Nonsmooth Analysis and Applications

Francis Clarke

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et Université de Lyon**

Classical Calculus

A basic technique in mathematics is linearization

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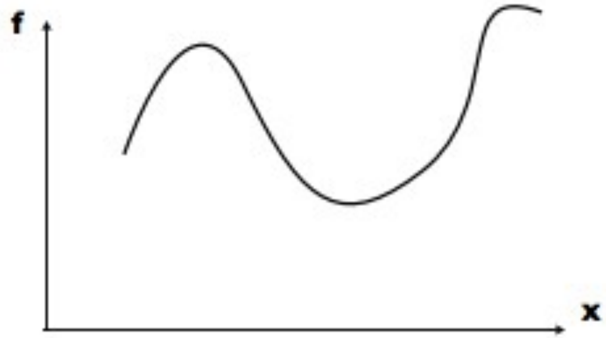

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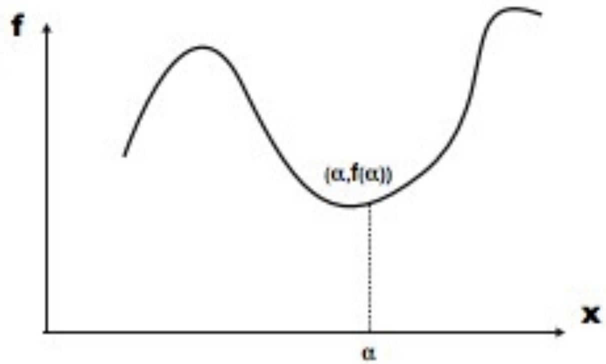

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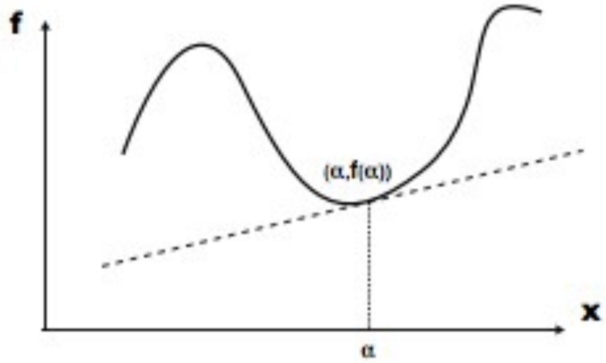

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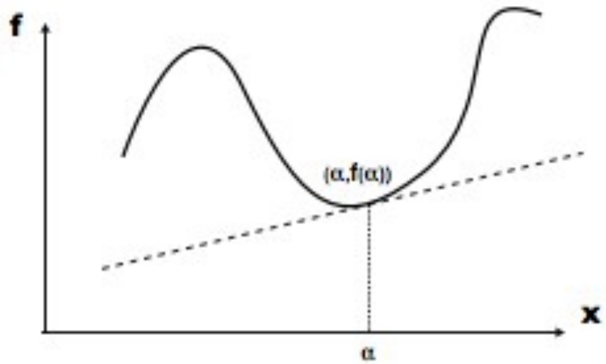

$f'(\alpha)$ = the slope of the tangent line to the graph of f through the point $(\alpha, f(\alpha))$

A. Minimizing a function f(x)

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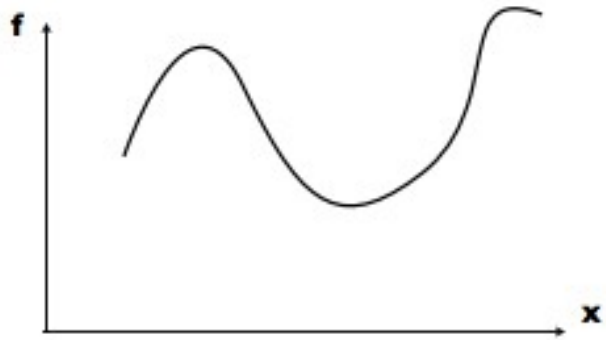

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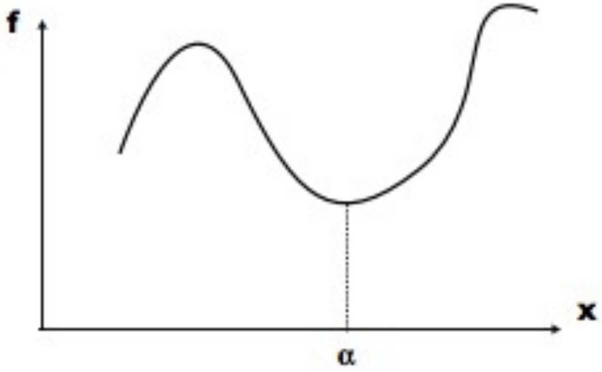

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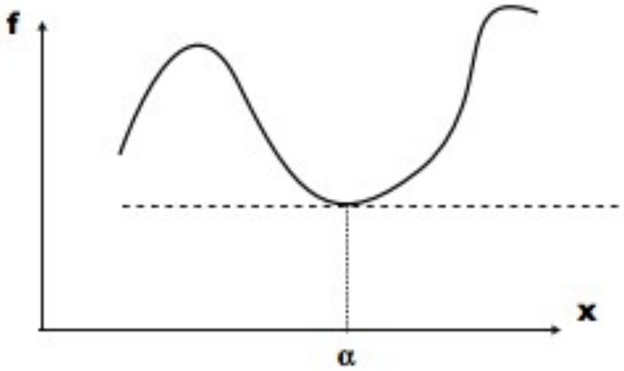

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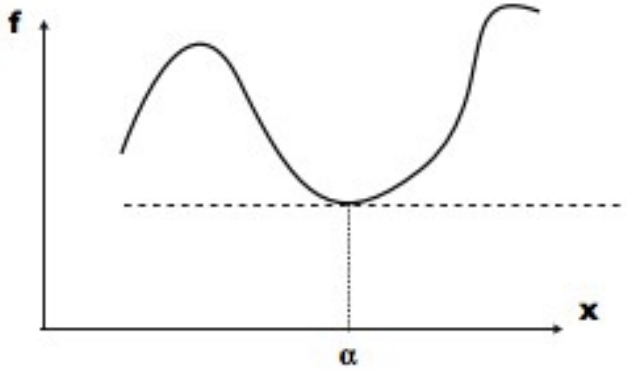
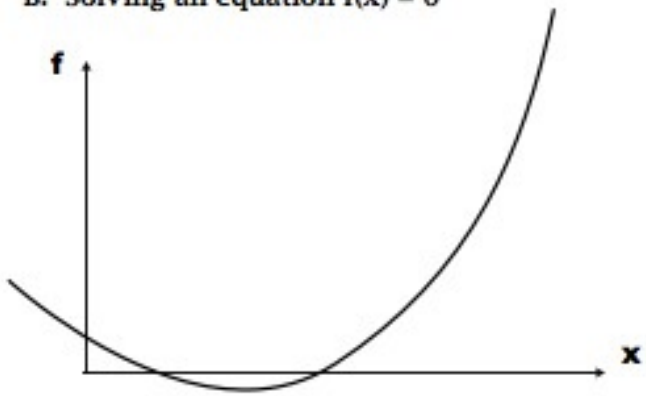


Fermat's rule : at a minimum α, we have $f'(\alpha) = 0$

B. Solving an equation $f(x) = 0$



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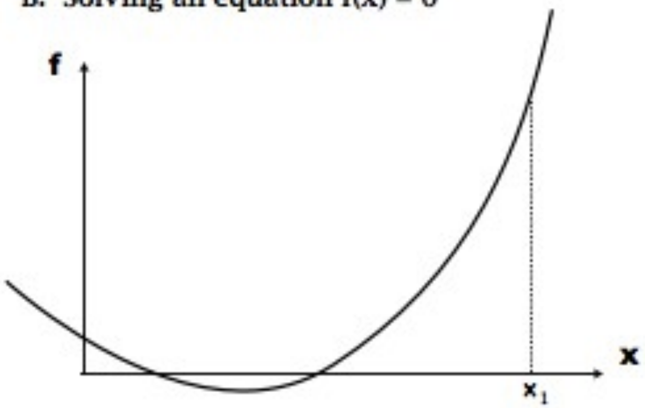

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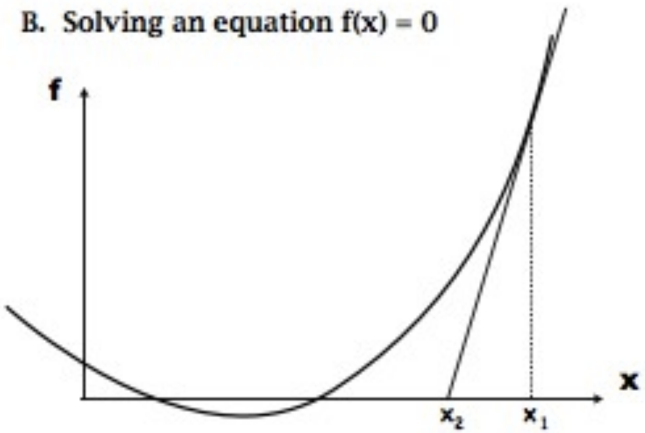

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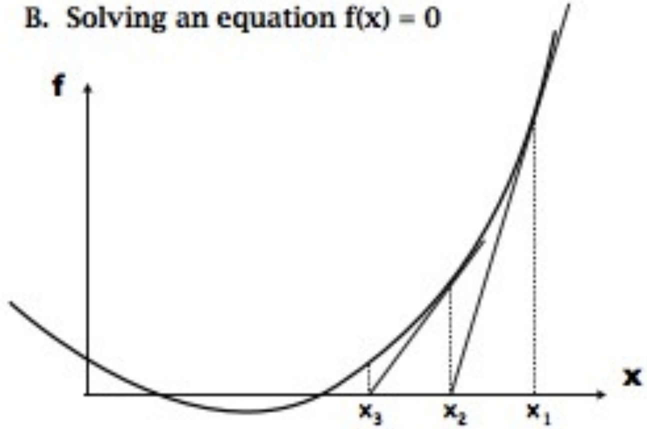

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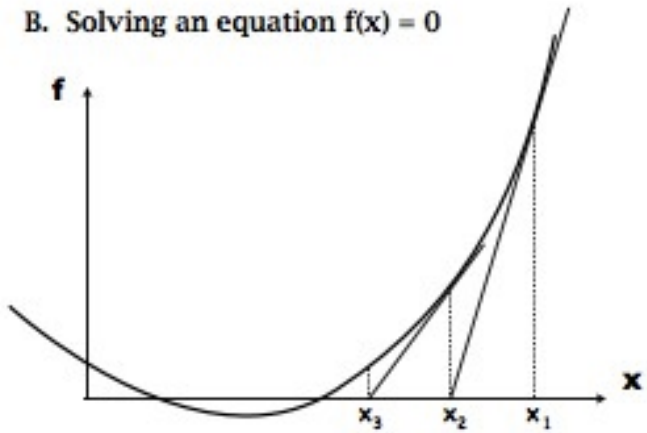

Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

C. Studying a system $\begin{array}{l} x'(t)=f(x(t),y(t)) \\ y'(t)=g(x(t),y(t)) \end{array}$

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Calculate

$$A := \begin{bmatrix} f_x(0,0) & f_y(0,0) \\ g_x(0,0) & g_y(0,0) \end{bmatrix}$$

Then study

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix}$$

via eigenvalues...

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But what if f is not differentiable?

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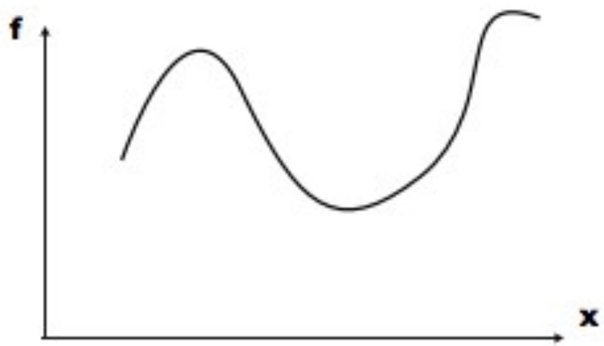

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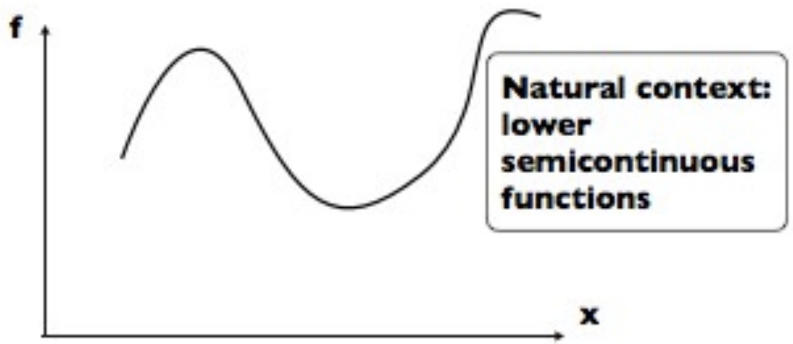

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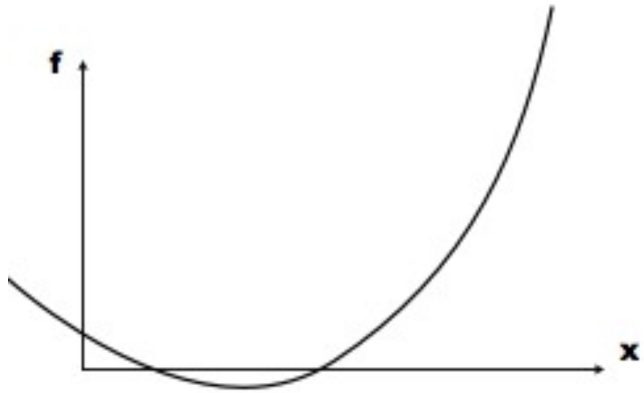

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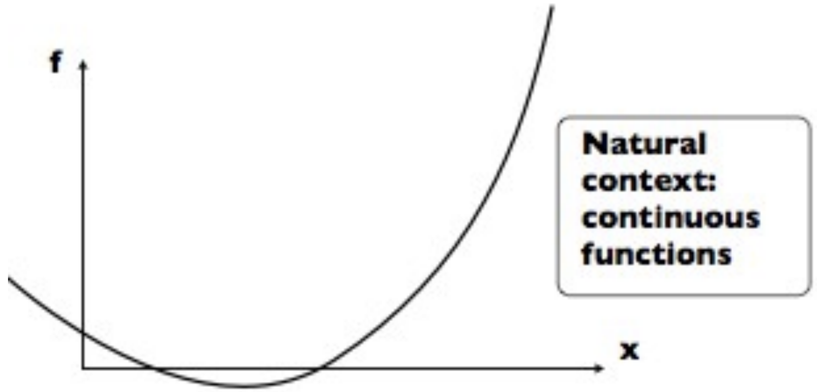

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Natural context: locally Lipschitz functions

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Nonsmooth Analysis began with "Dini Derivates" :

Fondamenti per la teorica delle funzioni di variabili reali Ulysse Dini 1878

The swing in your backyard

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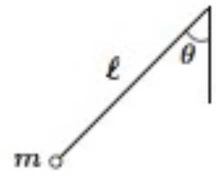
The nonlinear pendulum



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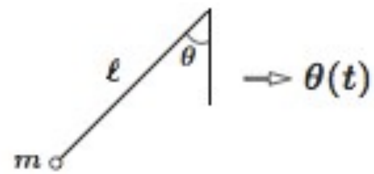

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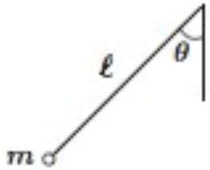

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Newton(-Euler)

$$m\,\ell\,\theta'' = -m\,g\sin\theta \implies \theta'' + (g/\ell)\sin\theta = 0$$

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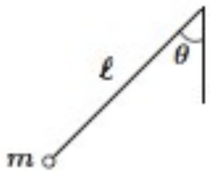

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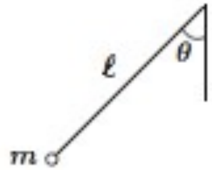

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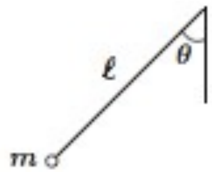

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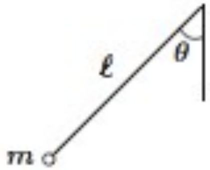

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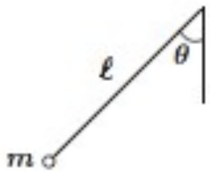

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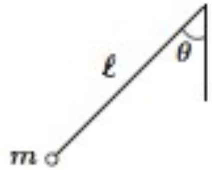

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equilibrium $\theta = \theta_0$, $\tan\theta_0 = -f/(mg)$

Let the mass be a thin seat of width w

Reality is more complicated...

