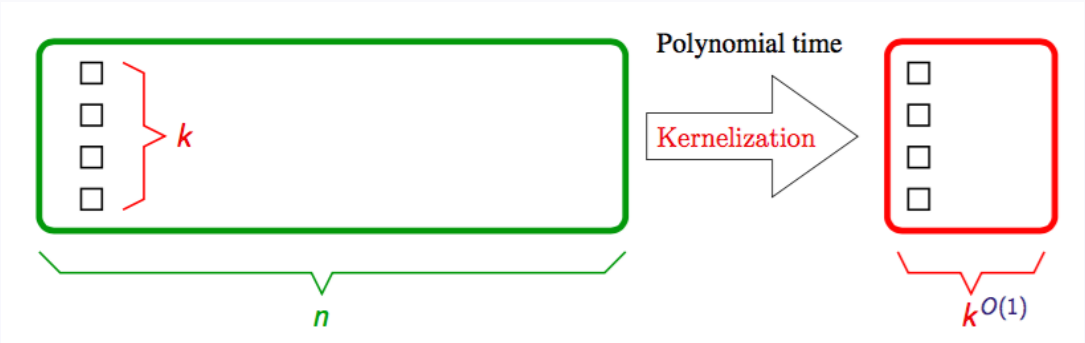

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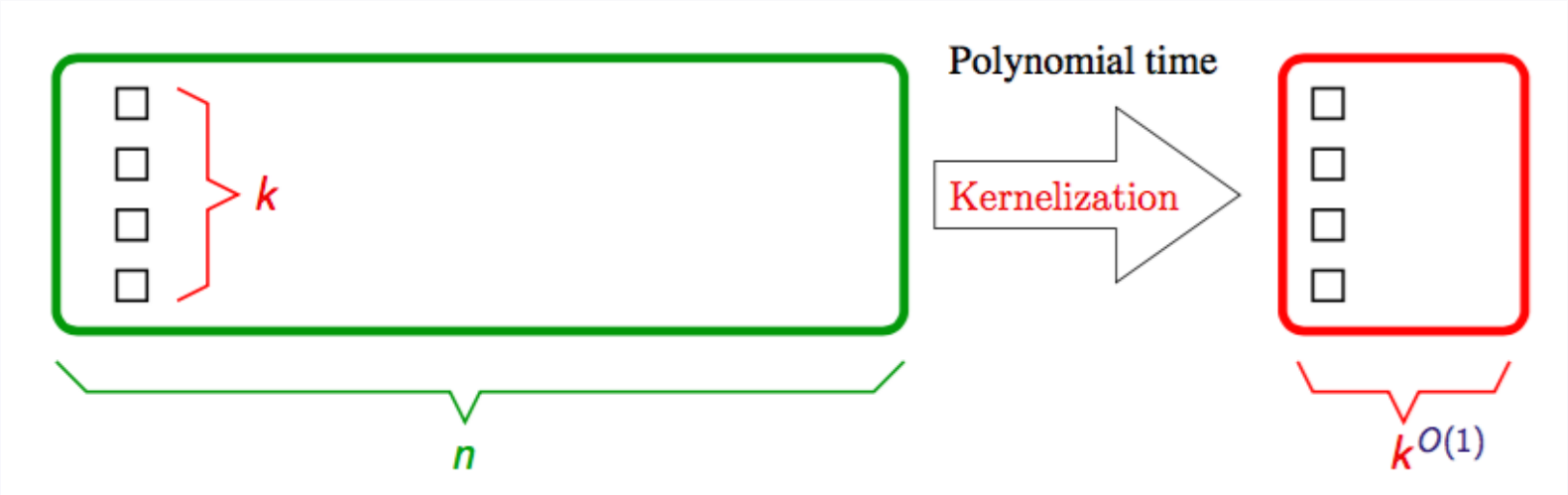
- *Kernelization can be seen as a paradigm for* **preprocessing**

p-Vertex Cover has kernelization algorithm that produces a kernel of $\leq 2k$ vertices.



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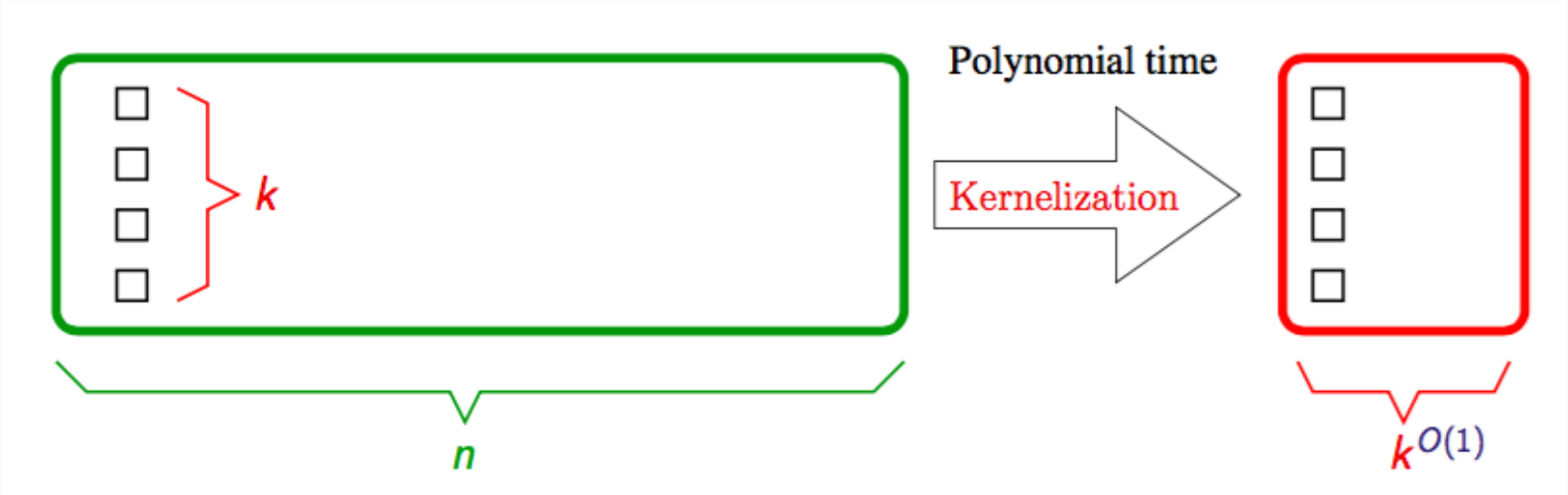

Alternatively, we say that p-Vertex Cover has a kernel of size $2k$.

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p-Path, p-Dominating Set, and p-Feedback Vertex Set are bidimensional.

Protrusions

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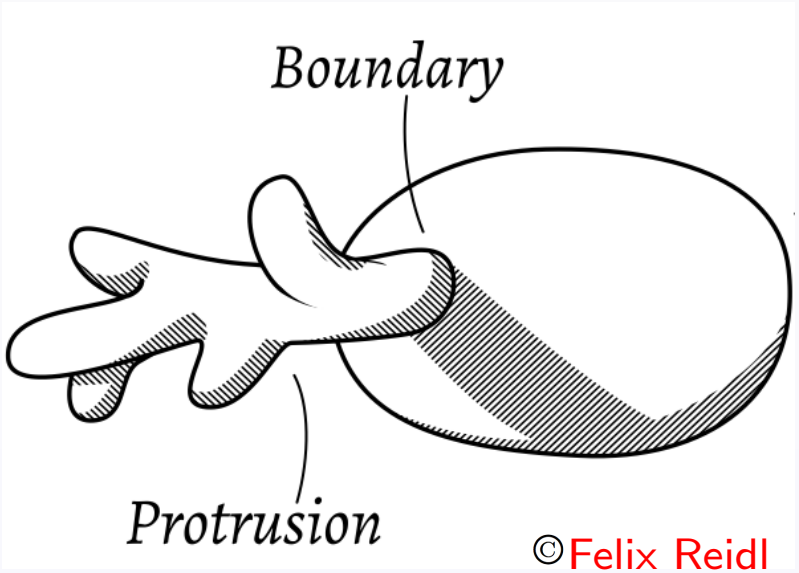

r-*protrusion*: a set $X \subseteq V(G)$ where

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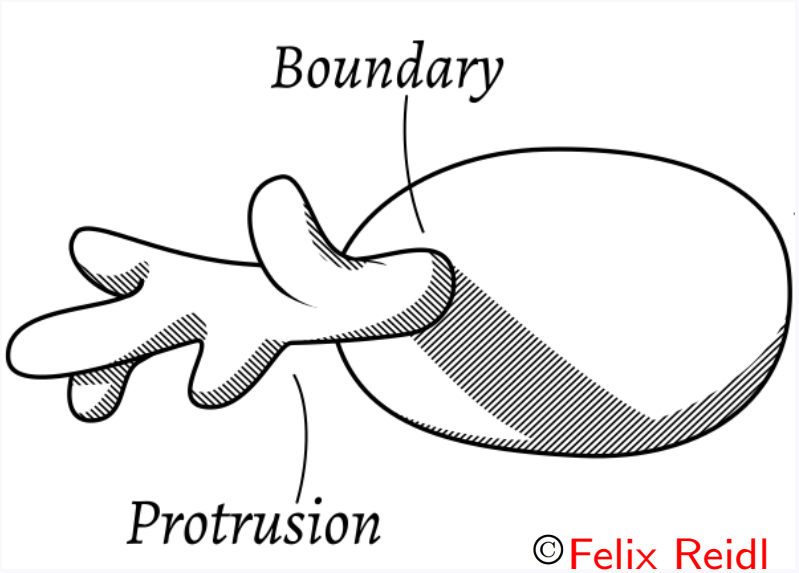

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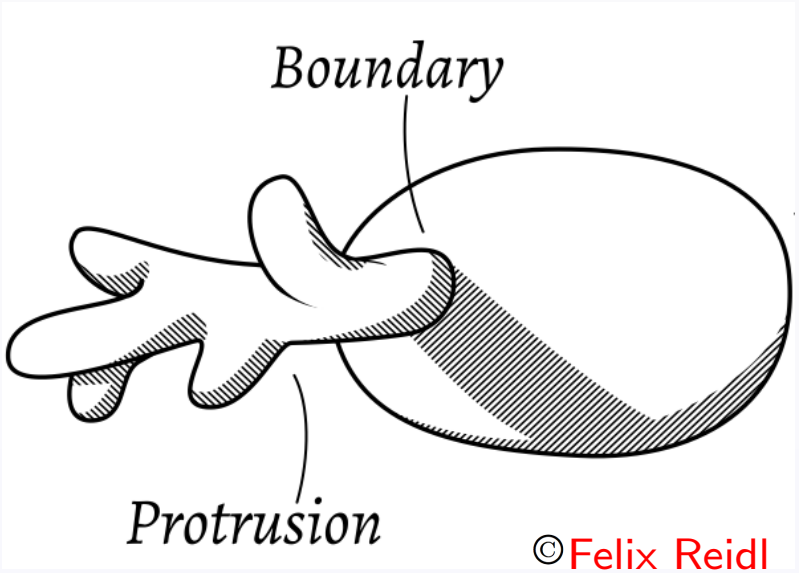

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is a partition $\mathcal{P} = \{R_0, R_1, \ldots, R_\rho\}$ of $V(G)$

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- for every $i \in \{1, \ldots, \rho\}$, $N_G(R_i) \subseteq R_0$.

  Remark: actually, this last condition is

  not necessary! But makes things more visualizable!

