Algorithms for embedded graphs

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Foreword and introduction

Foreword

This document is the overlapping union of some course notes that the author used in previous years for graduate courses. It is certainly not in final shape, and comments by e-mail are welcome.

Each exercise is labeled with one to three stars, supposed to be an indication of its importance (in particular, depending on whether the result or technique is used later), not of its difficulty.

Introduction

This is a partial introduction to the computational aspects of graphs drawn without crossings in the plane or in more complicated surfaces. This topic has been a subject of active research, especially over the last decade, and is related to rather diverse fields and communities:

- in graph algorithms: As we shall see, because planar graphs bear important properties, many general graph problems become easier when restricted to planar graphs (shortest path, flow and cut, minimum spanning trees, vertex cover, graph isomorphism, etc.). The same holds for graphs on surfaces, to some extent;
- in graph theory, the theory of graph minors founded by Robertson and Seymour makes heavy use of graphs embeddable on a fixed surface, as well as graphs excluding a fixed minor. Edge-width and face-width are closely related to the notion of shortest non-contractible closed curve;
- in topology, the classification of surfaces, as discovered in the beginning of the 20th century, is inherently algorithmic. Surfaces play a prominent role in the deep theories of knots and three-manifolds; there are also many algorithmic questions in these areas;
- in computational geometry, surfaces arise naturally in various applications. Operations in geometric spaces such as decomposition, extraction of important features, and shortest path computation are basic computational geometry tasks that are relevant in particular for surfaces, usually embedded in $\mathbb{R}^3$, or even planar surfaces.

Many graphs encountered in practice are geometric, and either are planar or have a few crossings (think of a road network with a few overpasses and underpasses). Thus it makes sense to look for efficient algorithms dedicated to such graphs. In addition, in computer graphics, one needs to efficiently process surfaces represented by triangular meshes, e.g., to cut them to make them planar; we shall introduce algorithms for such purposes.

Outline

The first chapter introduces planar graphs from the topological and combinatorial point of view. The second chapter considers the problem of testing whether a graph is planar, and, if so, of drawing it without crossings in the plane. Then we move on with some general graph problems, for which we give efficient algorithms when the input graph is planar.

Then, we consider graphs on surfaces (planar graphs being an important special case). In Chapter 4, we introduce surfaces from the topological point of view; in Chapter 5, we present algorithms using the cut locus to build short curves and decompositions of surfaces. In Chapter 6, we introduce two important topological concepts, homotopy and the universal cover. All these techniques are combined in Chapter 7 to provide algorithms to shorten curves up to deformation.

Some parts of these notes are not used in later sections, and can be safely skipped by the reader not interested in them. Such optional parts are Chapter 2, and Chapter 3 except Section 3.4 (which serves as a gentle introduction to the concept of cross-metric surface).

Only a part of the material covered in this course appeared in textbooks. For further reading or different expositions, mostly on the topological aspects, recommended books are Mohar and Thomassen [64], Armstrong [3], and Stillwell [73]. For more examples on recent algorithms for topological problems on graphs on surfaces, see[17]. For a wider perspective in computational topology, see the very recent course notes by Erickson [29].

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Chapter 1

Basic properties of planar graphs

1.1 Topology

1.1.1 Preliminaries on topology

We assume some familiarity with basic topology, but we recall some definitions nonetheless.

A topological space is a set $X$ with a collection of subsets of $X$, called open sets, satisfying the three following axioms:

- the empty set and $X$ are open;
- any union of open sets is open;
- any finite intersection of open sets is open.

There is, in particular, no notion of metric (or distance, angle, area) in a topological space. The open sets give merely an information of neighborhood: a neighborhood of $x \in X$ is a set containing an open set containing $x$. This is already a lot of information, allowing to define continuity, homeomorphisms, connectivity, boundary, isolated points, dimension. . .