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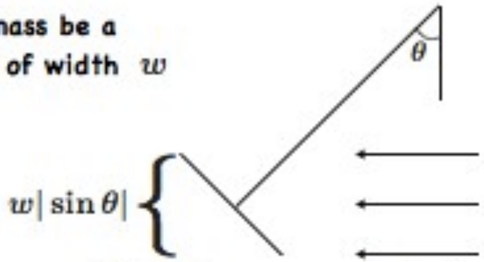

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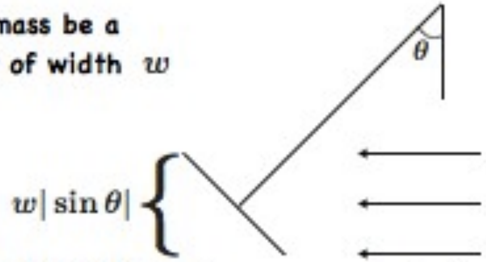

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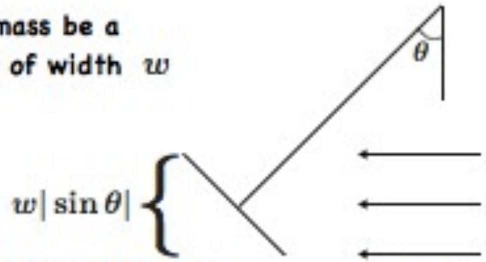

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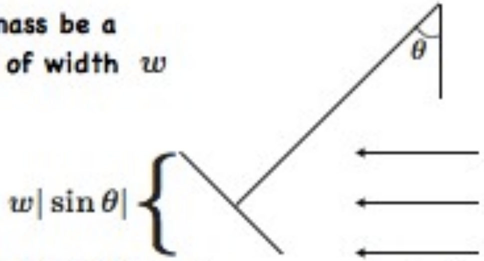

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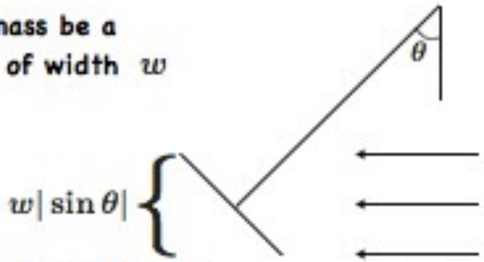

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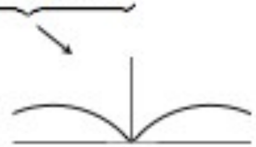

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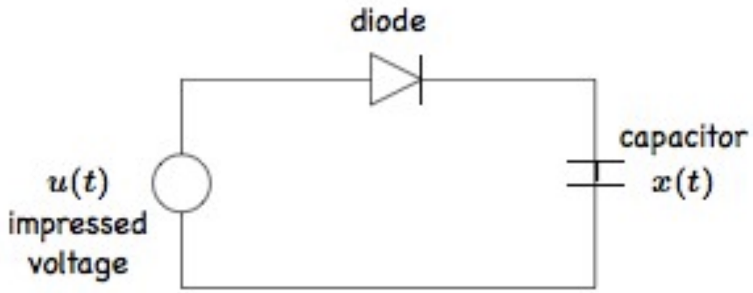

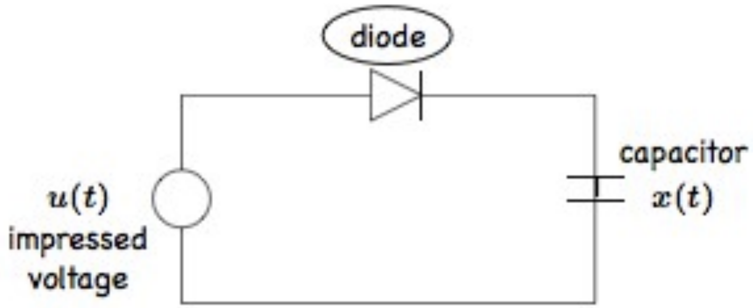

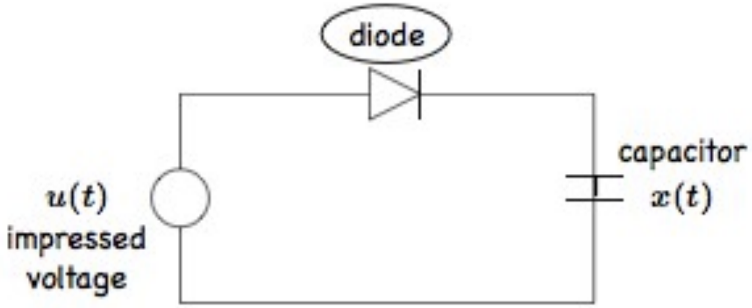

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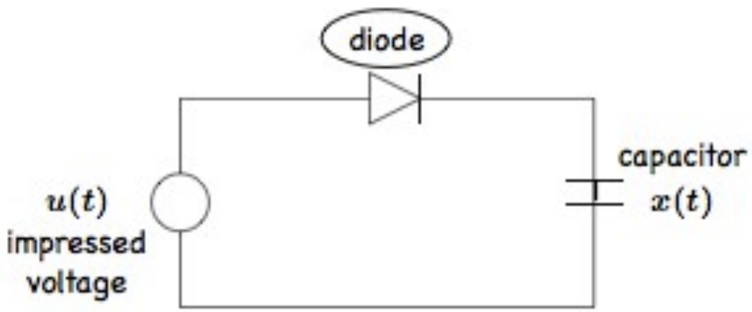

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$f(x,u)$ has a corner at $x = u$

Directional or threshold phenomena are often nonsmooth.
Also where there is presence of shapes... another example:

nonsmooth contact

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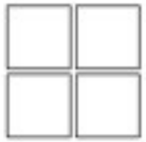

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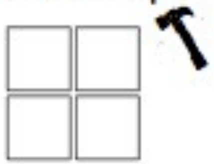

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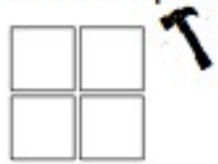

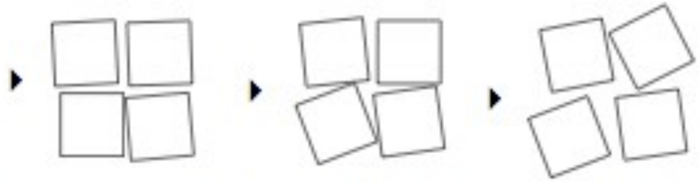

See Marsden et alii (generalized gradients + least action).
Other nonsmooth mechanics and elasticity: Brogliato, Moreau, Panagiotopoulos, Paoli, Schatzman, Schuricht ...

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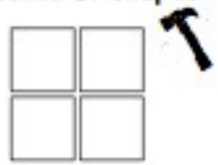

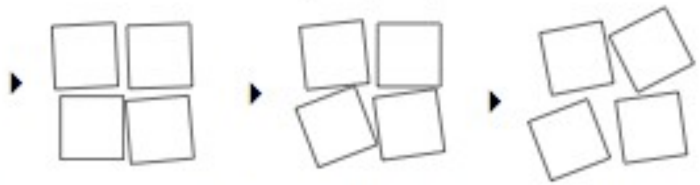

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hard-working dots

Optimization

Example: eigenvalue design

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$$A(x) = \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

$$\lambda = 1 \pm |x| \implies f(x) = 1 + |x|$$

Note that f attains its min at a "corner"

Example: the distance function

$$d_S(x) := \min_{s \in S} |x - s|$$

$S \subset \mathbb{R}^n$ (closed, nonempty)

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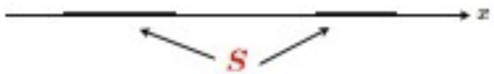

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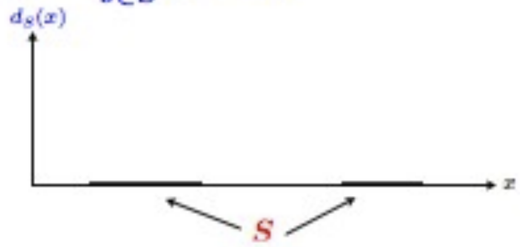

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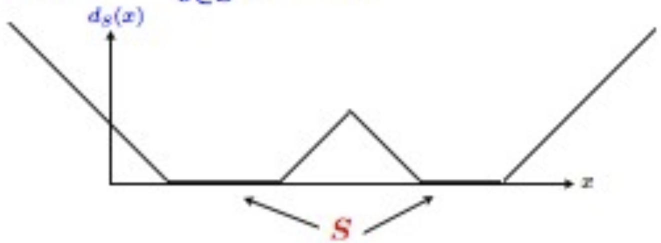

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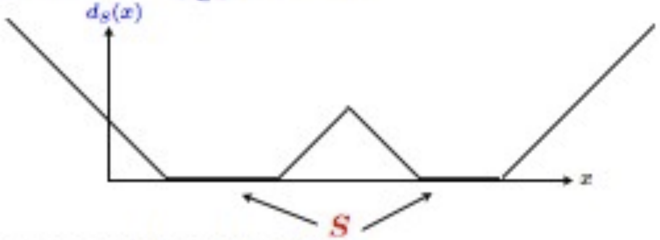

One use: exact penalization

$$\min_{x \in S} g(x) \quad \underset{?}{\Leftrightarrow} \quad \min_x g(x) + k\, d_S(x)$$

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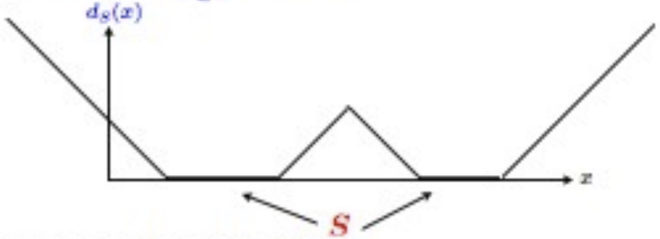
$$d_S(x) := \min_{s \in S} |x - s|$$



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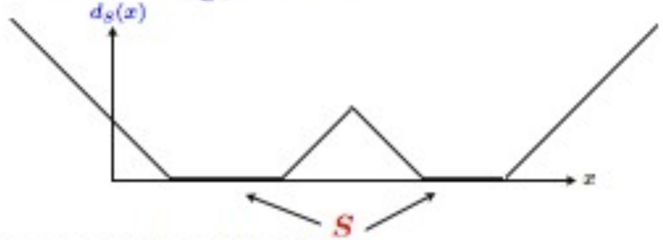

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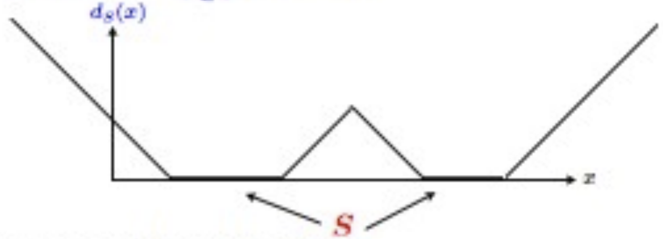

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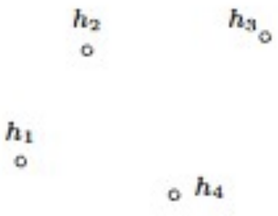
geometric interpretation via normals or (Euler-) Lagrange multipliers ?

Example: Problem of Torricelli/Steiner, n = 4



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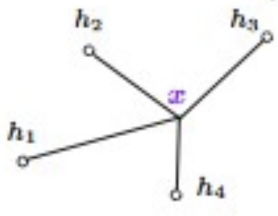
We seek a central point x relative to h_1, h_2, h_3, h_4: x minimizes $|h_1 - x| + |h_2 - x| + |h_3 - x| + |h_4 - x|$



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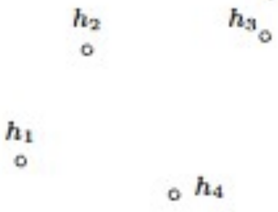

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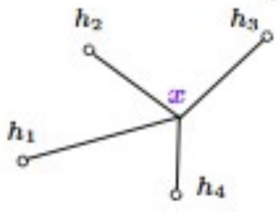

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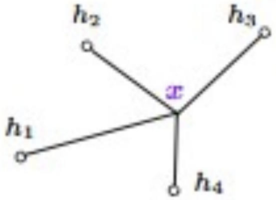

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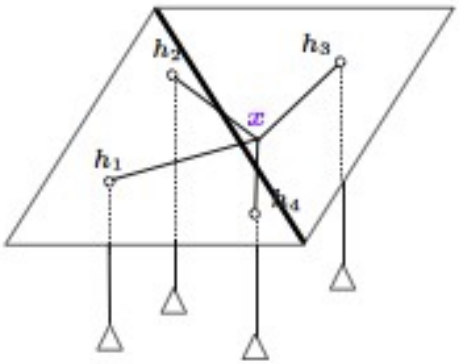

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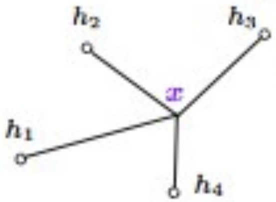

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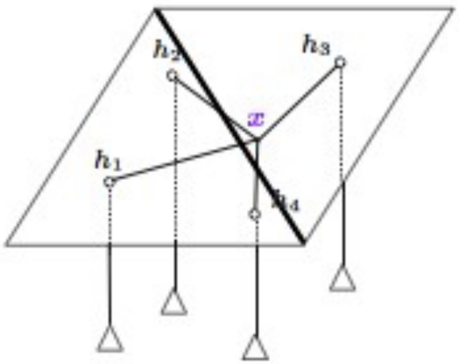

At equilibrium, the point minimizes the potential energy of the system (d'Alembert)

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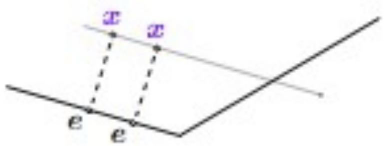

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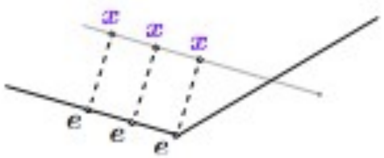

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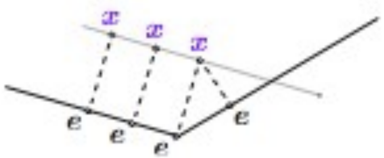

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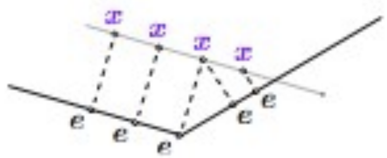

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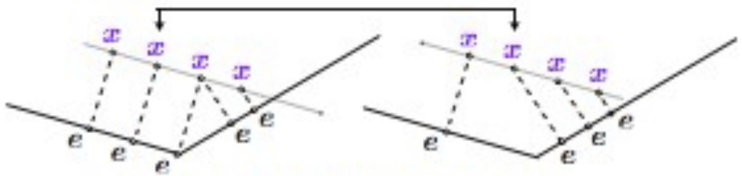

hysteresis (non-reversible dynamics)

Calculus of variations

The Basic Problem:

$$\min_{x(\cdot)} \int_a^b L(t, x(t), x'(t))\, dt,\ x(a) = A, x(b) = B$$

Euler (1744) defined the problem, found the basic necessary condition, introduced multipliers for constrained problems, postulated the principle of least action, and gave 100 examples.