Protrusion replacement

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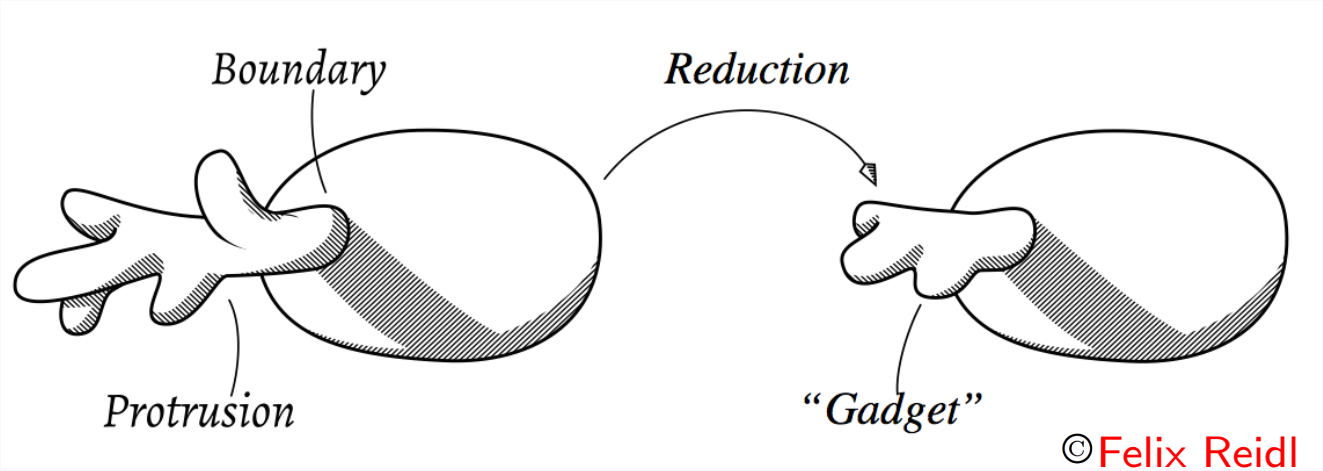

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To achieve Conditions **1** and **2.** we need some more definitions!

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- If ϕ is expressible in Monadic Second Order Logic, then we say that

$\boxed{\Pi_{\mathbf{p}}}$ is *CMSO-expressibile*

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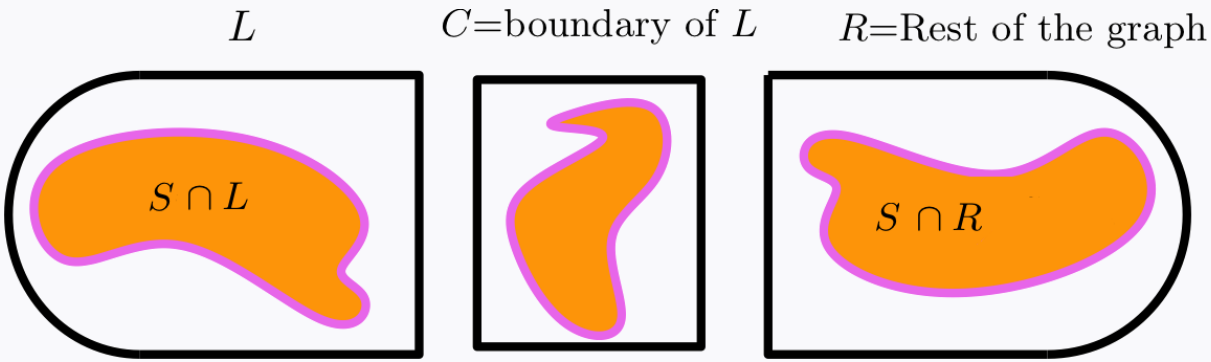

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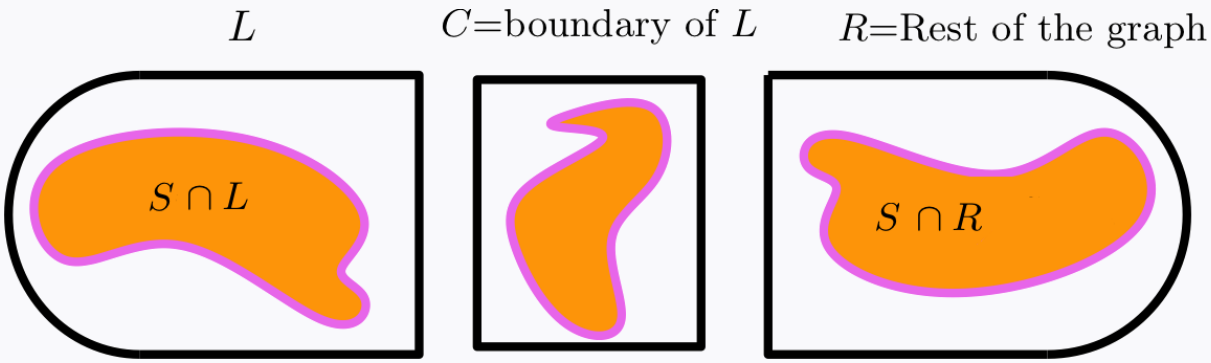

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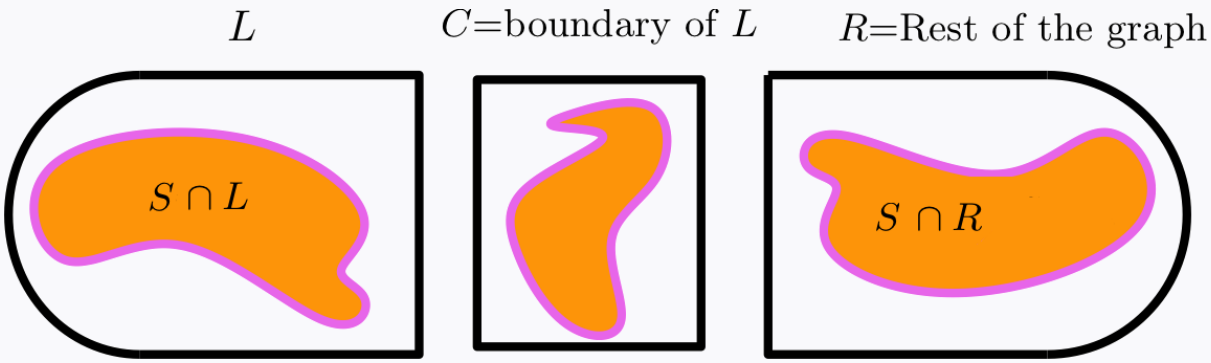

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where S is a solution certificate for $\mathbf{p}$

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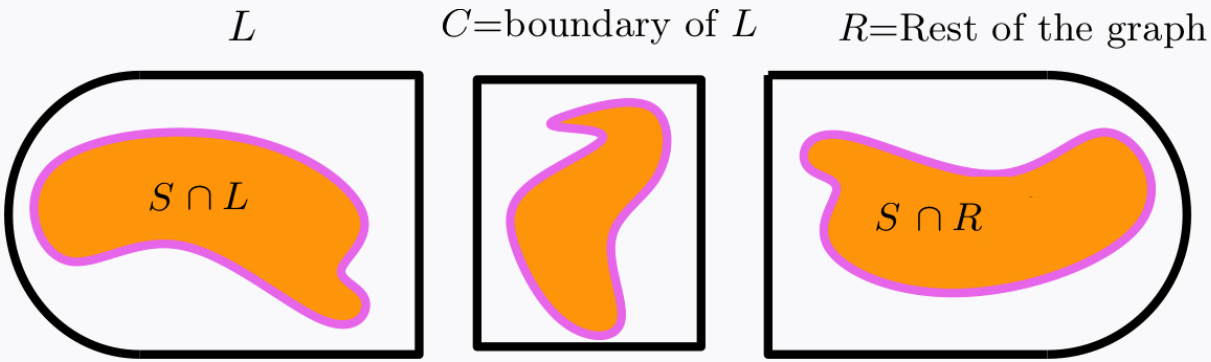

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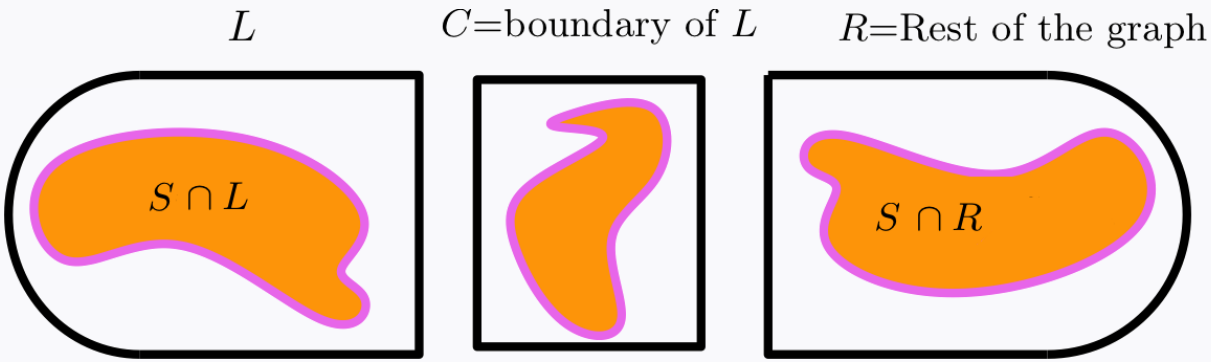

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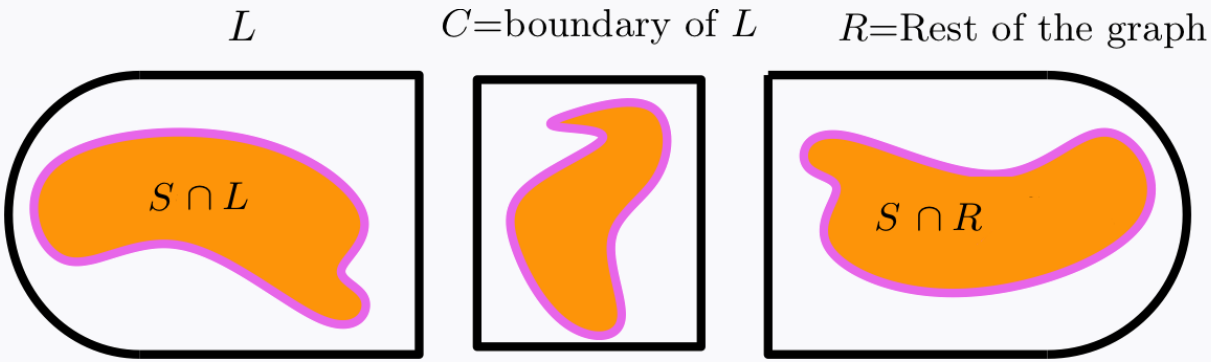

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EPTAS = (**E**fficient **P**olynomial-**T**ime **A**pproximation **S**cheme)

[Demaine, Hajiaghay, SODA 2005]
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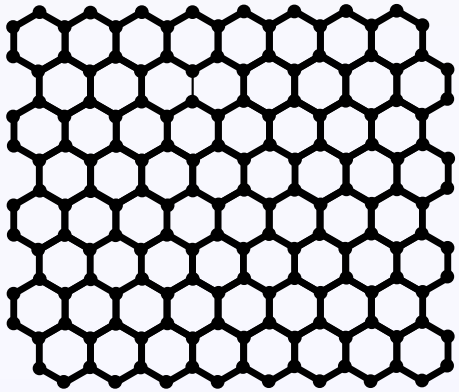

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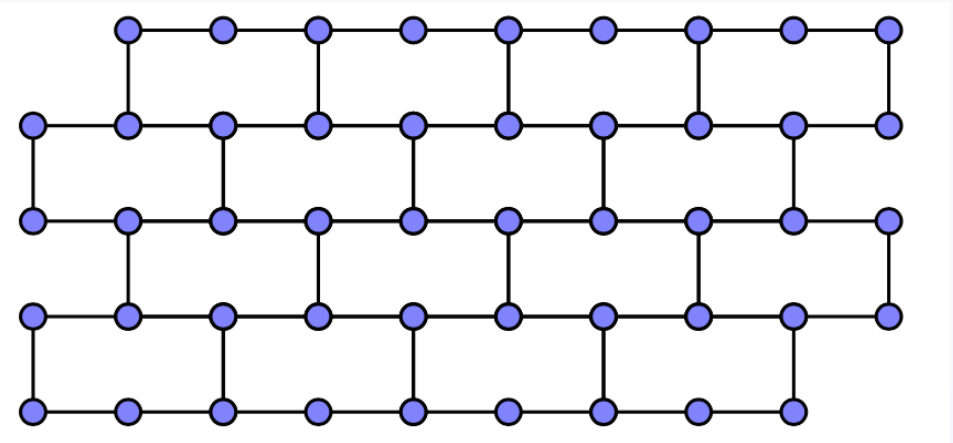

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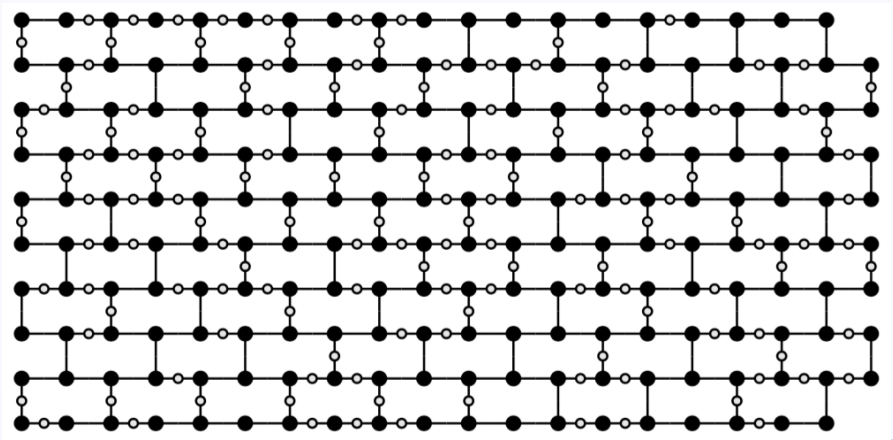

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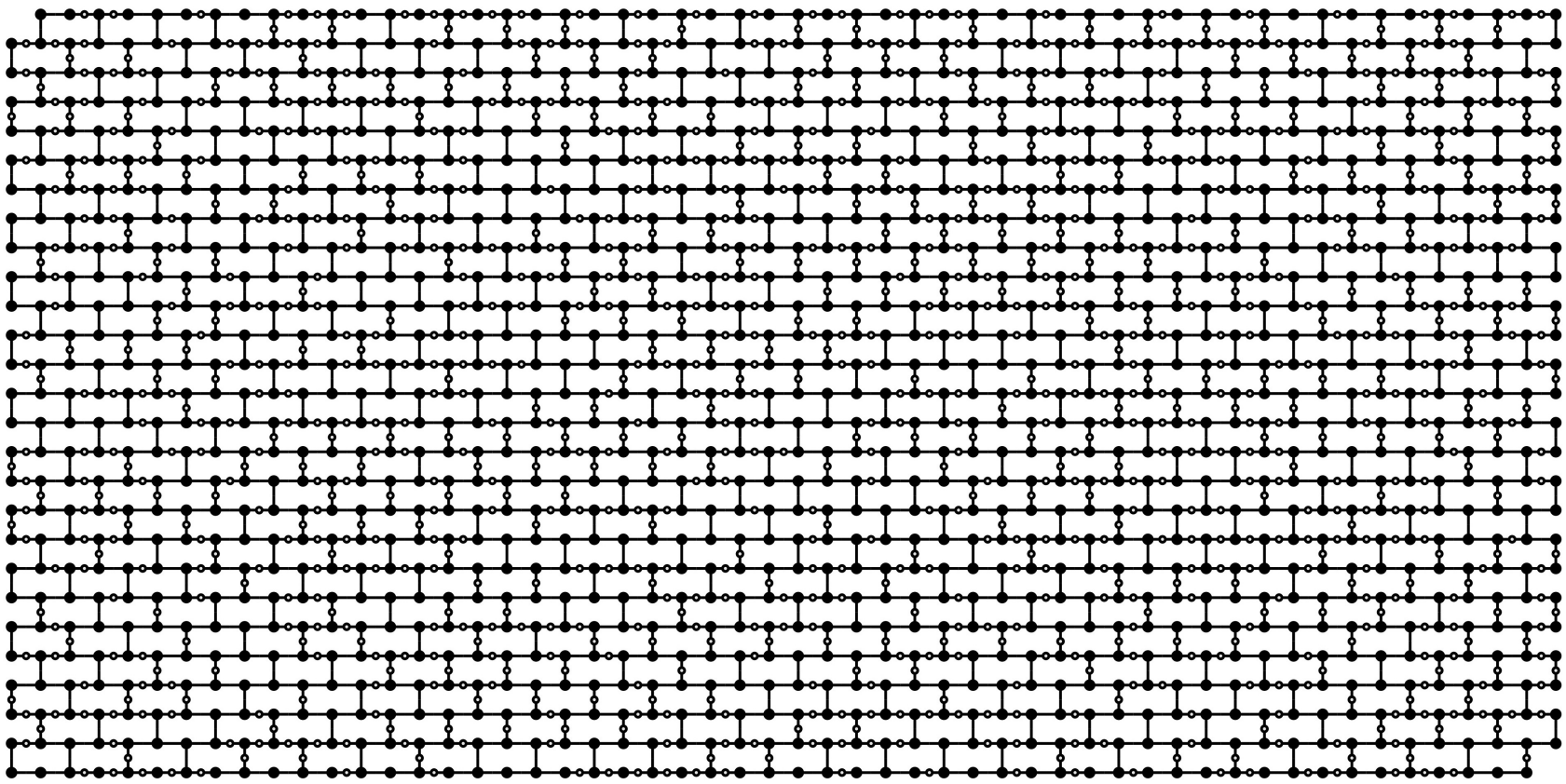

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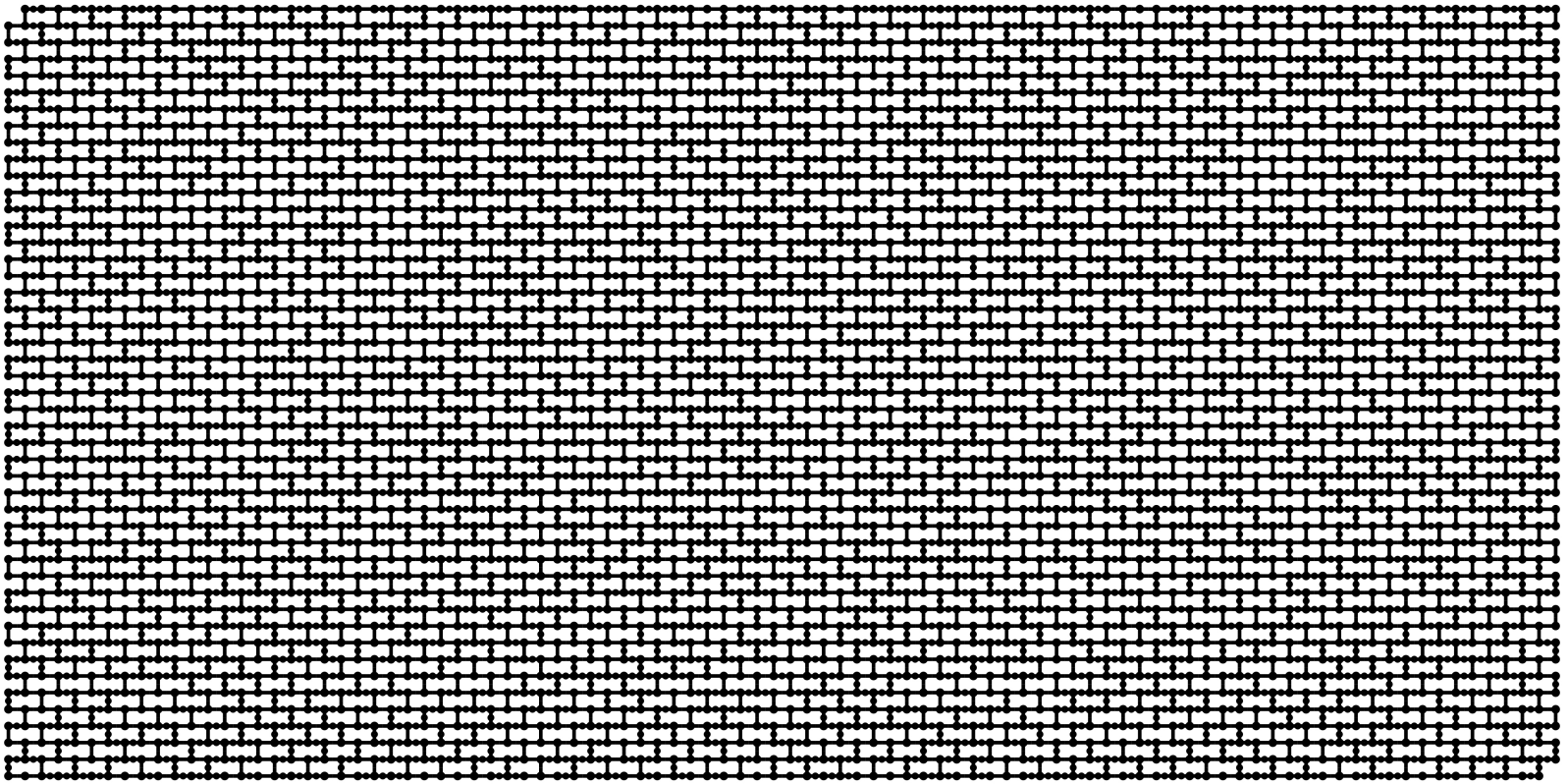

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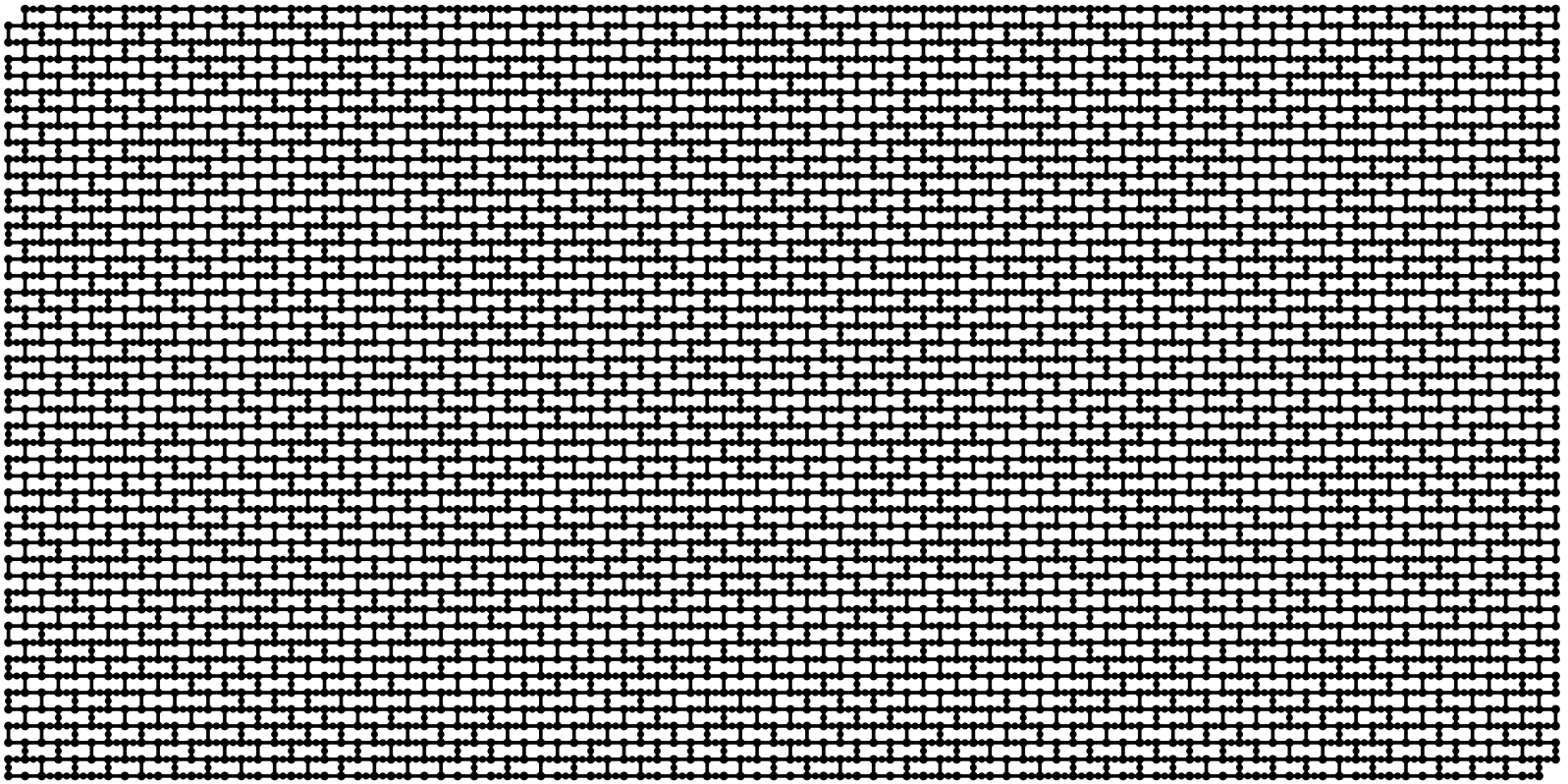