Leonhard Euler

1707-1783

Example: soap bubble

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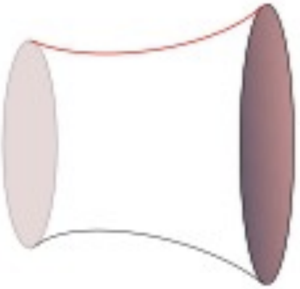

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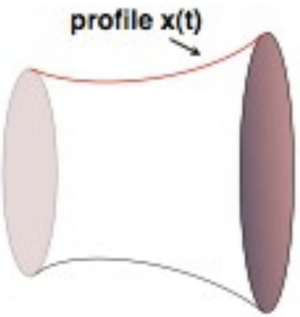

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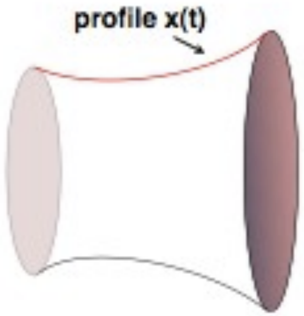
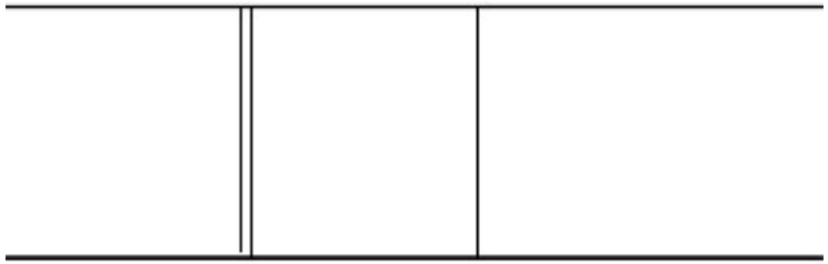


surface area | **Euler(-Lagrange) equation**

$$\min_{x(\cdot)} \int_a^b x(t)\sqrt{1+x'(t)^2}\,dt \quad \Rightarrow \quad \frac{d}{dt}\left\{\frac{x'(t)x(t)}{\sqrt{1+x'(t)^2}}\right\} = \sqrt{1+x'(t)^2}$$

Solutions with corners



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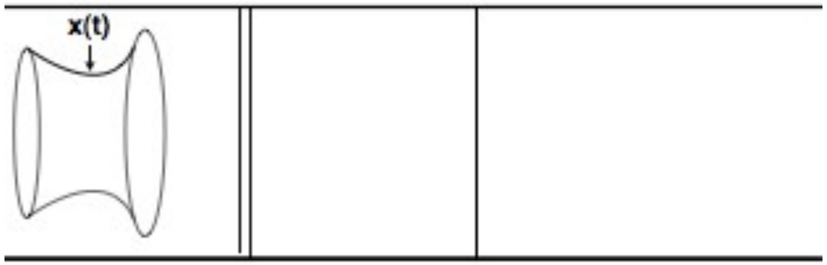

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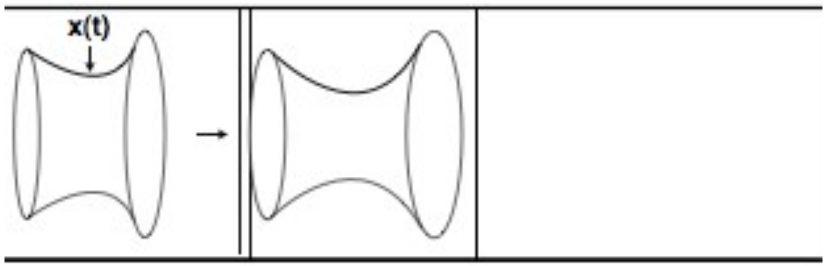

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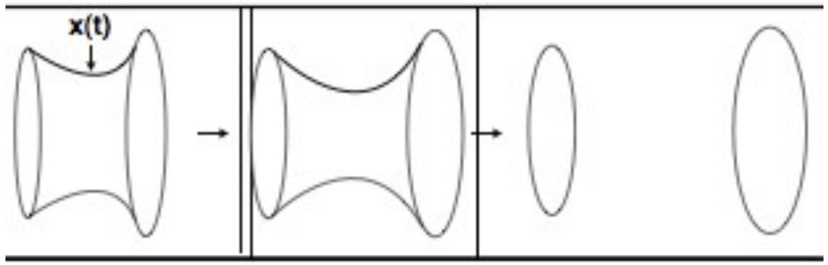

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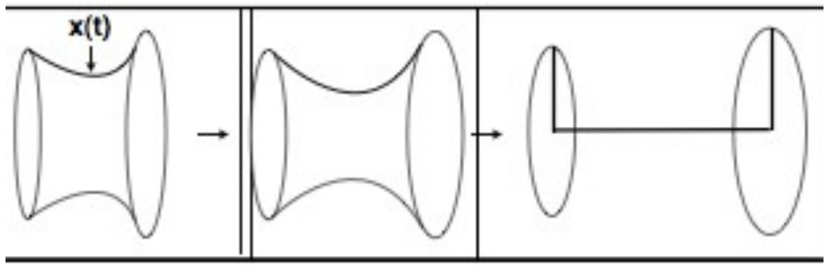

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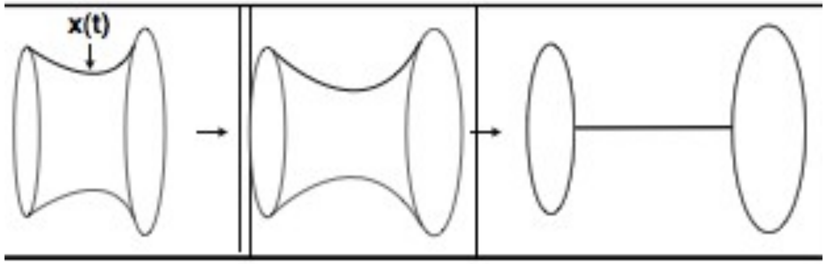

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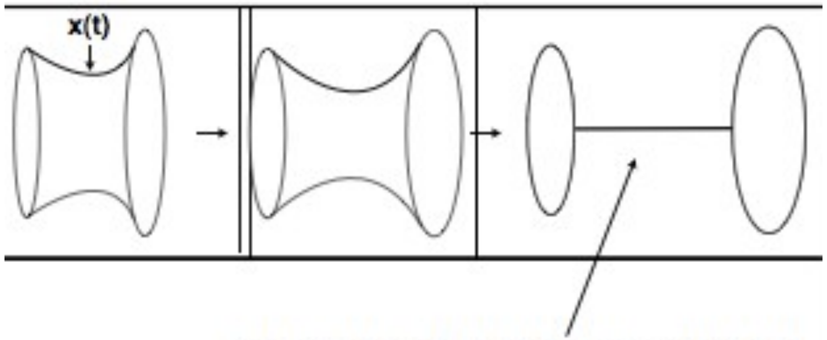

The Goldschmidt solution (1831)

A design problem

A design problem

Ainsi c'est un problème de maximis et minimis de déterminer la courbe qui, par sa rotation autour de son axe formera une colonne capable de supporter la plus grande charge possible, la hauteur et la masse de la colonne étant données.

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Lagrange (1770) *Sur la figure des colonnes*

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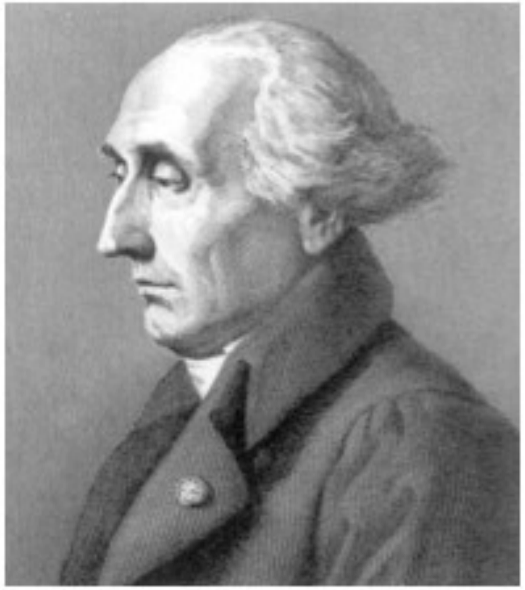
Lagrange (1770) *Sur la figure des colonnes*

To find the curve which by its revolution determines the column of greatest efficiency **Truesdell**



Joseph Louis
Lagrange

Born Turin 1736

- **Writes to Euler in 1755, describes the method of *variations***
- **Euler names the subject in his honor : *calculus of variations***
- **Euler is his mentor until his death**

Lagrange

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- Dies in Paris in 1813 at the age of 77

Designing an optimal column

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1. Choose a profile *x*

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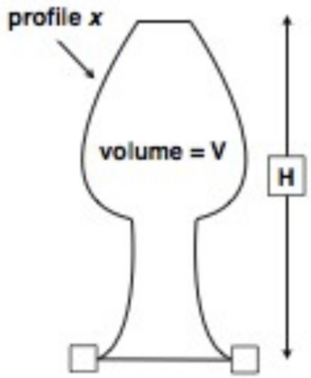

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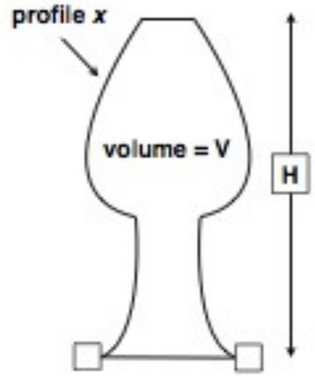

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4. Calculate (via Euler) the buckling strength $f(x)$ of the column $C(x)$

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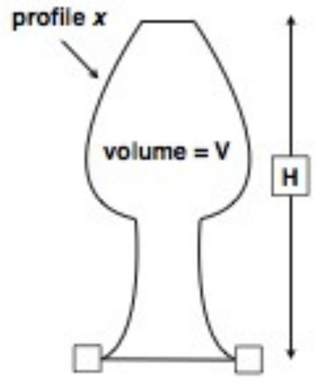

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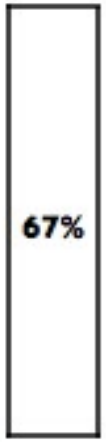
4. Calculate (via Euler) the buckling strength $f(x)$ of the column $C(x)$
5. Maximize $f(x)$ over x

Three solutions

Three solutions

Lagrange
1770

Three solutions



Lagrange
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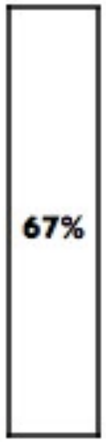
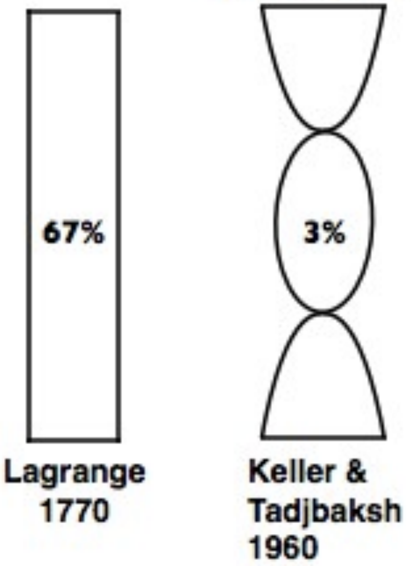


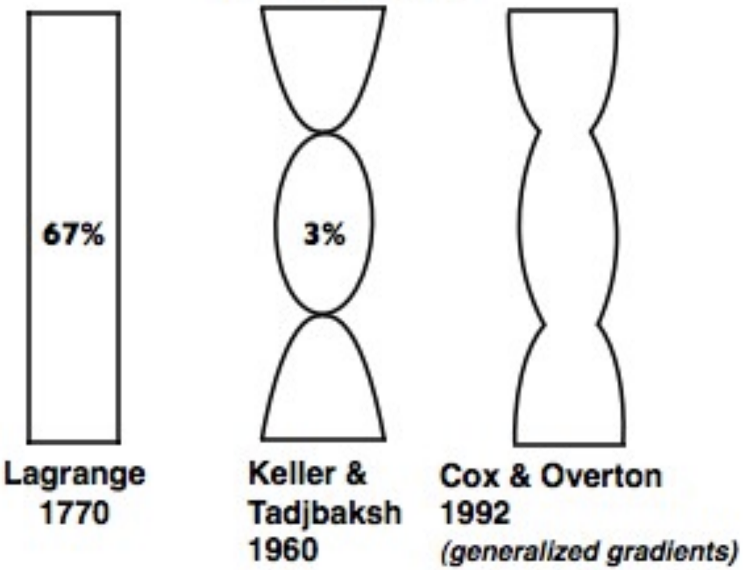
Lagrange
1770

Keller &
Tadjbaksh
1960

Three solutions



Three solutions



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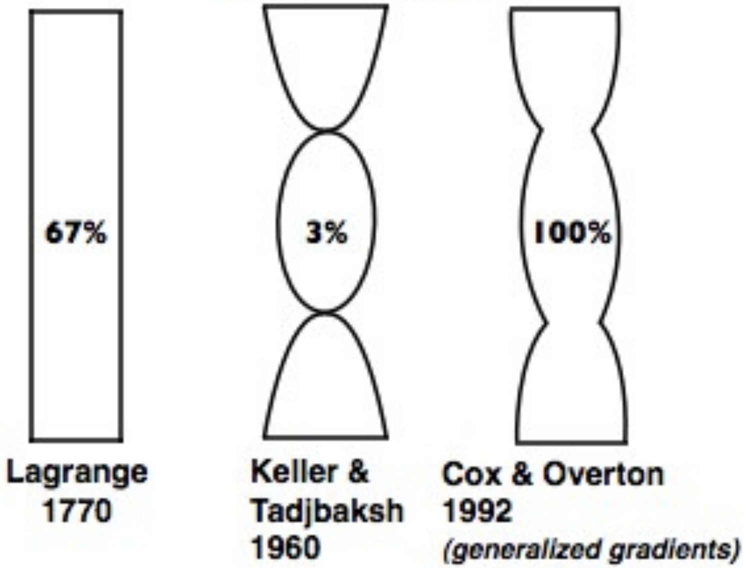

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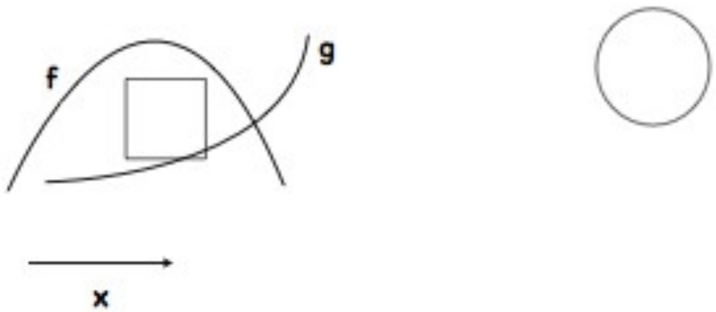

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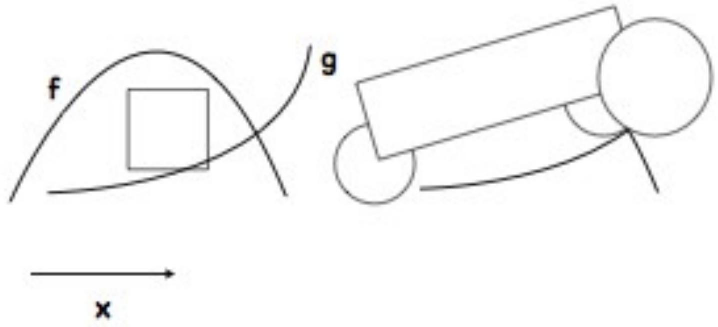

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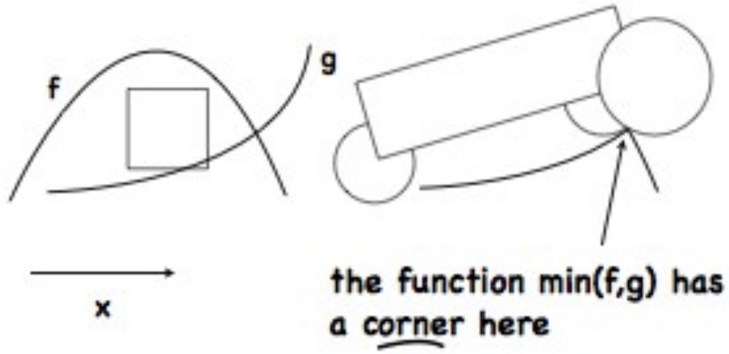

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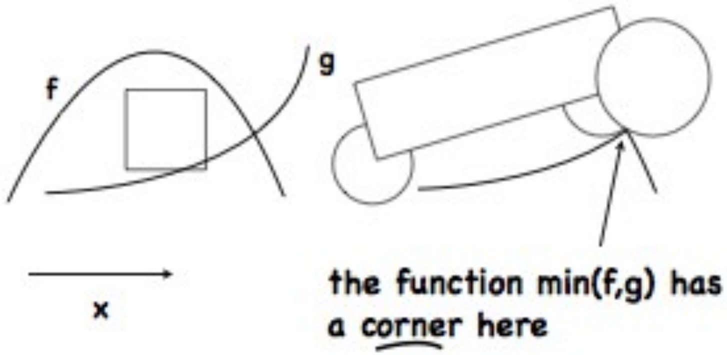

The function "maximal load supported by a column of profile x" is a nonsmooth function of x ... which is where the error was made

Generalized gradients and proximal normals

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Four definitions
Clarke 1973

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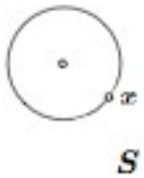

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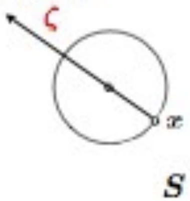

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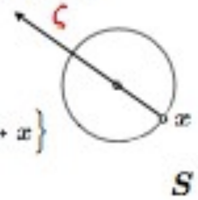
$$N_S^C(x) = \operatorname{cl} \bigcup_{\lambda \geq 0} \lambda \partial_C d_S(x)$$
$$= \operatorname{cl}\operatorname{co}\left\{ \lim_{i\to\infty} \zeta_i : \zeta_i \in N_S^P(x_i), x_i \to x \right\}$$



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These ***generalized gradients*** (1972)
apply on any Banach space.

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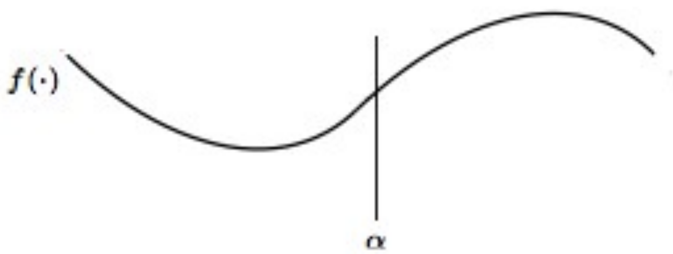

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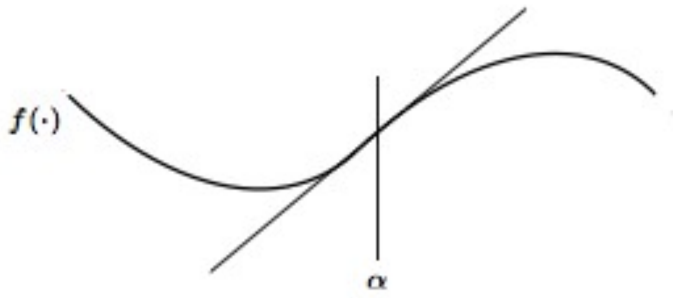

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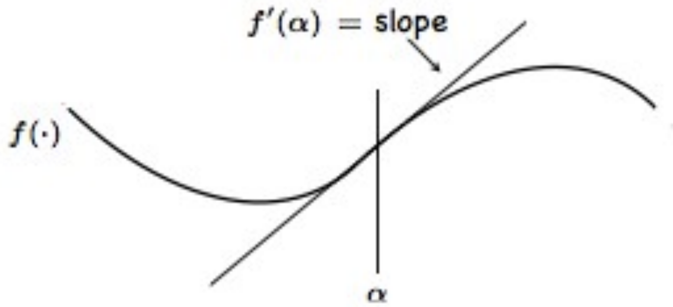

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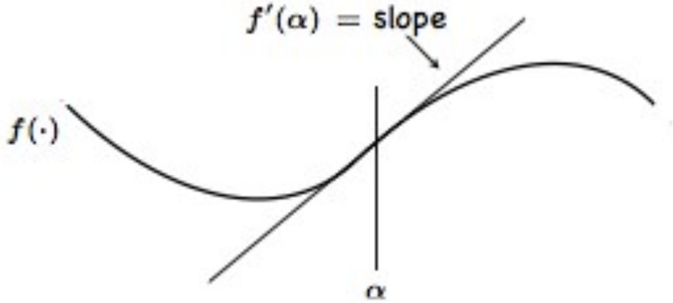
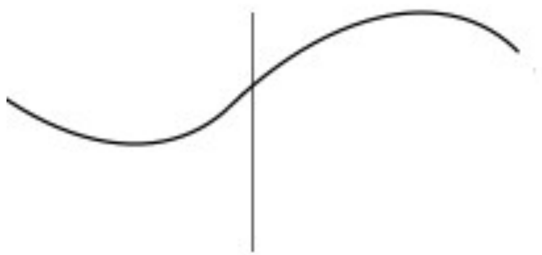
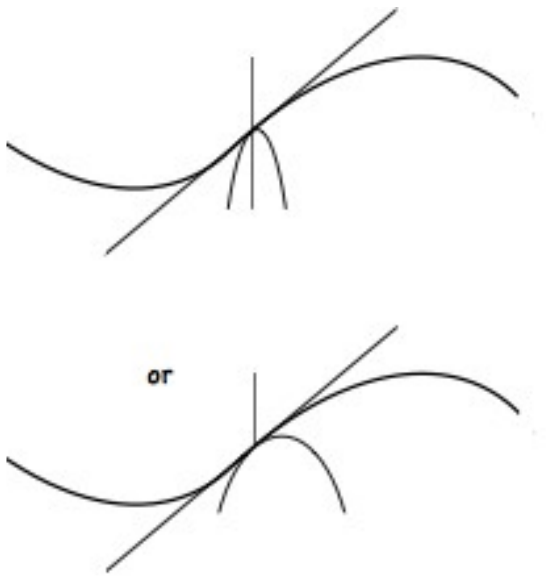
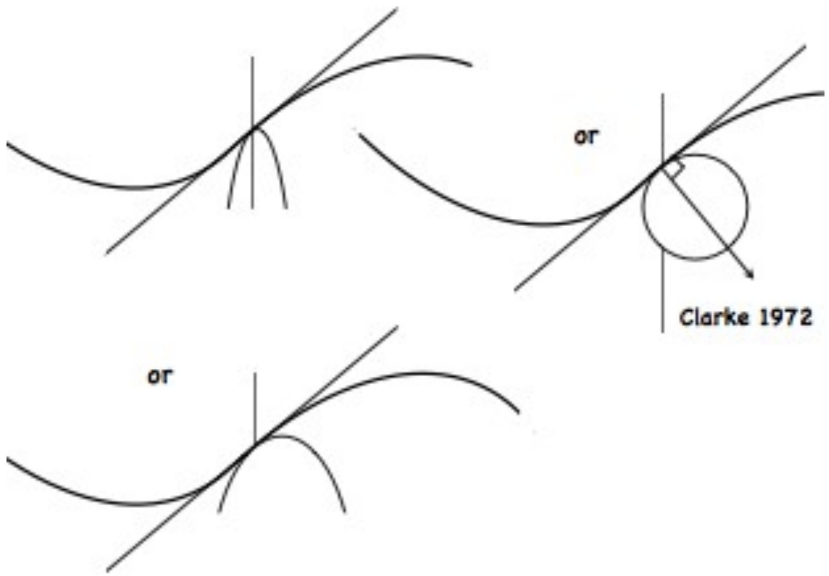
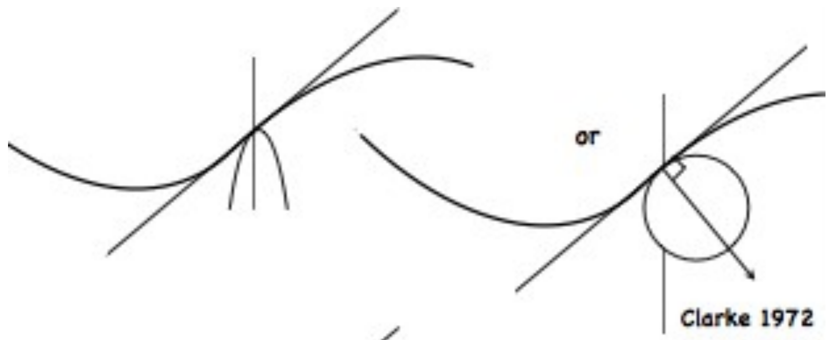


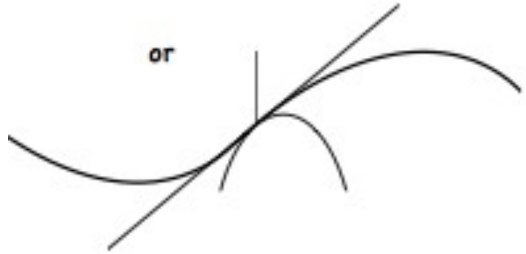
We may also approximate just from below, using nonlinear functions: proximal analysis











We can apply the `local lower-approximation by parabolas' idea to nonsmooth (lsc) functions

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(P = proximal)

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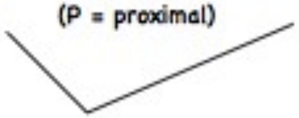

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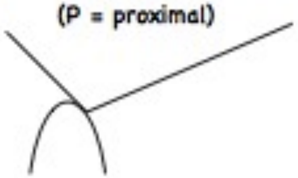

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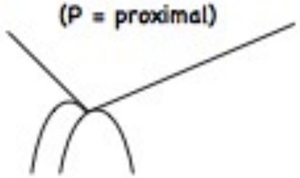

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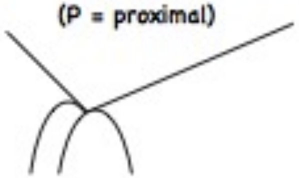

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$\partial_P f$ has a very complete (but fuzzy!) theory and calculus...
Borwein, Ioffe, Ledyaev, Loewen, Rockafellar, Vinter, Zeidan...

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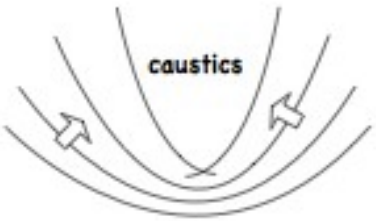

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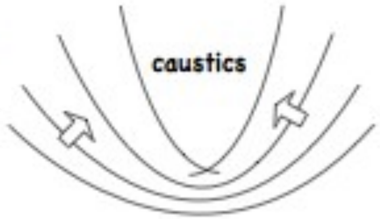

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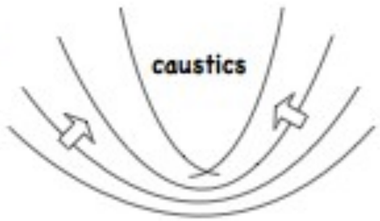

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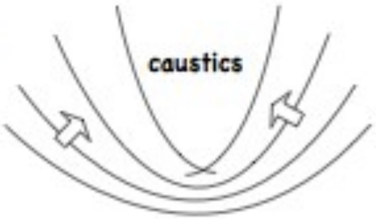

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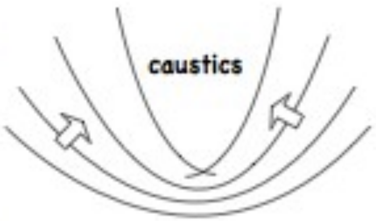

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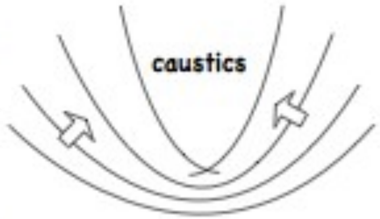

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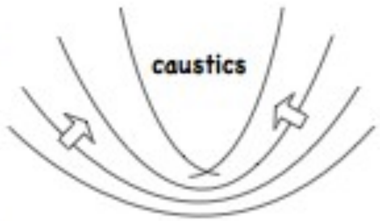

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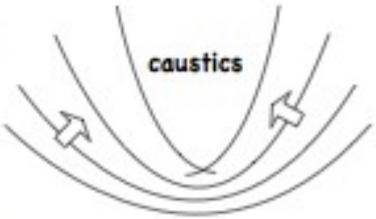

For linear pde's one can circumvent nonsmoothness by distributions... but in the nonlinear case, a careful analysis of the points of nondifferentiability is required.