The height of this wall is $h = \lceil \sqrt{k} \rceil \cdot (2(k+1)+1)$ (in the s-wall above $h = 46$)

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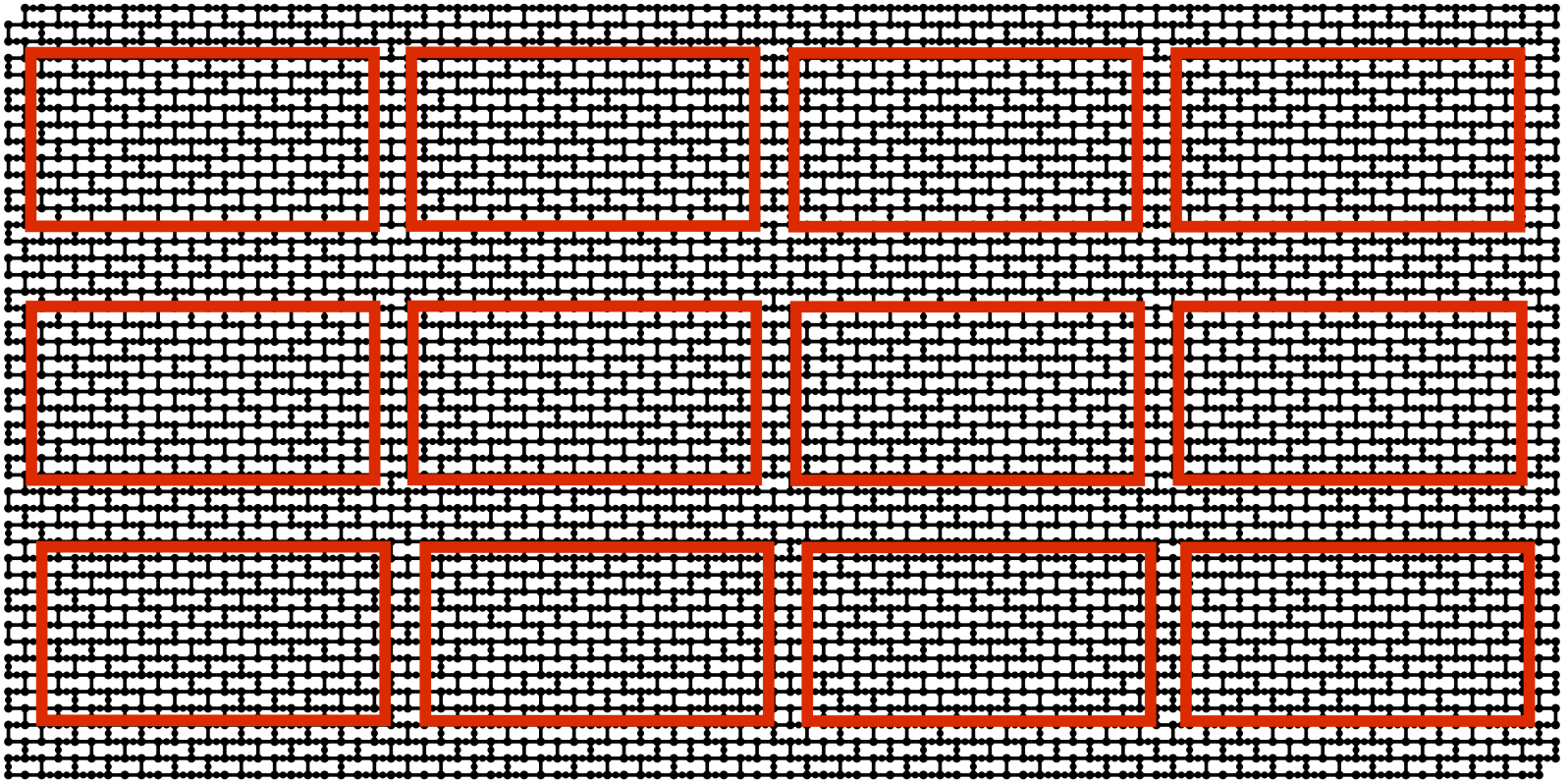

We locate k subwalls, each of heigh $2 \cdot (k+1)$.

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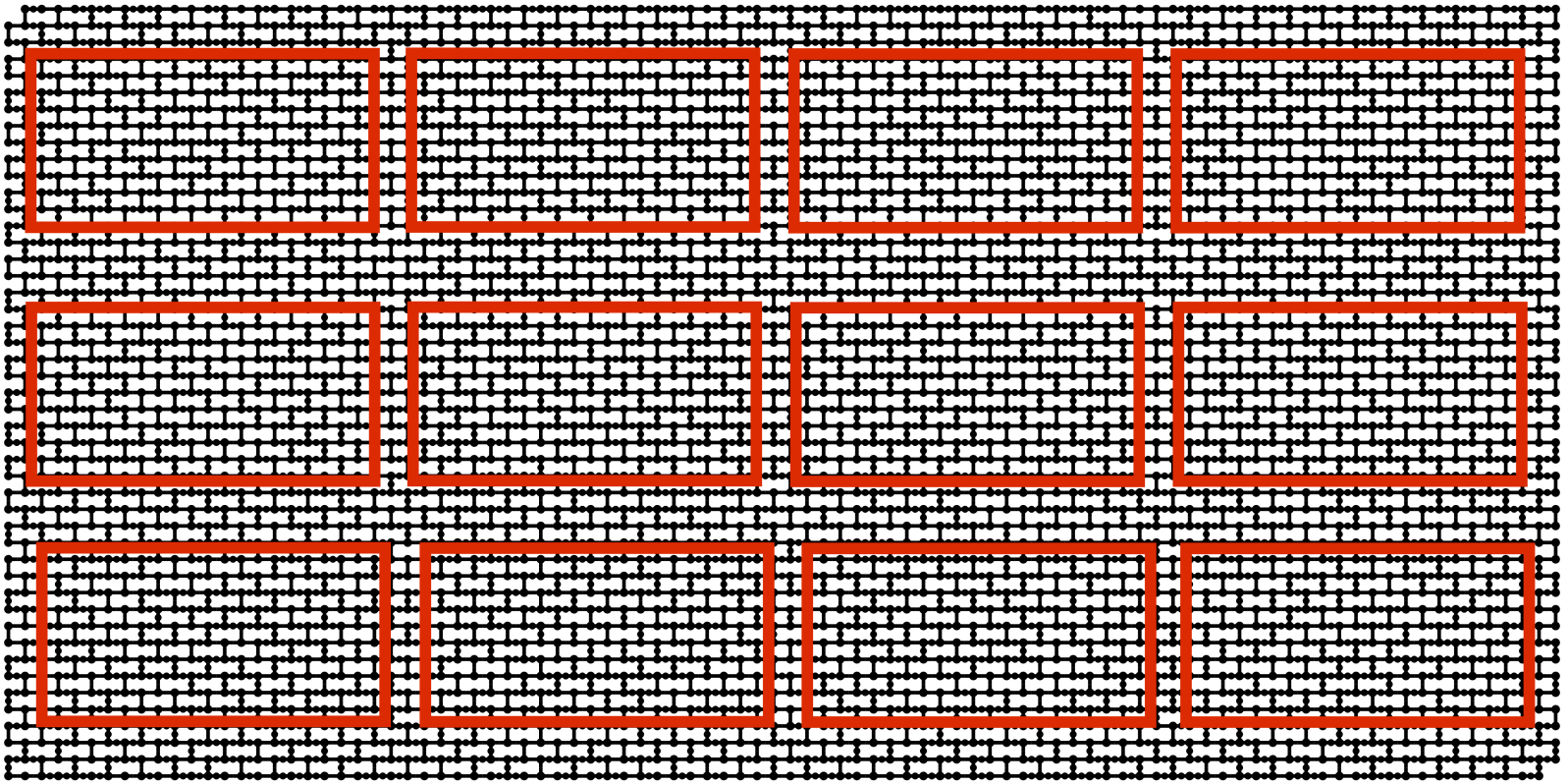

Let $G_1, \ldots, G_k$ be the graphs inside the perimetries of these subwalls

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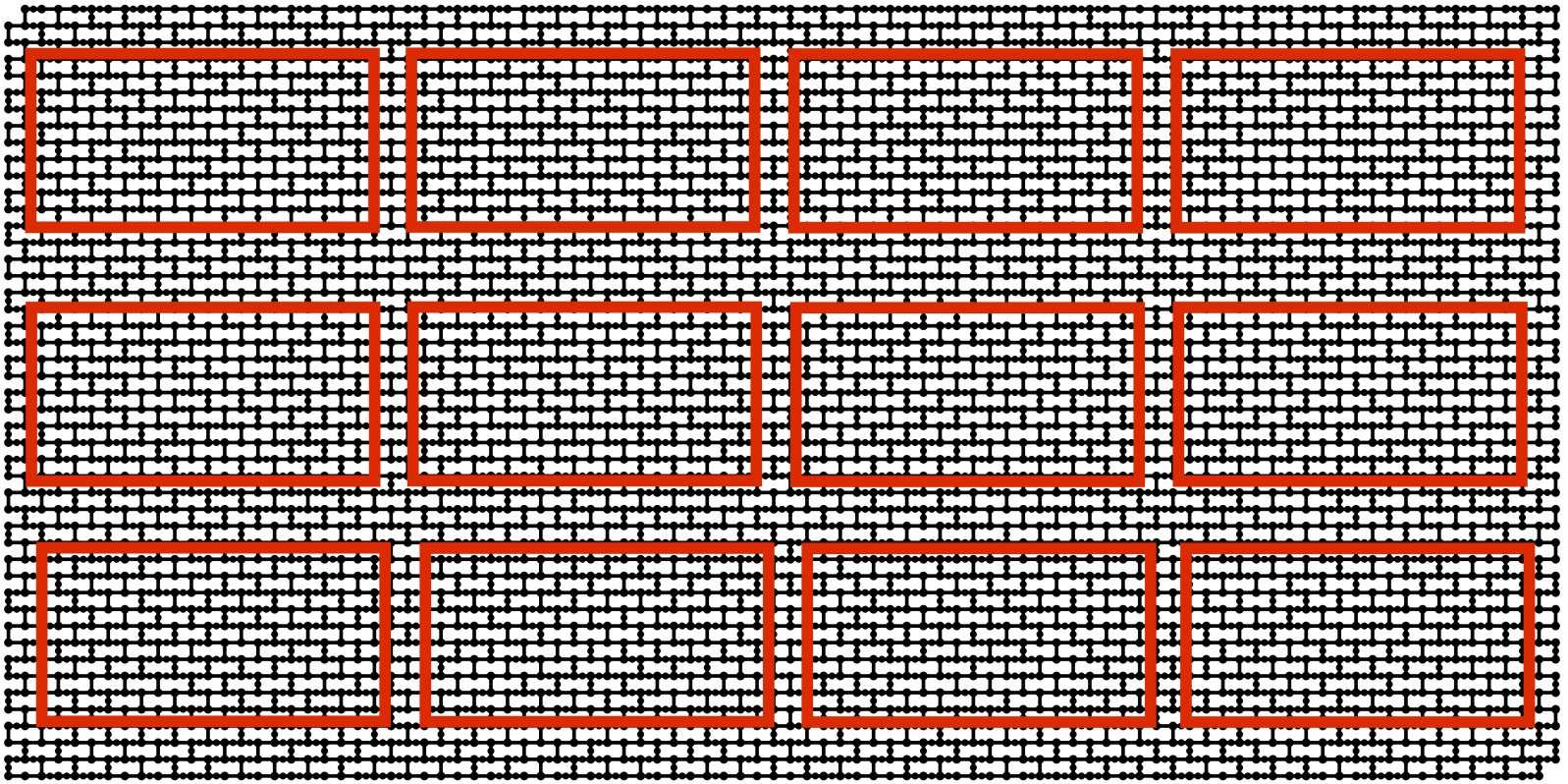

If all of them are non-bipartite then we answer YES and we are done!