Example (n = 1)

$$[\varphi'(x)]^2 - 1 = 0, \;\; \varphi(0) = \varphi(1) = 0$$

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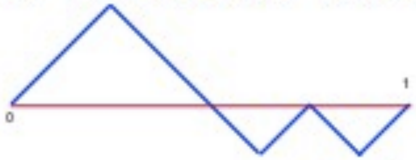

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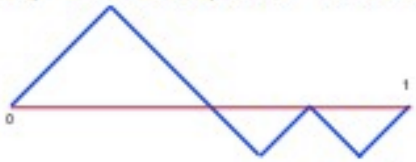

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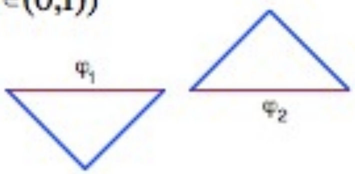
Hint: it is one of these two functions:



Optimal control : an example in bioeconomics

(Clark, Clarke, Munro / Econometrica)

$$x'(t)=G(x(t))-u(t)\,x(t)$$
$$\max\int_0^\infty e^{-\delta t}\{\pi\,x(t)-k\}\,u(t)\,dt$$
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u = fishing effort
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Optimal control : an example in bioeconomics

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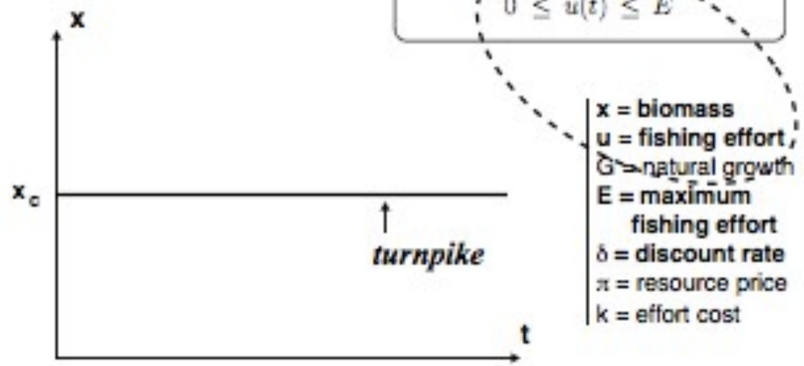

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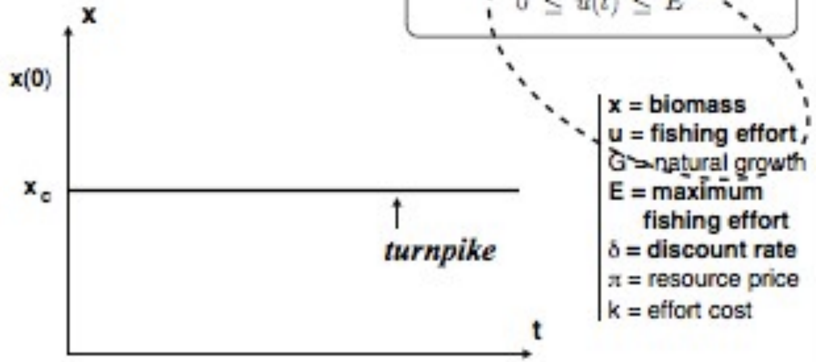

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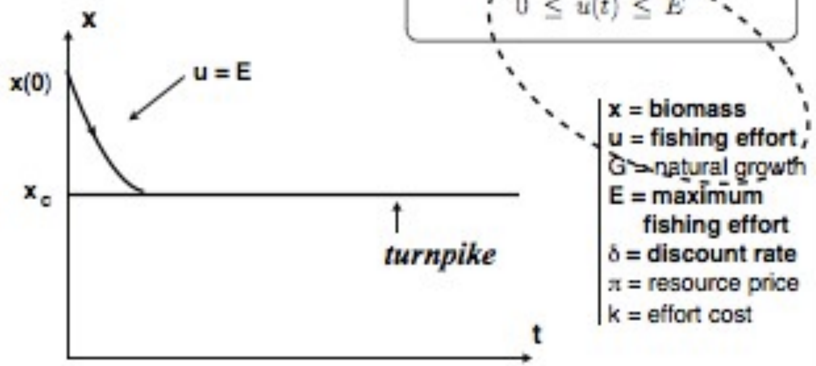

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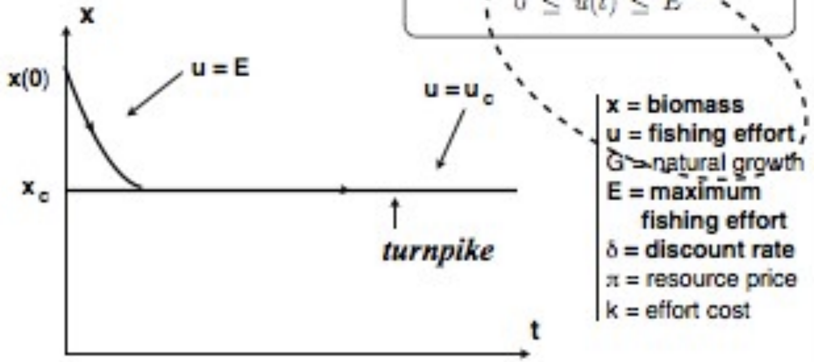

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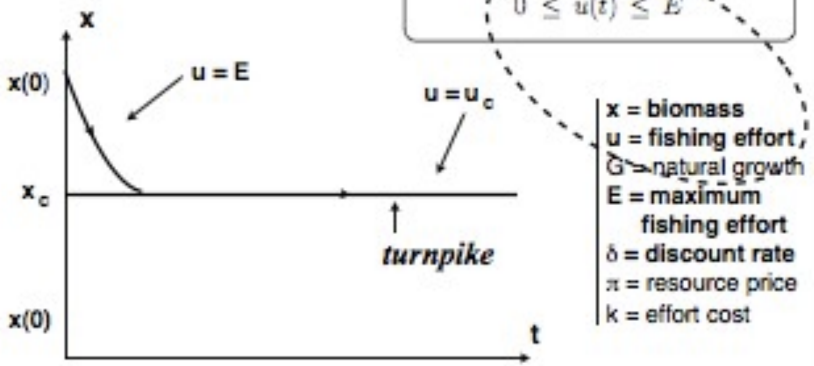

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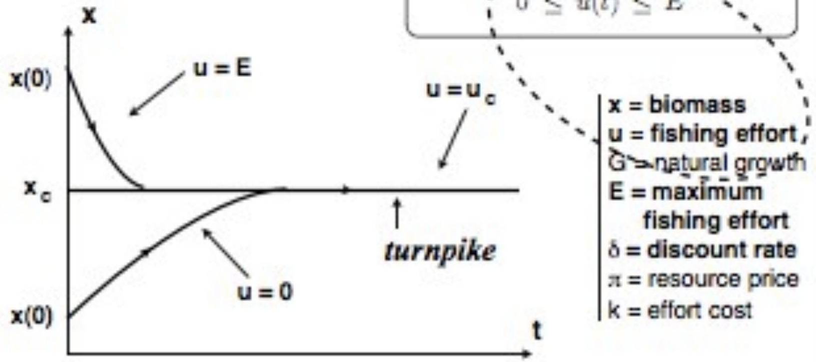

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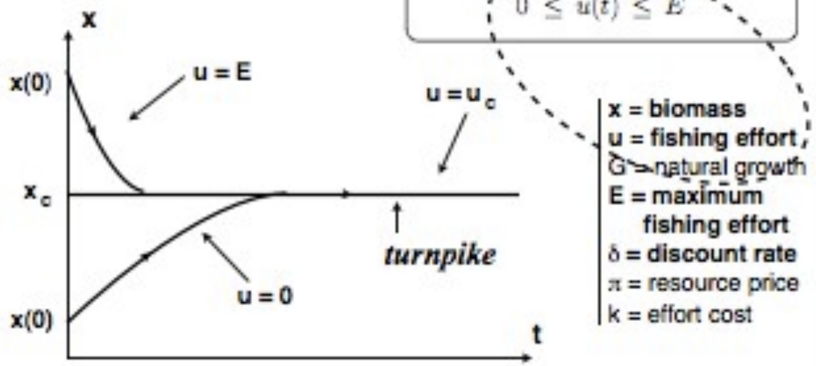
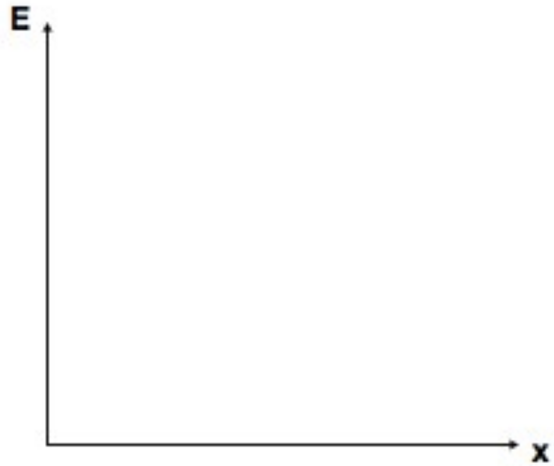


If δ is sufficiently large, we have $x_c = 0$ (extinction)

Example: Optimal fishing strategy in the presence of both investment and depreciation in boats
(Clark, Clarke, Munro / Econometrica)



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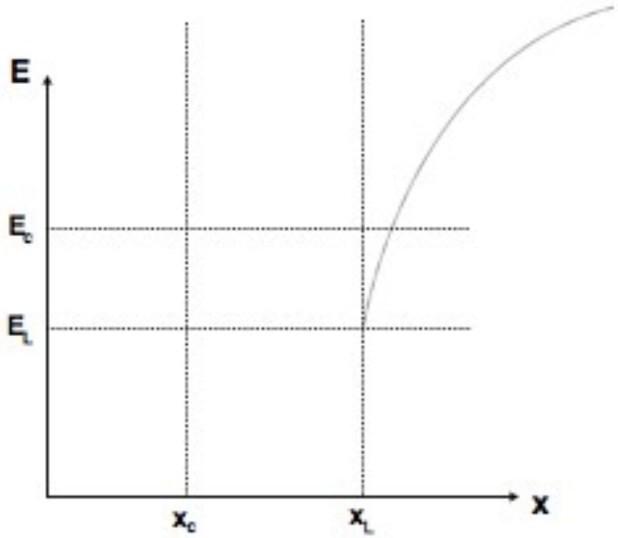
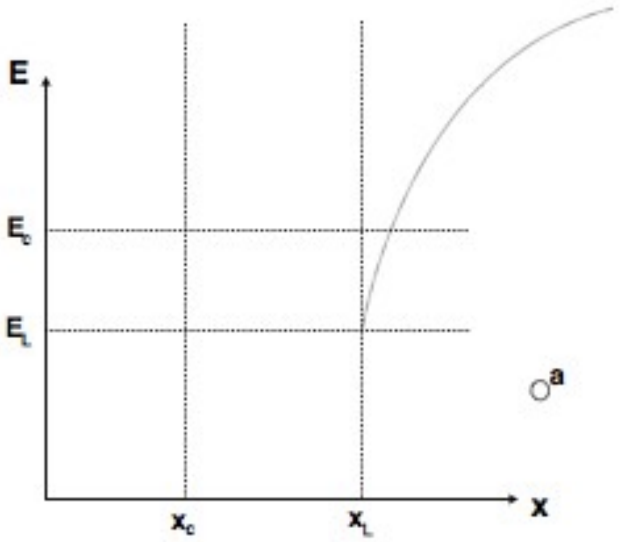
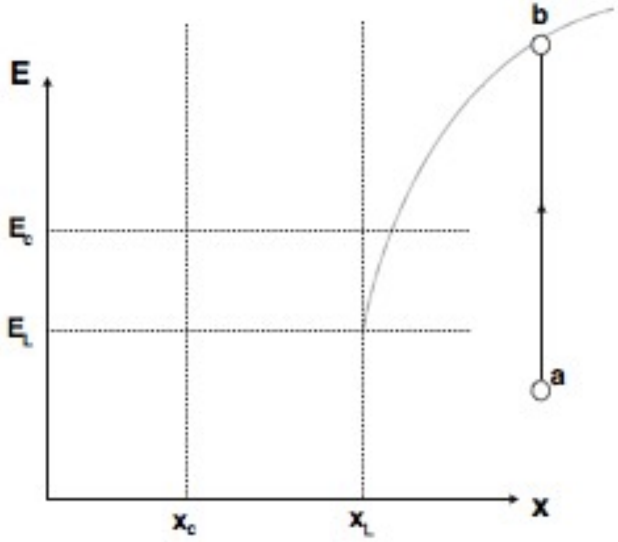
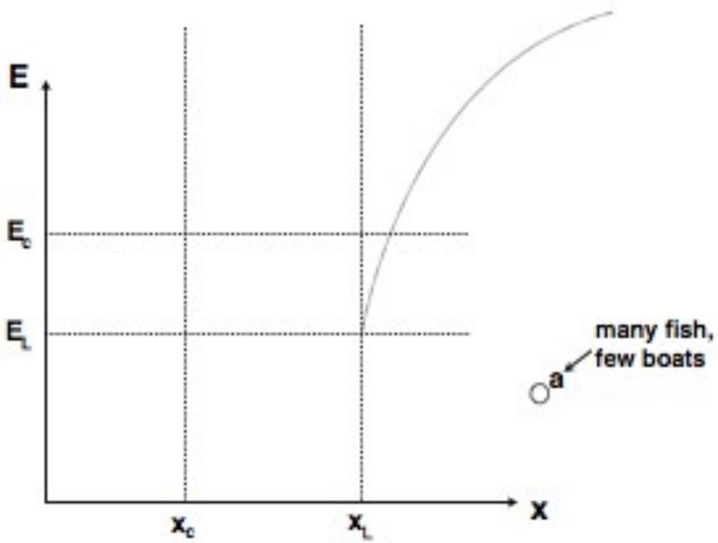
$$\max \int_0^\infty e^{-\delta t}\left\{(\pi x(t) - k)u(t) - cI(t)\right\} dt + \sum_i e^{-\delta t_i} \Delta E(t_i)$$

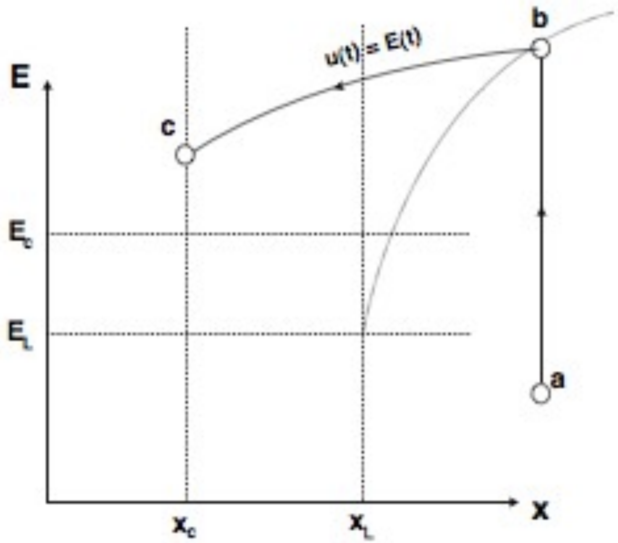
$$x'(t) = g(x(t)) - u(t)x(t),\ 0 \le u(t) \le E(t)$$
$$E'(t) = -\gamma E(t) + I(t),\ 0 \le I(t) \le +\infty$$