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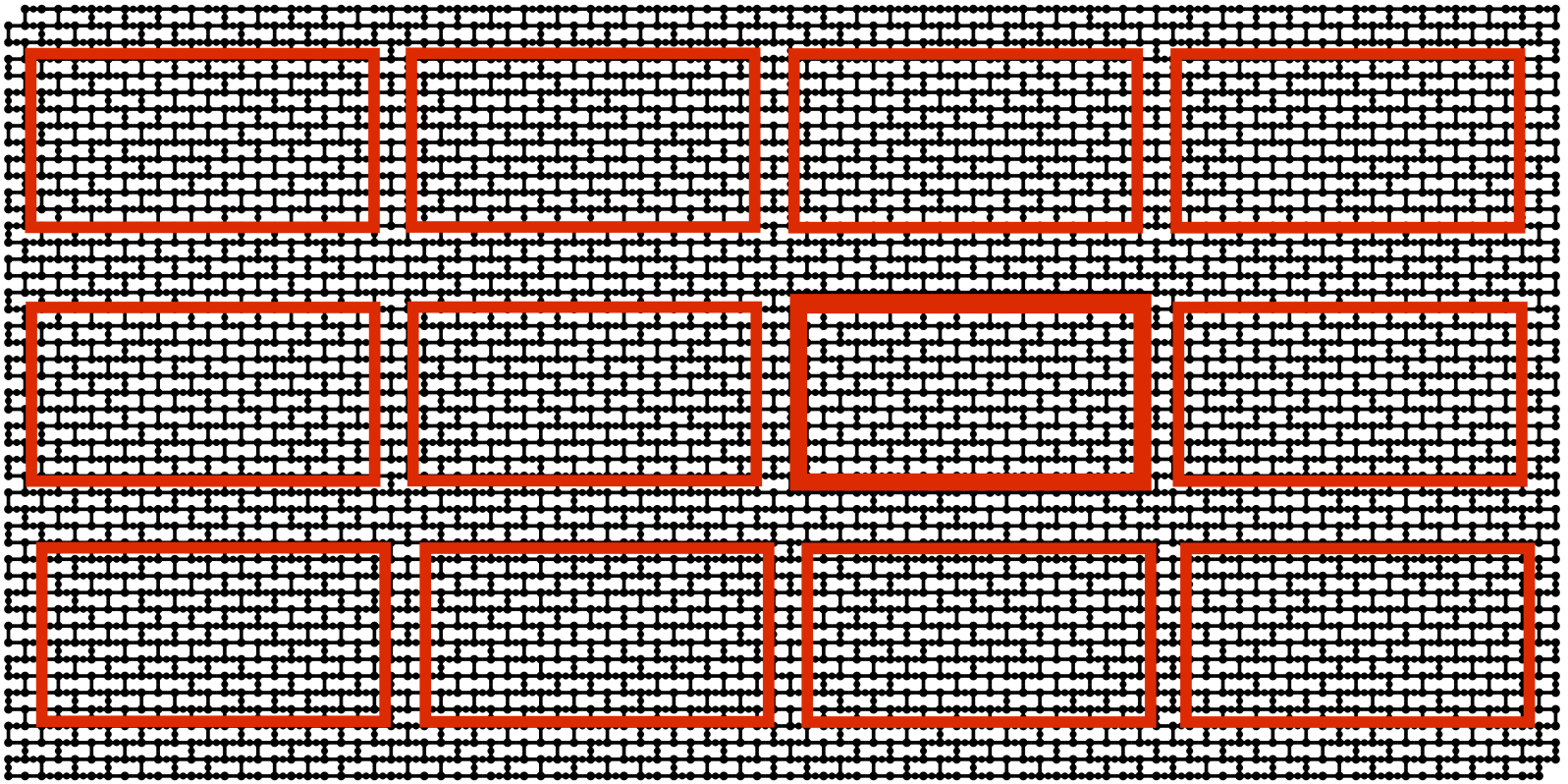

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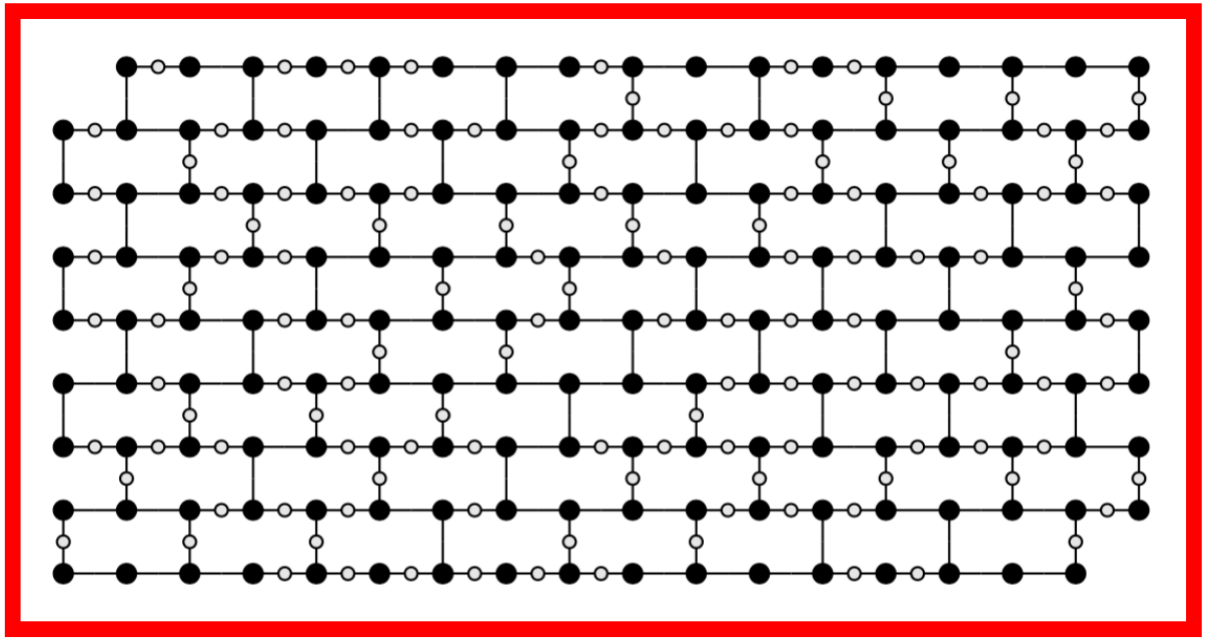

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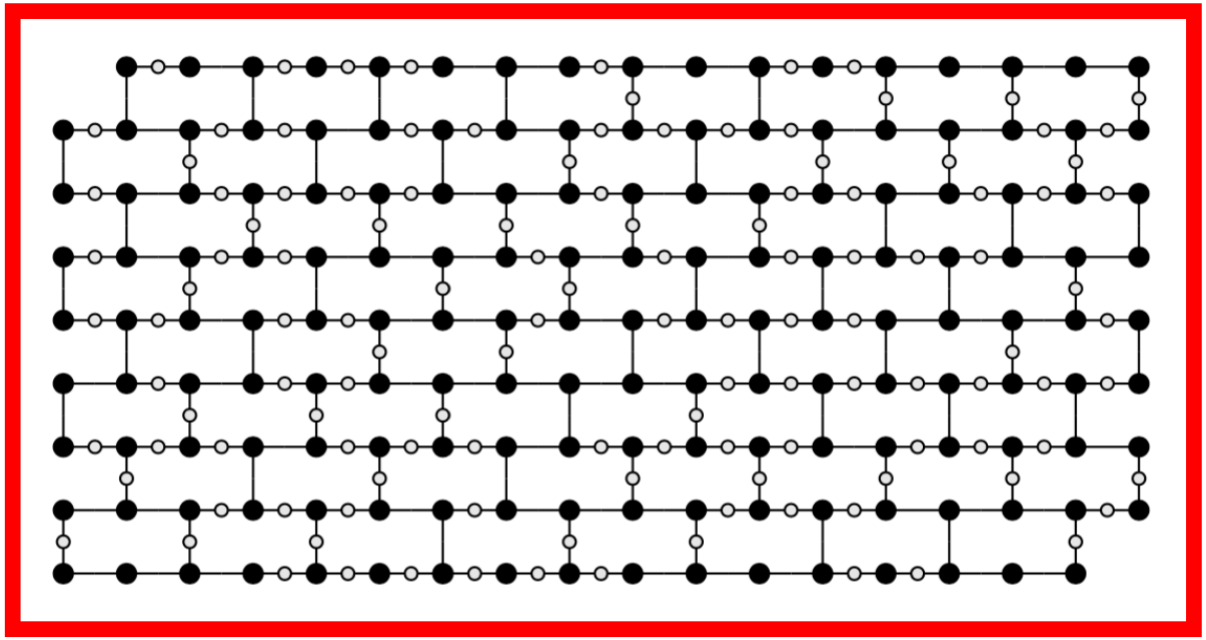

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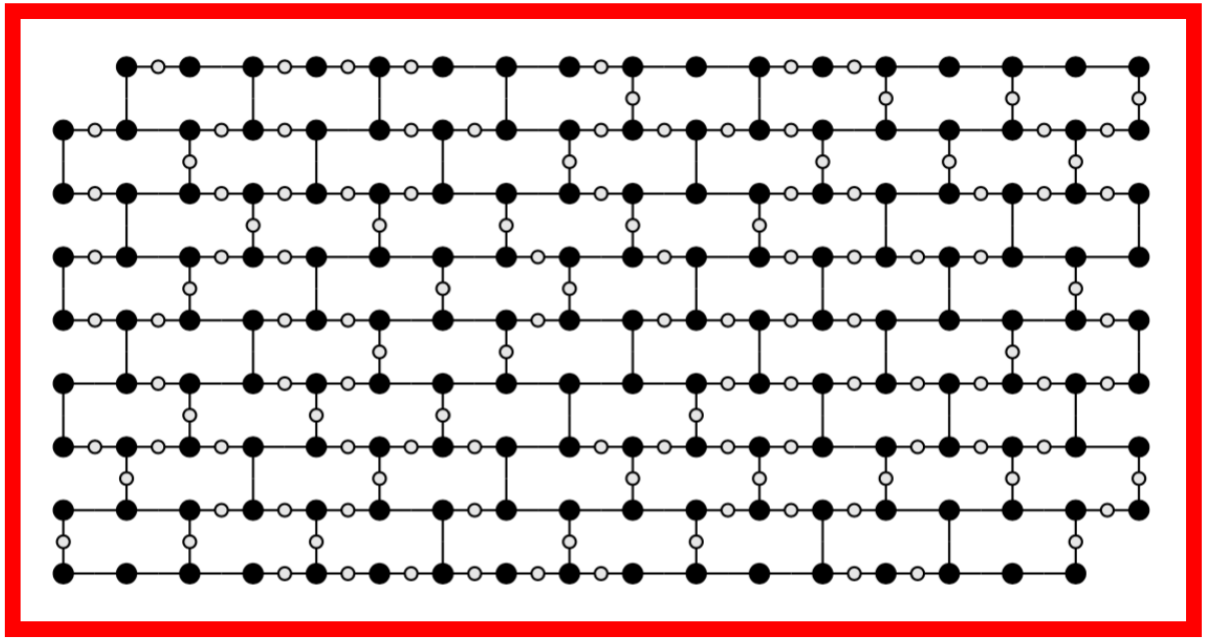