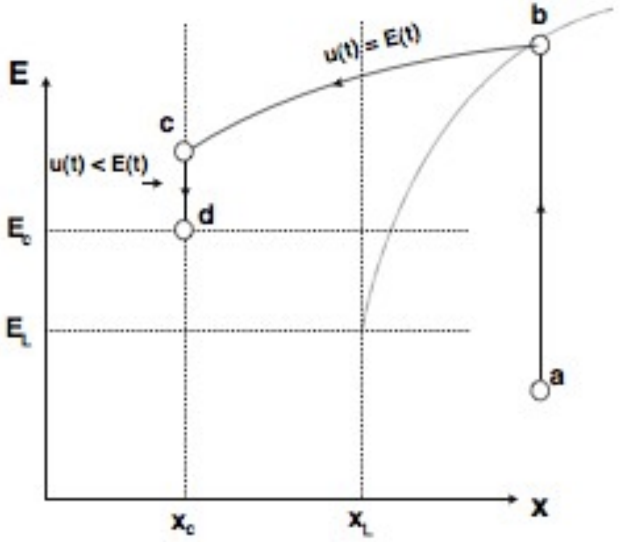


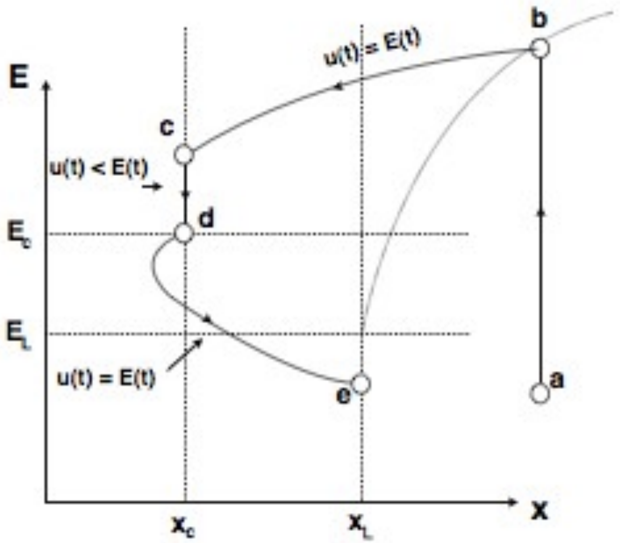

E
E_c
E_L
a
x_c
x_L
x

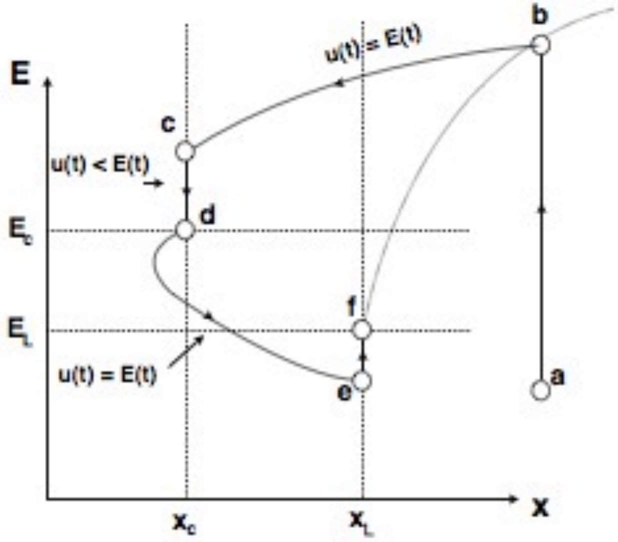

E
b
E_c
E_L
a
x_c
x_L
x

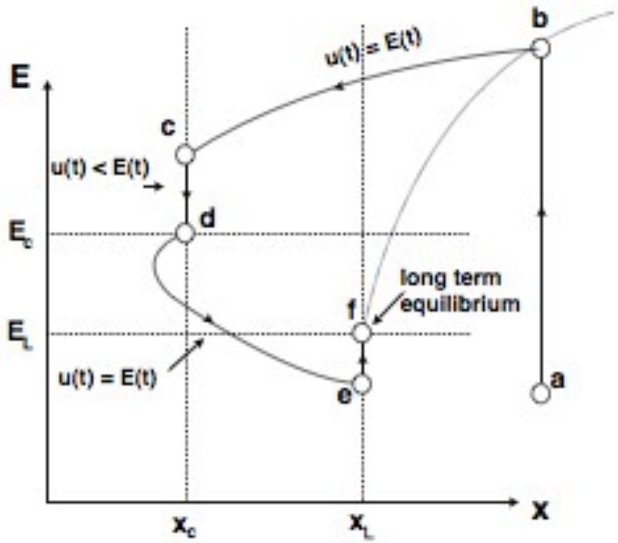

E
E_c
E_L
many fish,
few boats
a
x_c
x_L
x

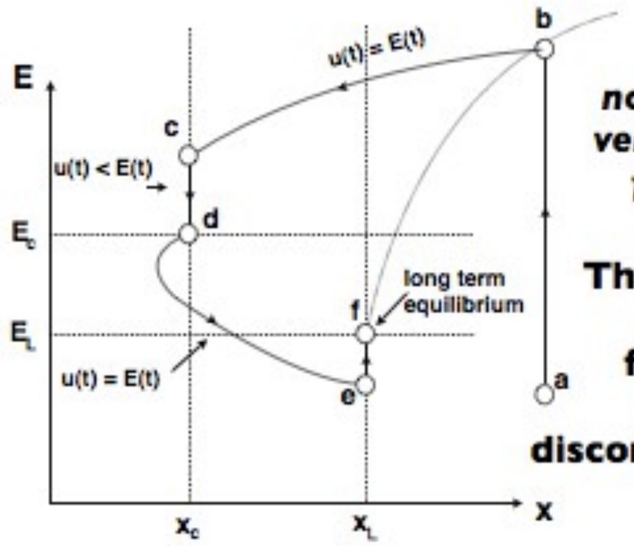

E
b
u(t) = E(t)
c
E_c
E_L
a
x_c
x_L
x


E
b
u(t) = E(t)
c
u(t) < E(t)
d
E_c
E_L
a
x_c
x_L
x


E
b
u(t) = E(t)
c
u(t) < E(t)
d
E_c
E_L
u(t) = E(t)
e
a
x_c
x_L
x


E
b
u(t) = E(t)
c
u(t) < E(t)
d
E_c
f
E_L
u(t) = E(t)
e
a
x_c
x_L
x


E
b
u(t) = E(t)
c
u(t) < E(t)
d
E_c
long term
equilibrium
f
E_L
u(t) = E(t)
e
a
x_c
x_L
x



Proof:
nonsmooth verification functions

There is an optimal feedback u(y,E)... discontinuous

Verification functions

$$\min J(x,u) := \int_a^b L(t,x(t),u(t))\,dt \qquad x(a) = A$$
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Goal: verify that a candidate (x_*, u_*) is optimal

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Method: exhibit a function ϕ satisfying

$$L(t,x,u) \ \ge\ \phi_t(t,x) + \langle \phi_x(t,x), f(t,x,u)\rangle \ \forall (t,x), u\in U(t)$$

$$(= \text{at } (t,x_*(t),u_*(t)))$$

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Proof: For any admissible (x,u) we have

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$$(= \text{for } (x,u) = (x_*,u_*)) \quad \text{QED}$$

Fact: smooth verification functions may not exist, but nonsmooth ones do (Clarke & Vinter, 1980's)

A reference

Nonsmooth Analysis and Control Theory
by
F. Clarke, Yu. Ledyaev, R. Stern, P. Wolenski

Graduate Texts in Mathematics
Springer-Verlag 1998

clarke@math.univ-lyon1.fr

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There are two kinds of mathematics books: the kind you can't read past the first sentence, and the kind you can't read past the first page.

Richard Feynman

clarke@math.univ-lyon1.fr

Generalized Gradients and Proximal analysis

Francis Clarke

Institut universitaire de France et Université de Lyon

Yesterday, we motivated the need for nonsmooth analysis. It appears that nonsmoothness is more common than one might have thought, and that the opposite of "linear" is often "nonsmooth".

Today, we examine the basic constructs and some elements of the calculus. We stress that difficult nonsmooth problems remain difficult even if one has mastered this theory! (But it can help...)

Generalized gradients and associated geometry

In an arbitrary Banach space, the starting point for <u>functions</u> is the generalized directional derivative:

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When f is locally Lipschitz, this is finite, and we find:

$$\begin{aligned} f^\circ(x; v+w) &\le f^\circ(x;v) + f^\circ(x;w)\ \forall v, w \\ f^\circ(x; tv) &= t f^\circ(x;v)\ \forall t \ge 0 \end{aligned}$$

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$$f^\circ(x;tv) \;=\; t f^\circ(x;v)\ \forall t \ge 0$$

These are properties of *support functions.*

If Z is a nonempty bounded set in X^*, then the support function has these properties:

$$H_Z(v) := \sup_{\zeta\in Z} \langle \zeta, v\rangle$$

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$$f^\circ(x;v) \;=\; H_Z(v)\ \forall v \in X$$

We denote this set by $\partial_C f(x)$, the generalized gradient. The following duality holds:

$$\partial_C f(x) = \left\{\zeta \in X^* : f^\circ(x;v) \ge \langle \zeta, v\rangle\ \forall v\right\}$$
$$f^\circ(x;v) \;=\; \max_{\zeta\in\partial_C f(x)} \langle \zeta, v\rangle$$

$\partial_C f(x)$ is convex, compact, and closed, which may explain the subscript C.

It is often referred to as the Clarke generalized gradient.

Other constructs will include:

$\partial_P f(x)$ (proximal subdifferential) and

$\partial_L f(x)$ (limiting subdifferential)

Let S be a nonempty closed subset of X.
Its **distance function** (Lipschitz) is given by

$$d_S(x) := \inf_{y \in S} \|x - y\|$$

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We define the generalized **normal and tangent cones** by

$$N_S^C(x) := \mathrm{cl}\,\{\, t\,\partial_C d_S(x) : t \geq 0\}$$

$$\begin{aligned} T_S^C(x) &= \left[N_S^C(x)\right]^\circ \\ &:= \{v : \langle \zeta, v\rangle \leq 0 \;\; \forall \zeta \in N_S^C(x)\} \\ &= \{v : d_S^\circ(x; v) = 0\} \end{aligned}$$

If we wish to start with geometry, the last shall be first:

$$\begin{aligned} T_S^C(x) := \{v : \forall\, x_i \to_S x,\ \forall\, t_i \downarrow 0, \\ \exists\, v_i \to v \;/\; x_i + t_i v_i \in S\} \end{aligned}$$

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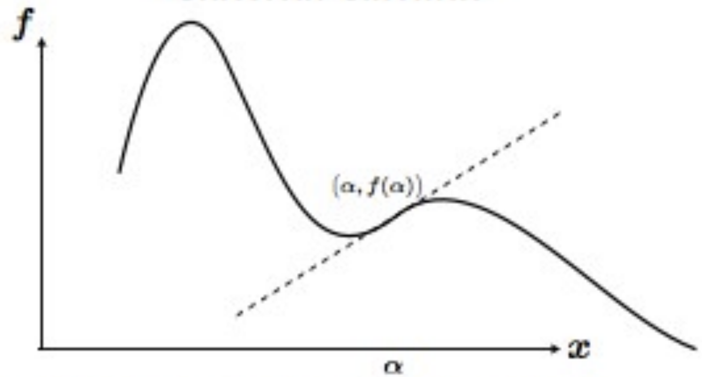
Classical Calculus



$f'(\alpha)$ = the slope of the tangent line to the graph of f through the point $(\alpha, f(\alpha))$.

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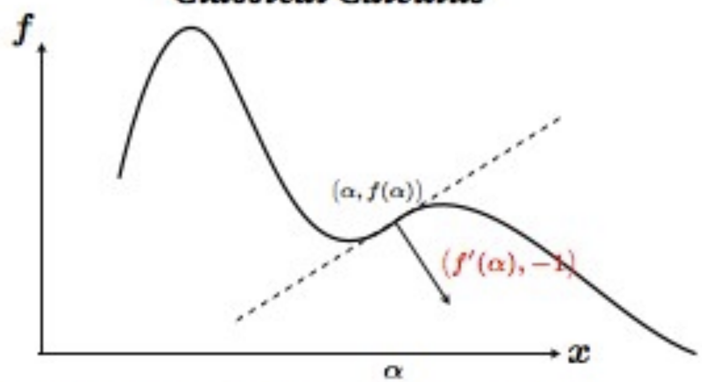
How do we recover the functional constructs?

Classical Calculus



$f'(\alpha)$ = the slope of the tangent line to the graph of f through the point $(\alpha, f(\alpha))$.

Dually, the value ζ such that $(\zeta, -1)$ is normal to the graph of f

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How do we recover the functional constructs?

$$\partial_C f(x) := \left\{\zeta : (\zeta, -1) \in N^C_{\mathrm{epi}\, f}(x, f(x))\right\}$$

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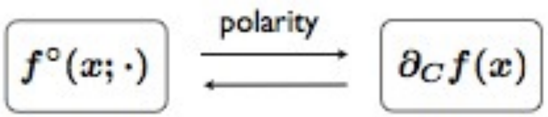
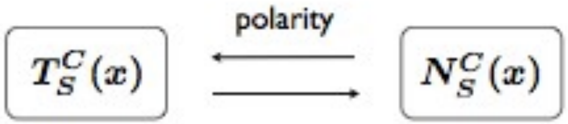
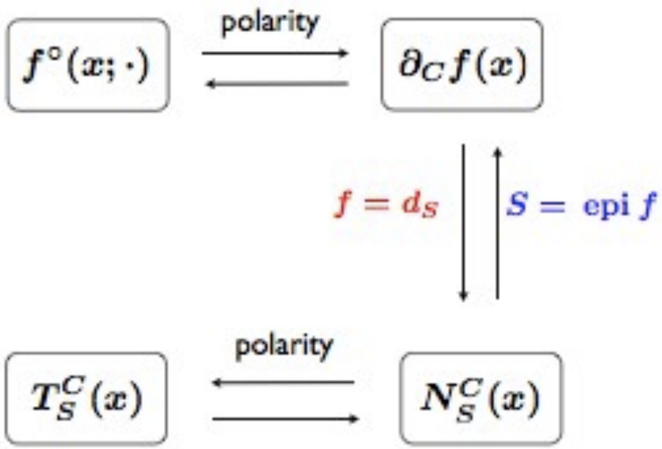
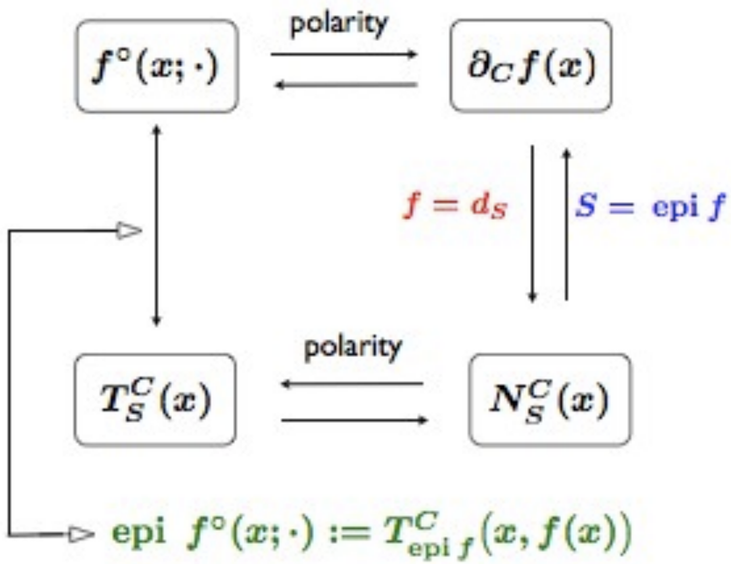
(and then $f^\circ(x;\cdot)$ is the support function of $\partial_C f(x)$)

$f^\circ(x;\cdot)$	$\partial_C f(x)$
$T_S^C(x)$	$N_S^C(x)$









The smooth case

The smooth case

If f is smooth, then $\partial_C f(x) = \{f'(x)\}$, since

$$\langle f'(x), v\rangle = f'(x;v) = f^\circ(x;v) = \max_{\zeta\in\partial_C f(x)} \langle \zeta, v\rangle$$

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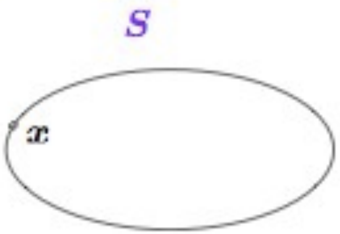

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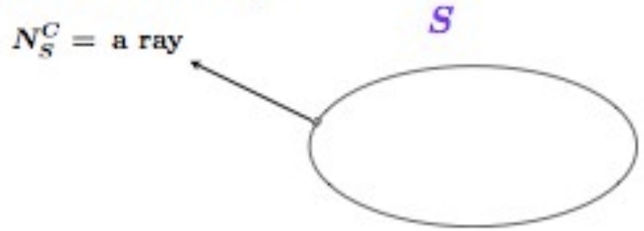

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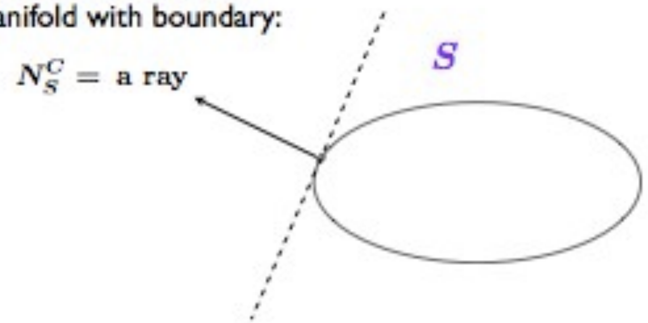

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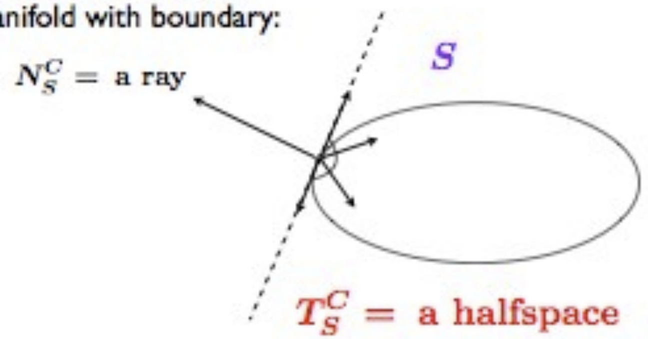

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$$\partial_C f(x) = \partial f(x) \quad \text{the subdifferential}$$
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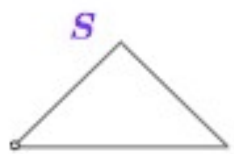

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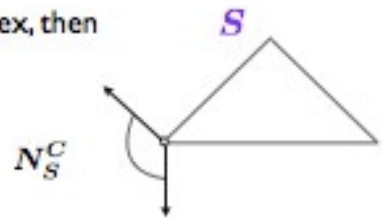

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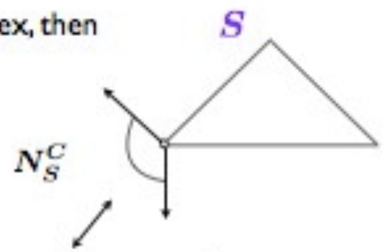

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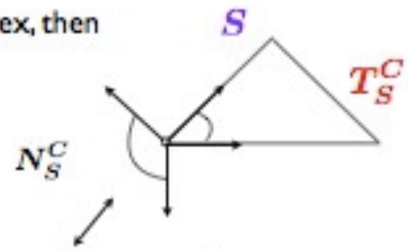

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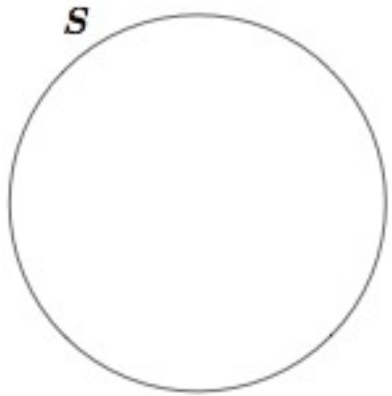

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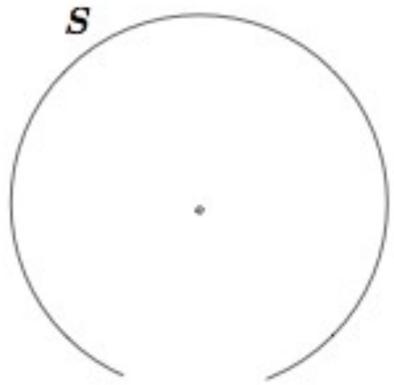

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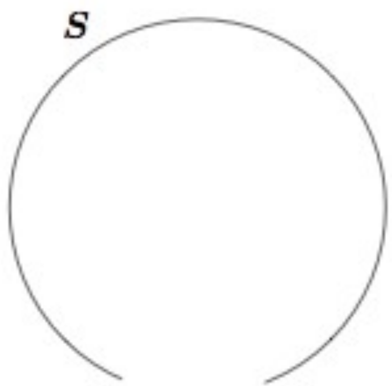

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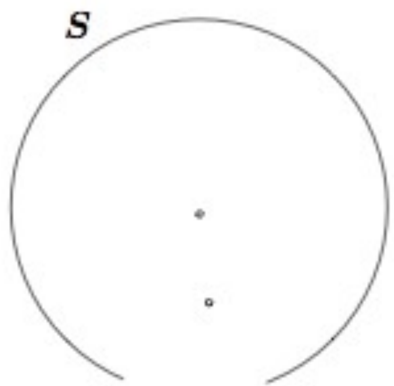

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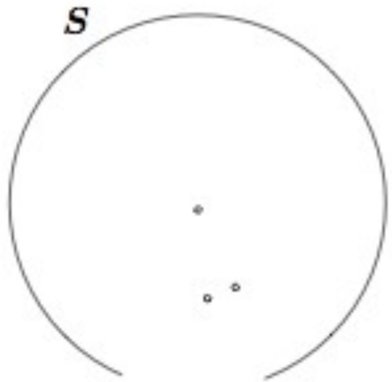

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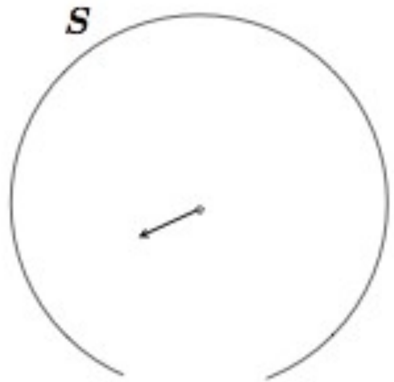

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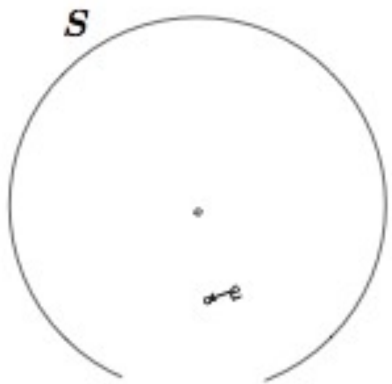

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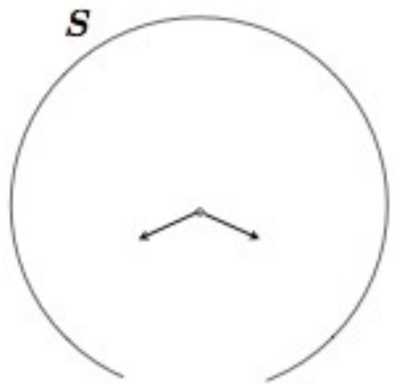

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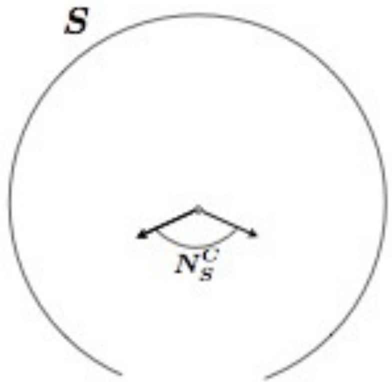

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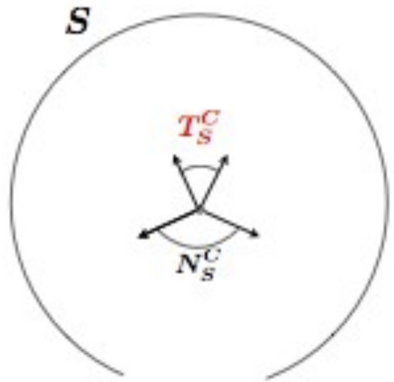

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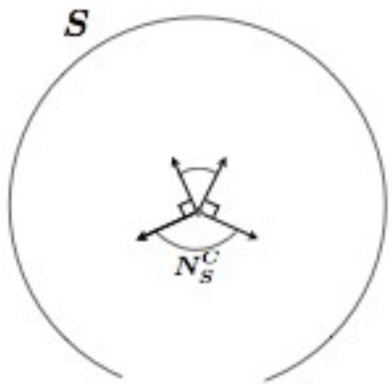

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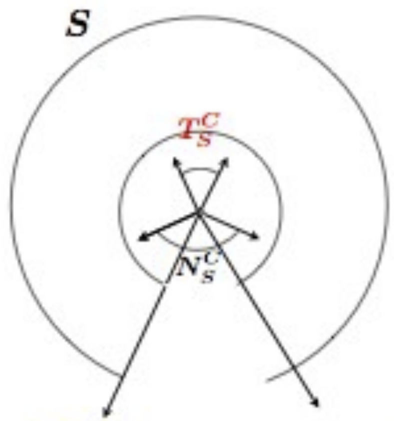

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Bouligand contingent cone

Some calculus

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Sums:

$$\partial_C(f_1 + f_2)(x) \subset \partial_C f_1(x) + \partial_C f_2(x)$$

(equality when f_1, f_2 ***regular***)

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When f if locally Lipschitz on $\mathbb{R}^n$, then f is differentiable a.e. (Rademacher). Let Ω be any set of measure 0 including the nondifferentiability points. Then

$$\partial_C f(x) = \mathrm{co}\,\{ \lim_{i \to \infty} \nabla f(x_i) : x_i \to x,\ x_i \notin \Omega \}$$

("blind to sets of measure 0").

This is a useful tool for calculation.

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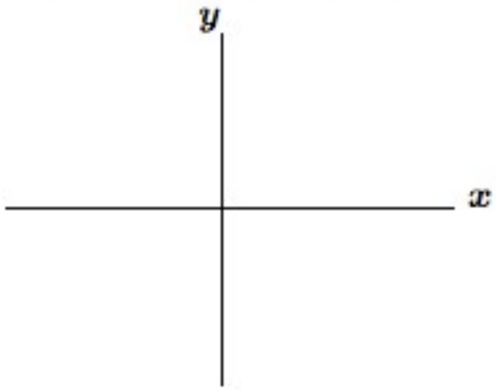

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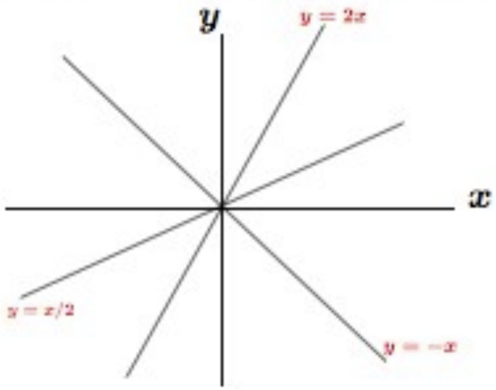

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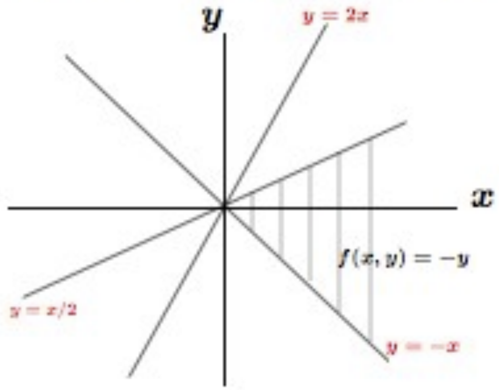

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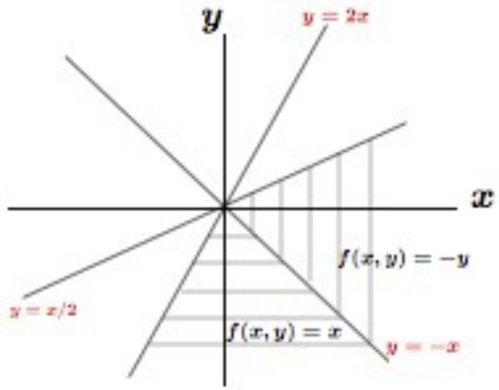

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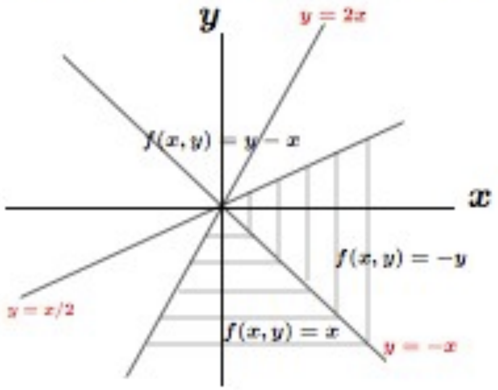

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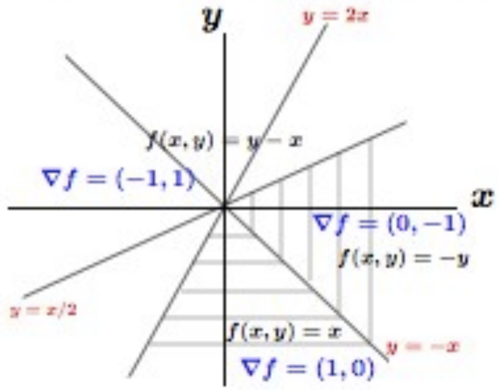

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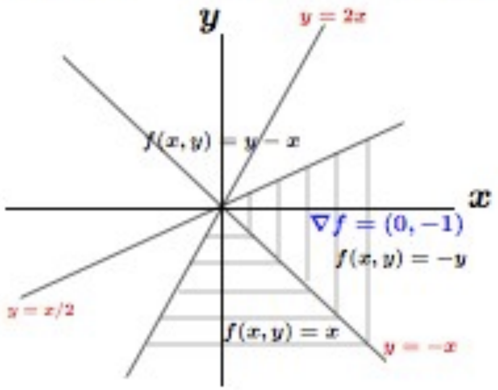