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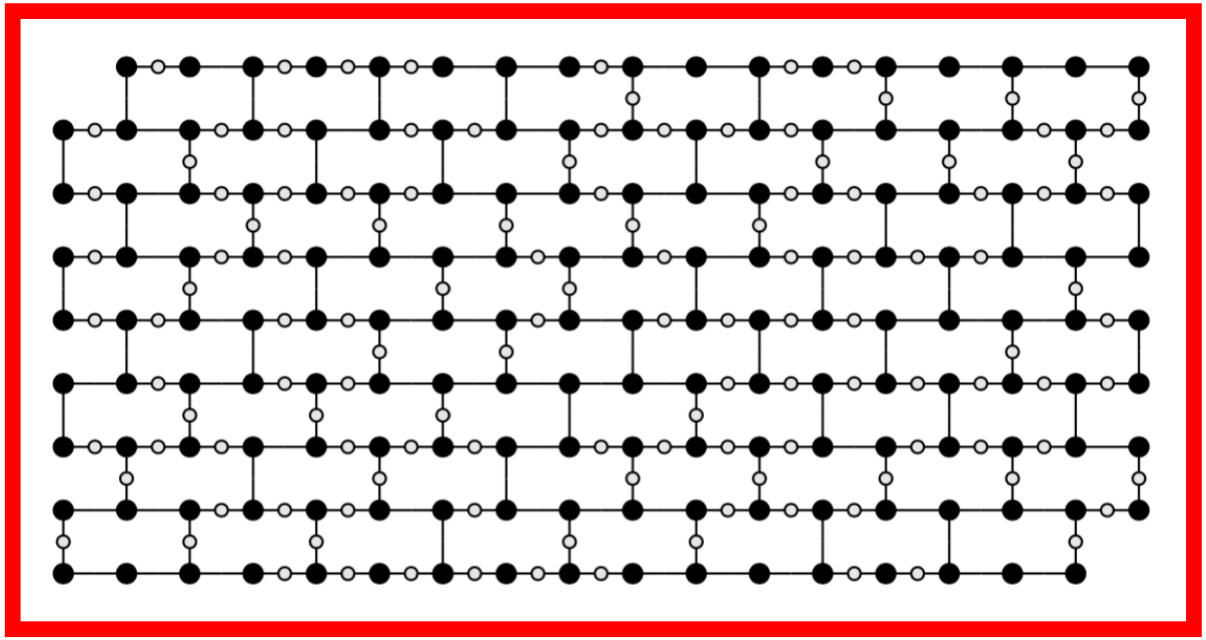

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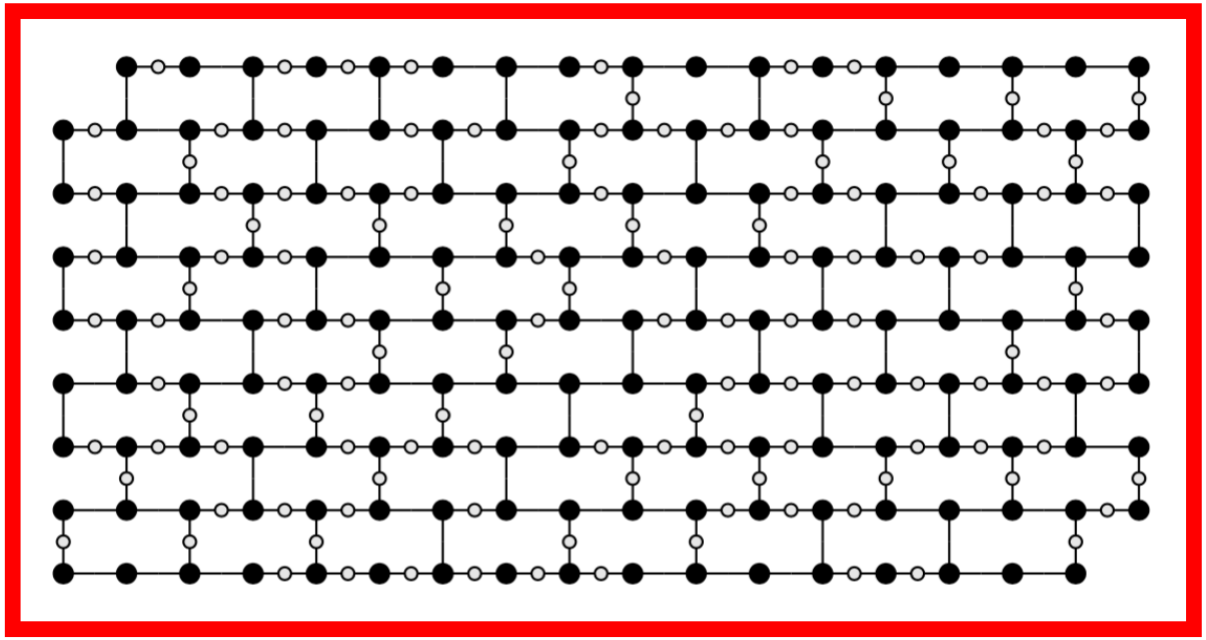

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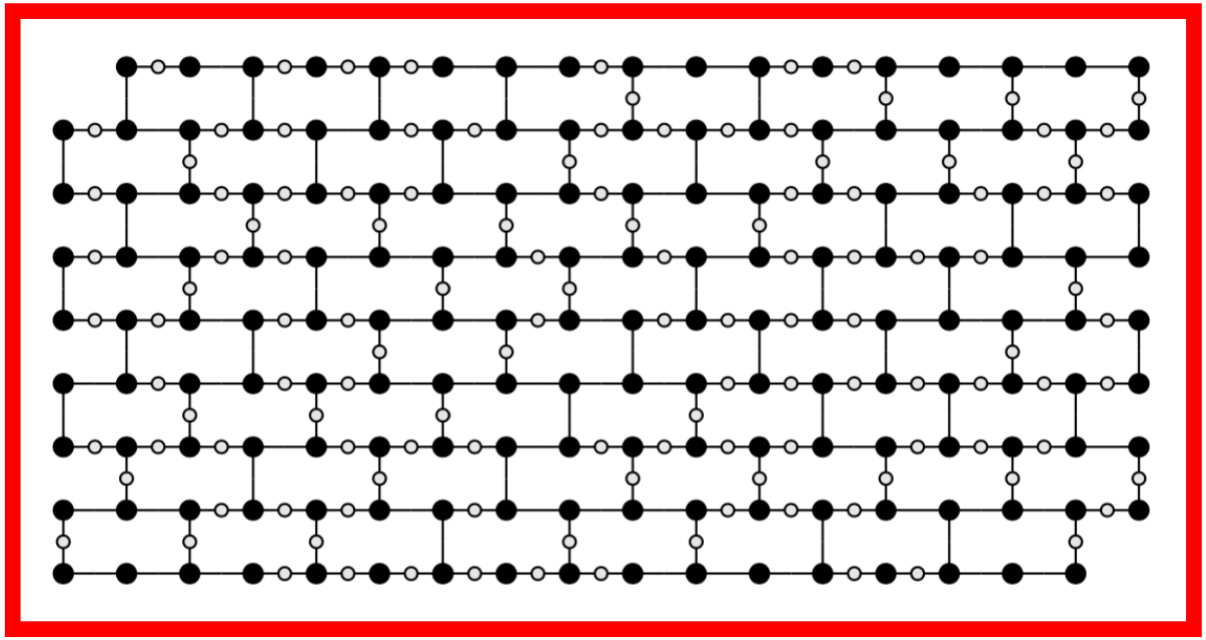

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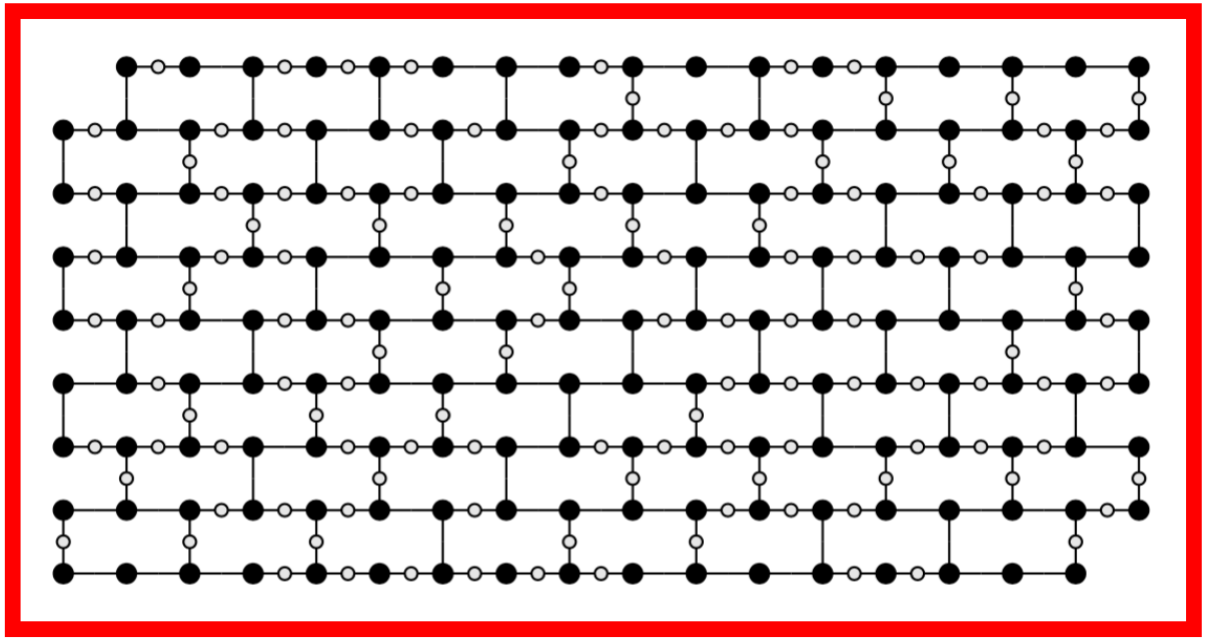

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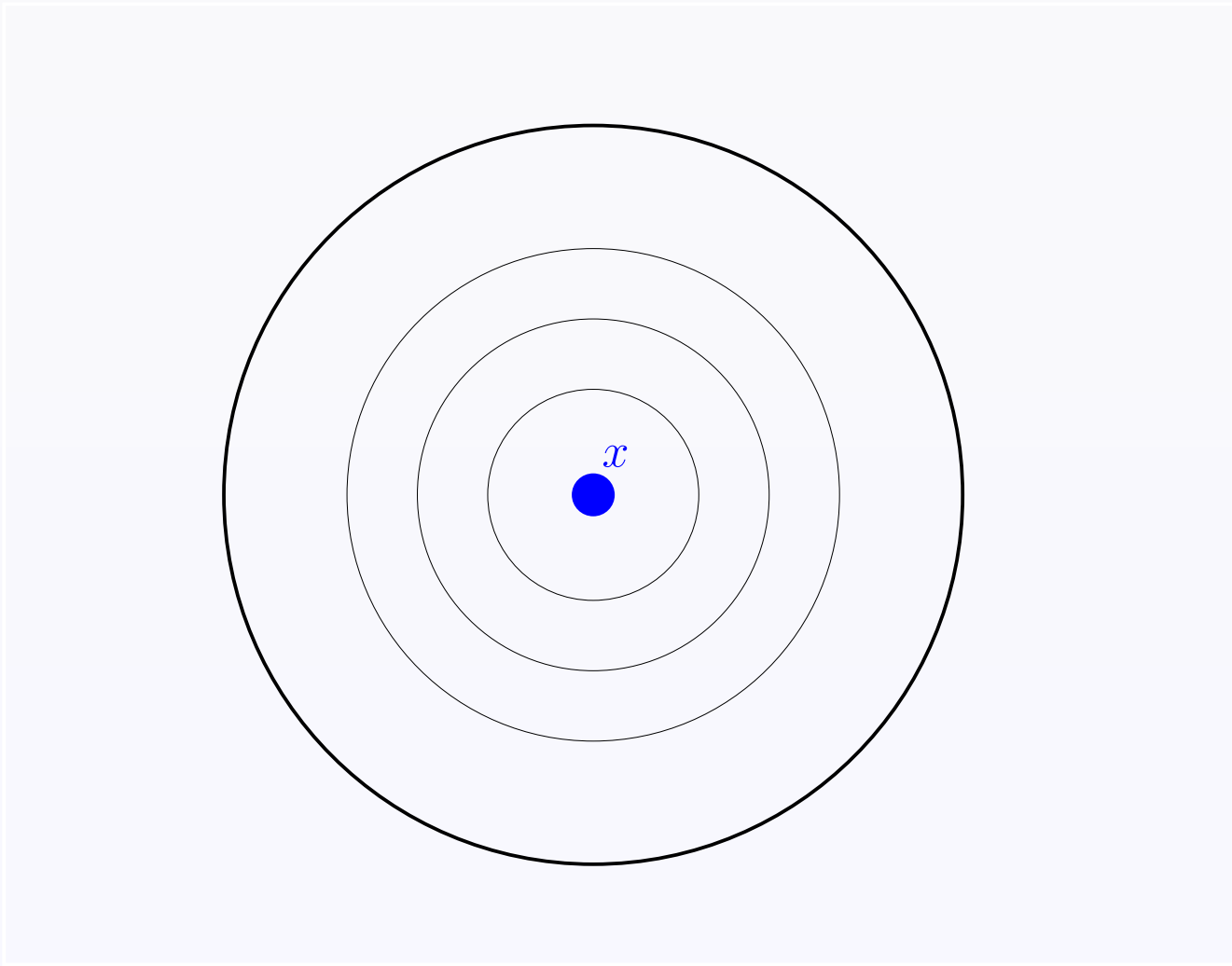
We will prove that $G \setminus x$ has k odd disjoint cycles avoiding x.

We detect, using the layers of the wall, $k+1$, homocentric cycles around x

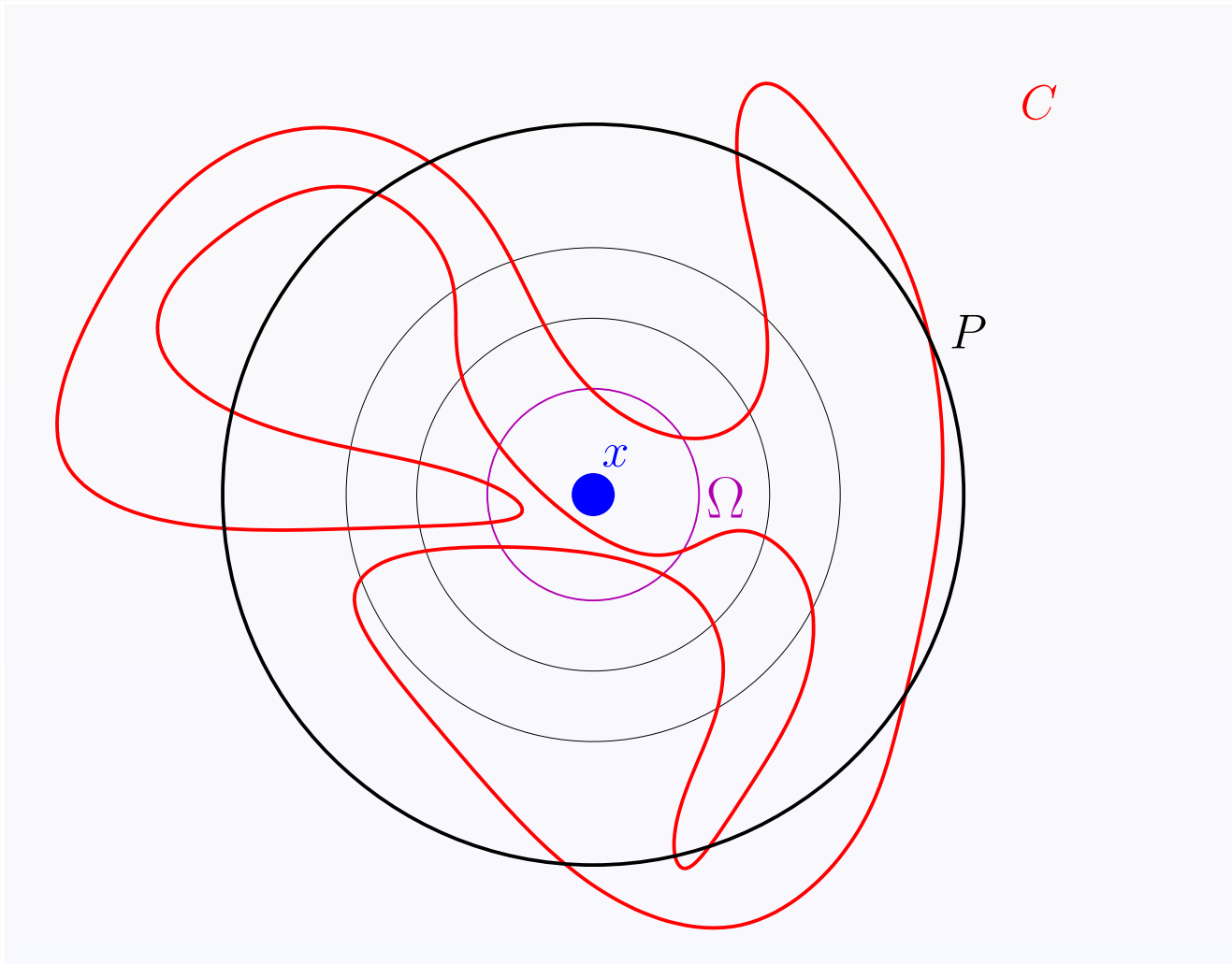
If G has $k+1$ disjoint odd cycles we are done (x meets only one of them)

Therefore G has at exactly k disjoint odd cycles.



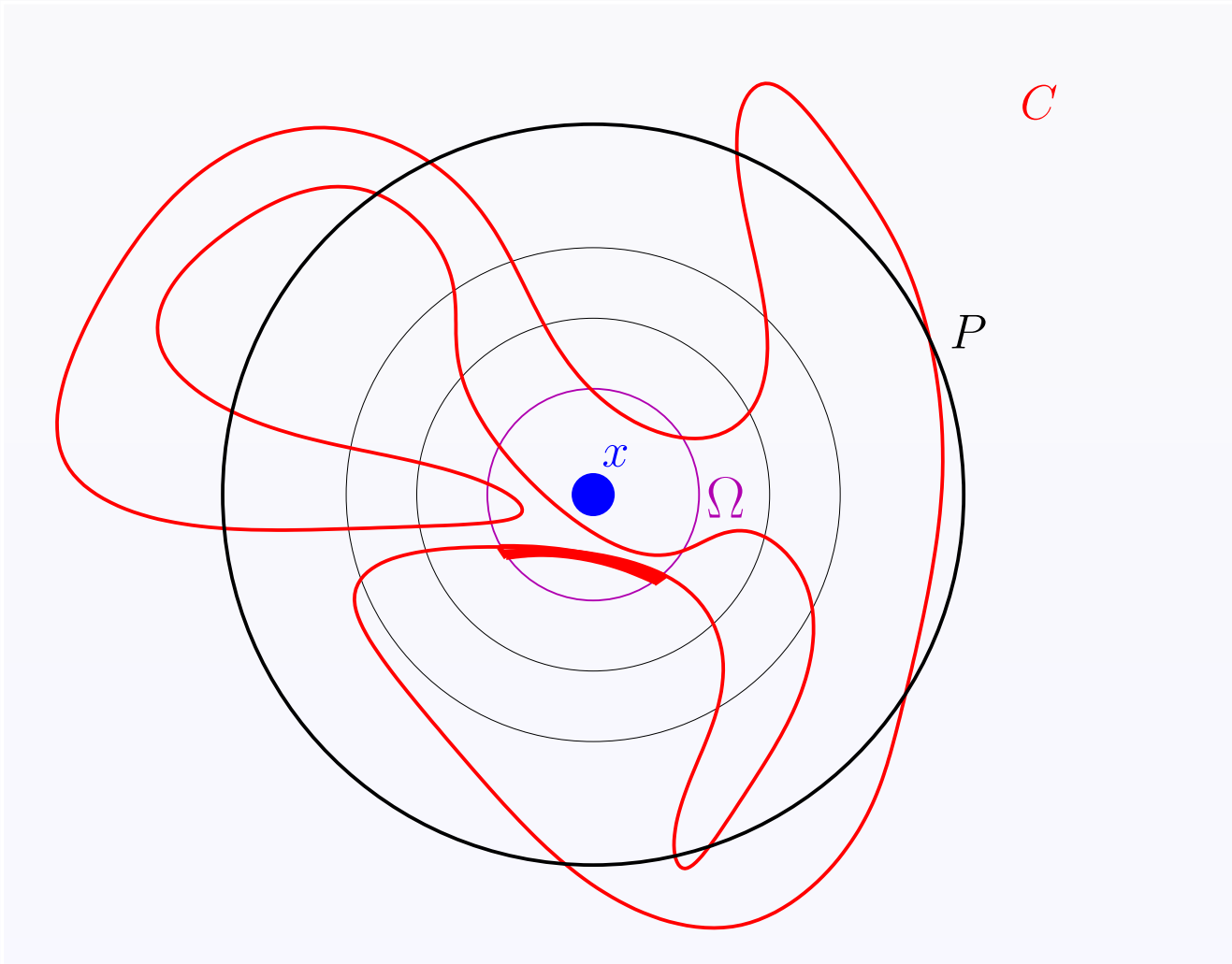
Assume # chords of the k disjoint cycles "cropped" by the homocentric cycles is minimized. For example C crosses Ω 4 times and C crosses perimetry P 5 times.

We argue that none of these k cycles can cross the inner cycle Ω, thus x is irrelevant!



Suppose, to the contrary, that some cycle C crosses the inner cycle Ω.

Consider an "extremal" chord X: one of the two paths of Ω does not contain any other endpoint of a chord.



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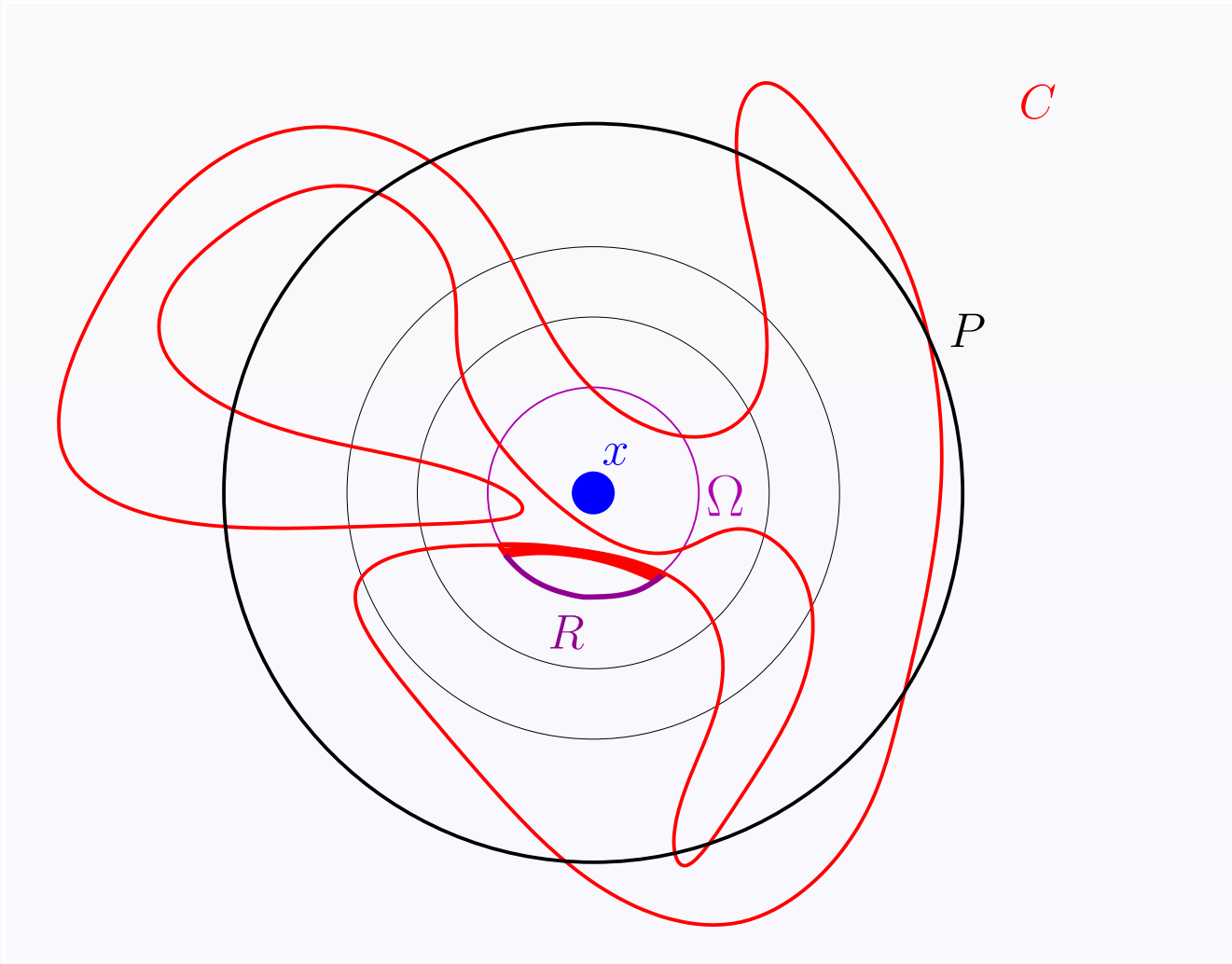

Suppose, to the contrary, that some cycle C crosses the inner cycle Ω.