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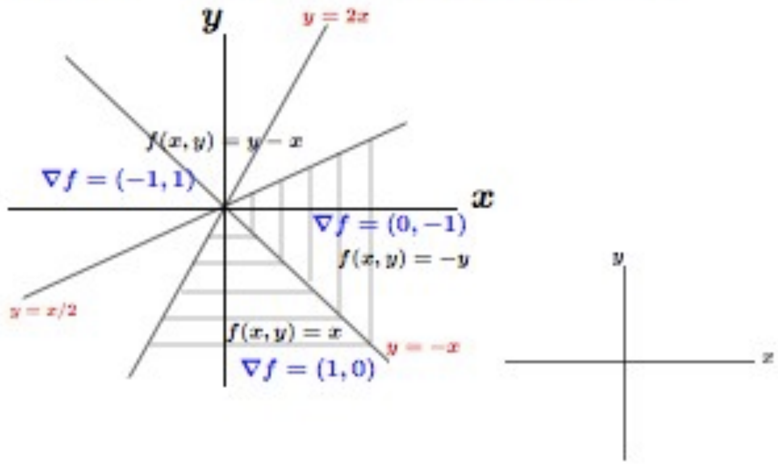

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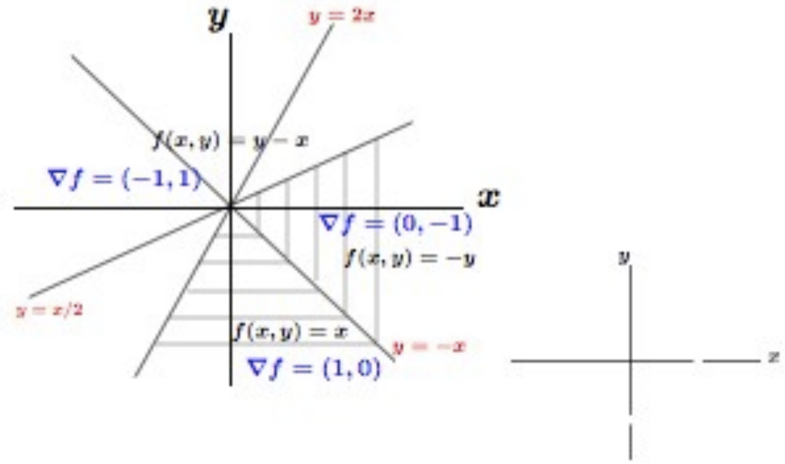

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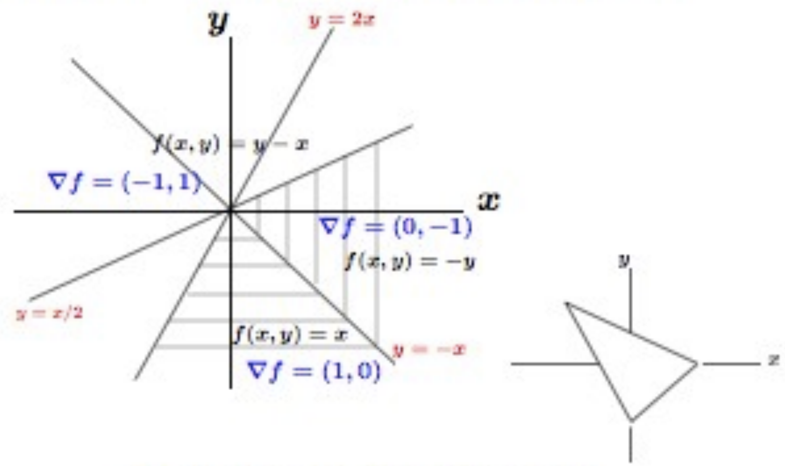

$$\partial_C f(0,0) = \text{co}\{(1,0),(0,-1),(-1,1)\}$$

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When $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz, we can ***define*** the **generalized Jacobian** this way:

$$\partial_C f(x) := \{\lim_{i\to\infty} Df(x_i) : x_i \to x,\ x_i \notin \Omega\},$$

A convex set of $m \times n$ matrices. Then: inverse function theorem, Sard, etc. [General case $f : X \to Y$: Pales/Zeidan]

Theorem (1973)

Let $\partial_C F(x_0)$ be of maximal rank, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz near x_0. Then there exist neighborhoods U of x_0 and V of $F(x_0)$ and a Lipschitz function $G : V \to \mathbb{R}^n$ such that

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Example

$$F(x,y) = \left[|x| + y, 2x + |y|\right]$$

$$\partial_C F(0,0) = \left\{ \begin{bmatrix} s & 1 \\ 2 & t \end{bmatrix} : -1 \le s \le 1,\ -1 \le t \le 1 \right\}$$

$$\det \begin{bmatrix} s & 1 \\ 2 & t \end{bmatrix} = st - 2 \ne 0$$

Calculus of sets

Let $x_0 \in S := \{x : f(x) \le 0\}$. If $0 \notin \partial_C f(x_0)$, then

$$T_S^C(x_0) \supset \{v \in X : f^\circ(x_0; v) \le 0\}.$$

If in addition f is regular at x_0, then

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Let Y be another Banach space, and $F : X \to Y$ a continuously differentiable function. Set

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If $N_{S_1}^C(x) \bigcap -N_{S_2}^C(x) = \{0\}$, then

$$N_{S_1 \cap S_2}^C(x) \subset N_{S_1}^C(x) + N_{S_2}^C(x) \text{ and}$$
$$T_{S_1 \cap S_2}^C(x) \supset T_{S_1}^C(x) \cap T_{S_2}^C(x),$$

with equality when S_1 and S_2 are regular.

Wedged (or epi-Lipschitz) sets

A set S is said to be ***wedged*** if:

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Let $S \subset \mathbb{R}^n$ be wedged. Then

- $\text{int } S \neq \emptyset$
- $S = \text{cl}\,\{\text{int } S\}$
- $T_S^C(x) = \mathbb{R}^n$ iff $x \in \text{ int } S$
- S is locally the epigraph of a Lipschitz function

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function satisfying

$$\phi(x) \in T_{B(0,1)}(x) \; \forall \, x \in B(0,1).$$

Then there exists $x_0 \in B(0,1)$ such that $\phi(x_0) = 0$.

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Boundary analysis: Inner and outer sphere conditions, lower C^2 property, reach, semiconcavity, ϕ-convexity, packing, etc. (Federer, Stern, Colombo, Nour, Cannarsa...)

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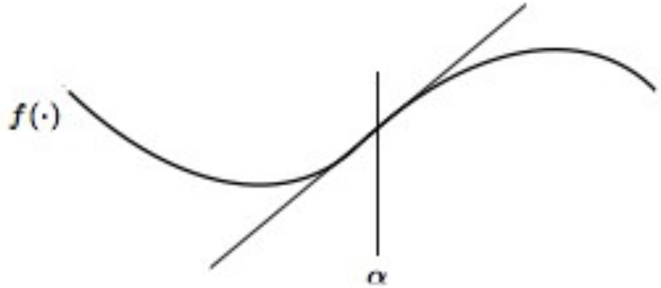
Does $r = \dfrac{nR}{2\sqrt{n^2-1}}$ suffice?

Proximal theory

The classical derivative corresponds to a two-sided local approximation by an affine function.



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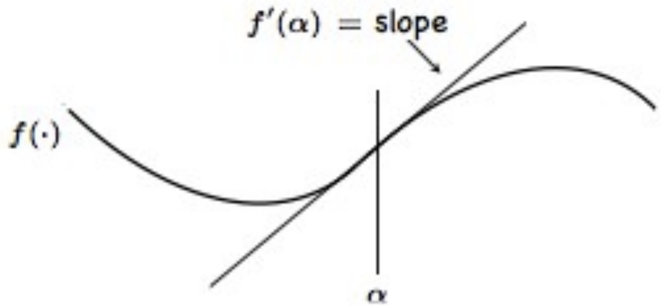

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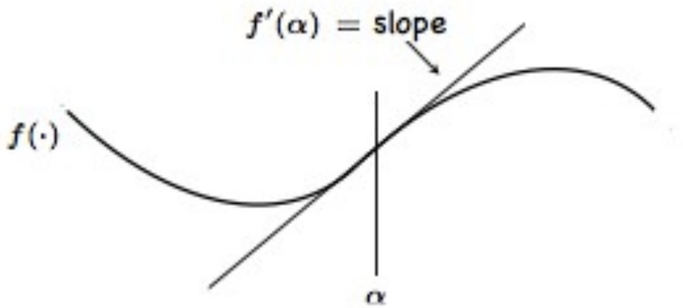

We may also approximate just from below, using nonlinear functions: proximal analysis

The set of all `contact slopes' of lower locally supporting parabolas is the proximal subdifferential $\partial_p f(\alpha)$

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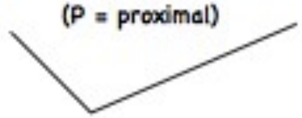

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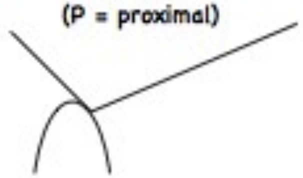

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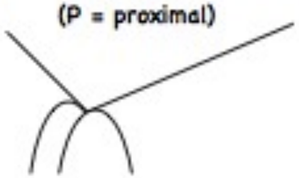

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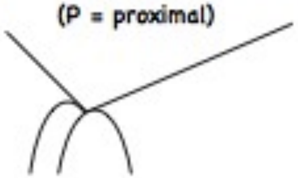

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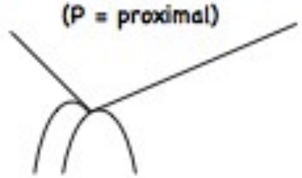

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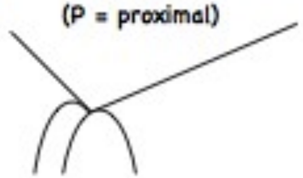

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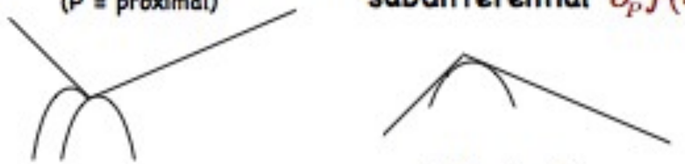

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$\partial_P f$ has a very complete (but fuzzy!) theory and calculus...
Borwein, Ioffe, Ledyaev, Loewen, Rockafellar, Vinter, Zeidan...

We cannot expect to have, in general:

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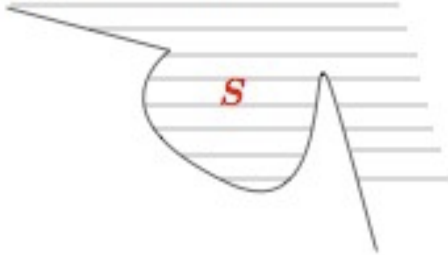
The geometry of proximal normals



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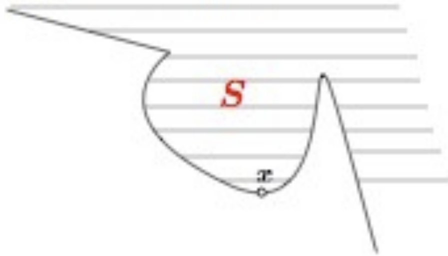

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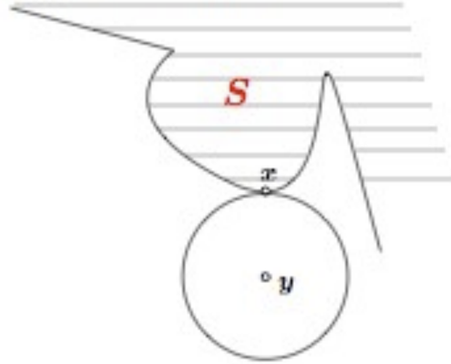

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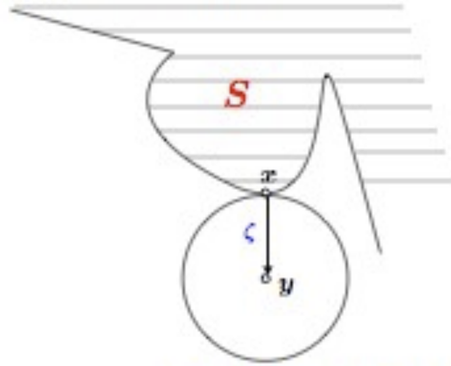

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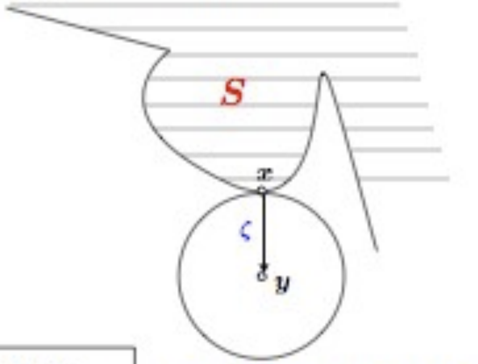

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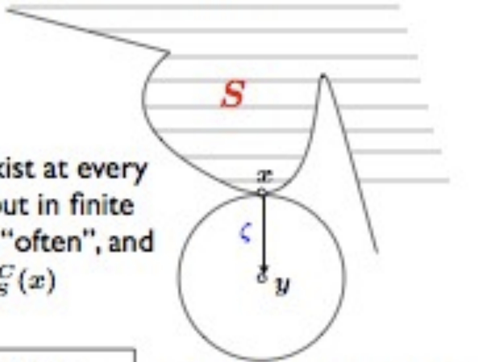

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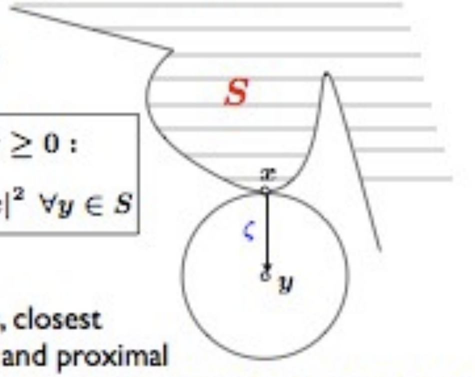

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In infinite dimensions, closest points may not exist, and proximal normals may be scarce. But they exist "densely" in a Hilbert space (Lau's Theorem), or, more generally, in *smooth Banach spaces*

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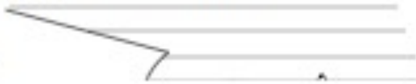

Fact:
if f is lower semicontinuous, finite at x, then

$$\zeta \in \partial_P f(x) \iff (\zeta, -1) \in N^P_{\mathrm{epi}\, f}(x, f(x))$$

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Limiting constructs

When proximal normals exist densely, as in a Hilbert space, we define

$$N_S^L(x) = \left\{ \lim_{i \to \infty} \zeta_i : \zeta_i \in N_S^P(x_i), x_i \to_S x \right\}$$

$$\partial_L f(x) = \left\{ \lim_i \zeta_i : \zeta_i \in \partial_P f(x_i), x_i \to x, f(x_i) \to f(x) \right\}$$

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These constructs inherit a calculus that is "less fuzzy". For example:

In finite dimensions, if f and g are lower semicontinuous, and if one of them is Lipschitz near x, then

$$\partial_L(f+g)(x) \subset \partial_L f(x) + \partial_L g(x)$$

$\partial_C f$ vis-à-vis $\partial_P f$ / $\partial_L f$

f

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- For a Lipschitz function on a Hilbert space we have

$$\partial_C f = \mathrm{co}\, \partial_L f$$

Two references chosen at random:

Optimization and Nonsmooth Analysis
Clarke, 1983

Nonsmooth Analysis and Control Theory
Clarke, Ledyaev, Stern and Wolenski,
Graduate Texts in Mathematics 1998

clarke@math.univ-lyon1.fr
(or web site)

THE END