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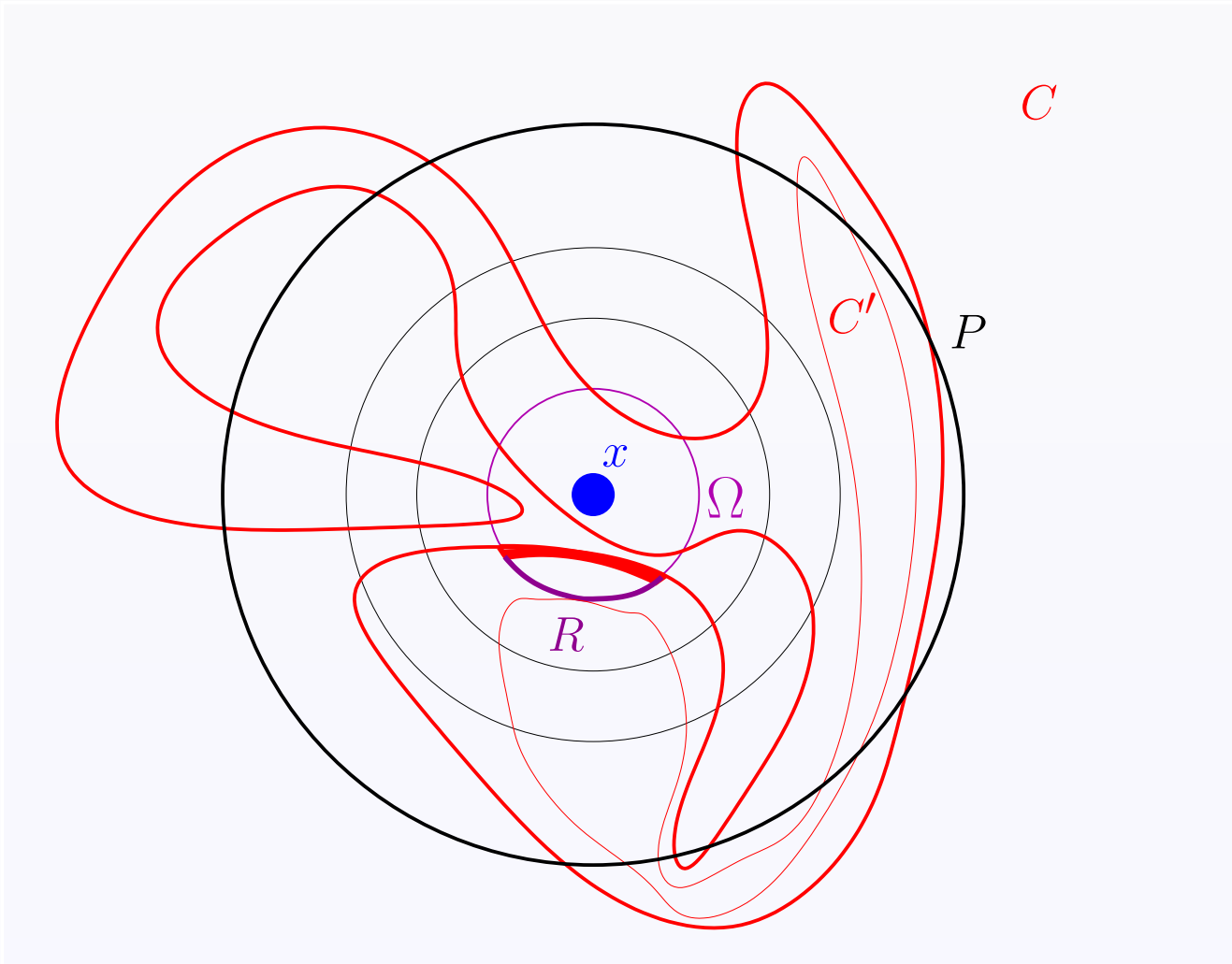
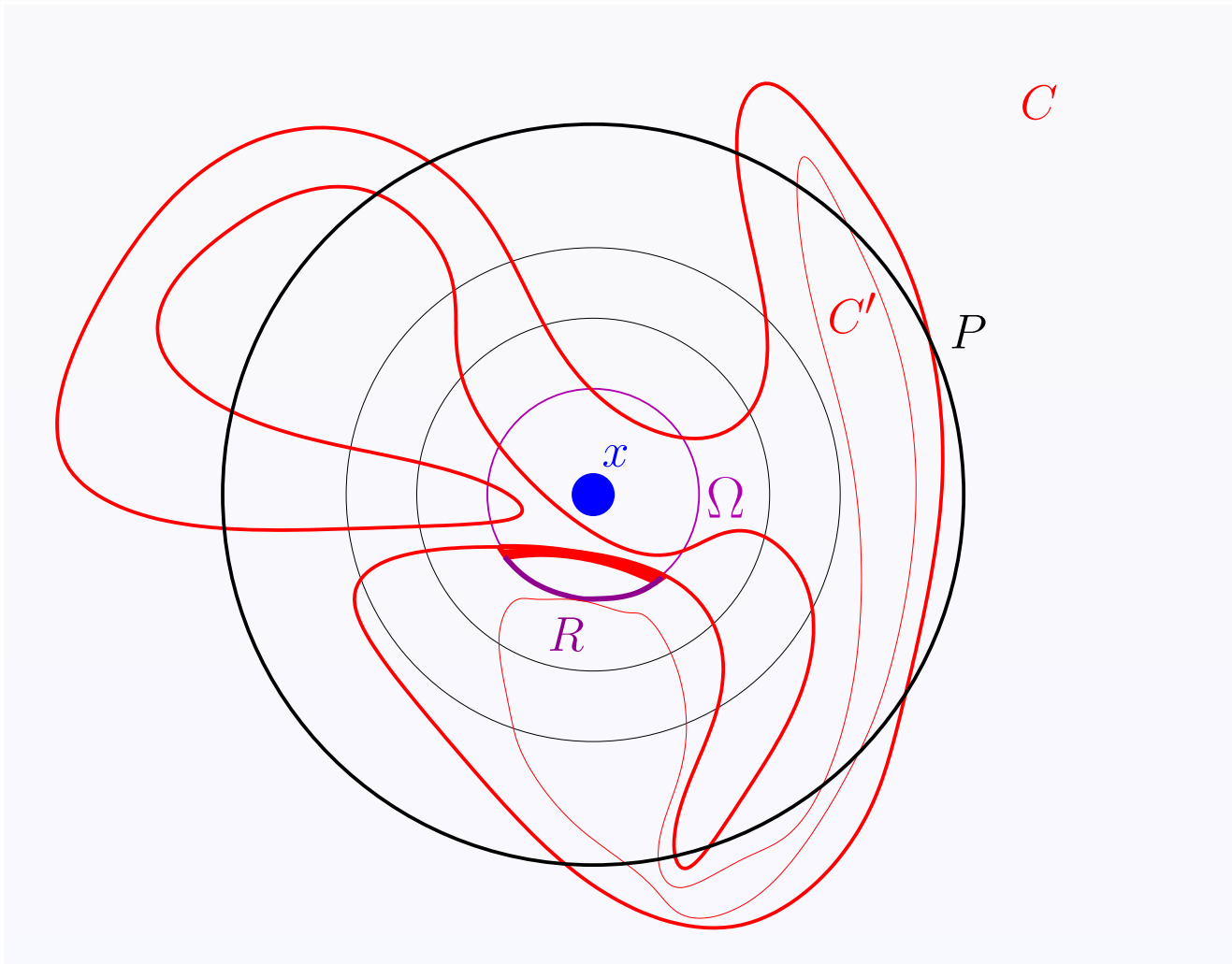
By minimality R should be met by some other cycle $C' \neq C$.

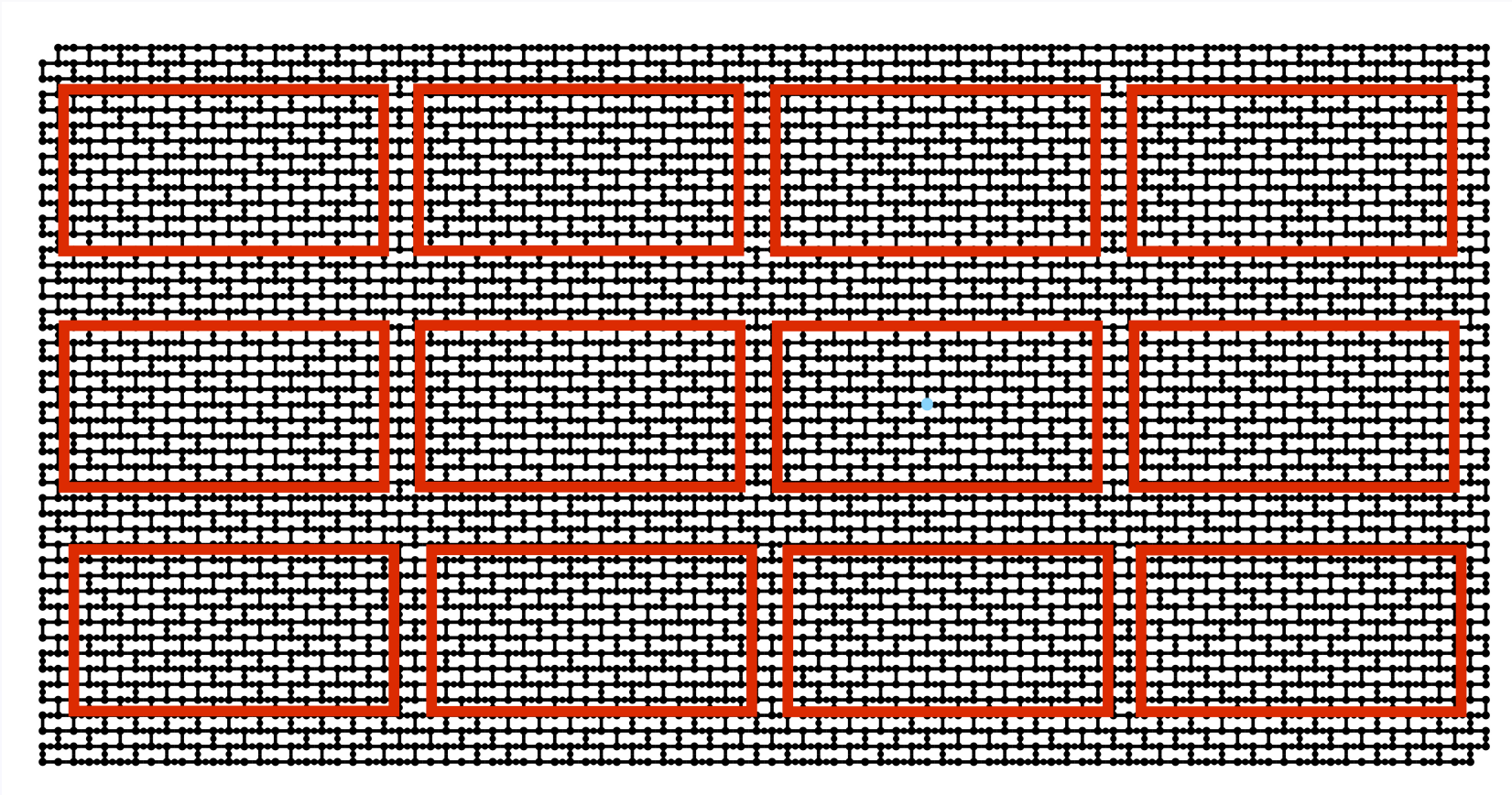


By repetitively applying this argument we find in G has as many disjoint odd cycles as its homocentric cycles that are $k+1$, a contradiction.

Therefore none of the k disjoint odd cycles crosses Ω. Thus x is irrelevant.



To find the irrelevant vertex x can be done in polynomial time!



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- The same ideas prove: p-Planar Odd Induced Cycle Packing$\in 2^{O(k^{3/2})}$-FPT

while general p-Odd Induced Cycle Packing problem is para-NP-hard.

[Golovach, Kamiński, Paulusma, Thilikos, TCS 2012]

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Question: $H \leq G$?

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Merci beaucoup!