Specifically, a map $f : X \to Y$ is continuous if the inverse image of any open set by $f$ is an open set. If $X$ and $Y$ are two topological spaces, a map $f : X \to Y$ is a homeomorphism if it is continuous, bijective, and if its inverse $f^{-1}$ is also continuous. A point of detail, ruling out pathological spaces: the topological spaces considered in these notes are assumed to be
**Example 1.1.** Most of the topological spaces here are endowed with a natural metric, which should be “forgotten”, but define the topology:

- \(\mathbb{R}^n\) (\(n \geq 1\));
- the \(n\)-dimensional sphere \(S^n\), i.e., the set of unit vectors of \(\mathbb{R}^{n+1}\);
- the \(n\)-dimensional ball \(B^n\), i.e., the set of vectors in \(\mathbb{R}^n\) of norm at most 1; in particular \(B^1 = [-1, 1]\) and \([0, 1]\) are homeomorphic;
- the set of lines in \(\mathbb{R}^2\), or more generally the Grassmannian, the set of \(k\)-dimensional vector spaces in \(\mathbb{R}^n\).

**Exercise 1.2** (stereographic projection). Prove that the plane is homeomorphic to \(S^2\) with an arbitrary point removed. (Indication: see Figure 1.1.)

A **path** in \(X\) is a continuous map \(p : [0, 1] \to X\); its **endpoints** are \(p(0)\) and \(p(1)\). Its **relative interior** is the image by \(p\) of the open interval \((0, 1)\).

A path is **simple** if it is one-to-one. A space \(X\) is **connected**\(^1\) if it is non-empty and, for any points \(a\) and \(b\) in \(X\), there exists a path in \(X\) whose endpoints are \(a\) and \(b\). The **connected components** of a topological space \(X\) are the classes of the equivalence relation on \(X\) defined by: \(a\) is equivalent to \(b\) if there exists a path between \(a\) and \(b\). A topological space \(X\) is **disconnected** (or **separated**) by \(Y \subseteq X\) if and only if \(X \setminus Y\) is not connected; points in different connected components of \(X \setminus Y\) are **separated** by \(Y\).

### 1.1.2 Graphs and embeddings

We will use standard terminology for graphs. Unless noted otherwise, all graphs are undirected and finite but may have loops and multiple edges. A **circuit** in a graph \(G\) is a closed walk without repeated vertices.\(^2\)

A graph yields naturally a topological space:

- for each edge \(e\), let \(X_e\) be a topological space homeomorphic to \([0, 1]\);
- let \(X\) be the disjoint union of the \(X_e\); identify (quotient topology), in \(X\), endpoints of \(X_e\) and \(X_{e'}\) if these endpoints correspond to the same vertex in \(G\).

An **embedding** of \(G\) in the plane \(\mathbb{R}^2\) is a continuous one-to-one map from \(G\) (viewed as a topological space) to \(\mathbb{R}^2\). Said differently, it is a “crossing-free drawing” of \(G\) on \(\mathbb{R}^2\), being the data of two maps:

- \(\Gamma_V\), which associates to each vertex of \(G\) a point of \(\mathbb{R}^2\);
- \(\Gamma_E\), which associates to each edge \(e\) of \(G\) a path in \(\mathbb{R}^2\) between the images by \(\Gamma_V\) of the endpoints of \(e\),

in such a way that:

\(^1\)In this course, the only type of connectivity considered is path connectivity.

\(^2\)This is often called a cycle; however, in the context of these notes, this word is also used to mean a homology cycle or a closed curve, so it seems preferable to avoid overloading it again.
• the map $\Gamma_V$ is one-to-one (two distinct vertices are sent to distinct
two distinct points of $\mathbb{R}^2$);
• for each edge $e$, the relative interior of $\Gamma_E(e)$ is one-to-one (the image
of an edge is a simple path, except possibly at its endpoints);
• for all distinct edges $e$ and $e'$, the relative interiors of $\Gamma_E(e)$ and
$\Gamma_E(e')$ are disjoint (two edges cannot cross);
• for each edge $e$ and for each vertex $v$, the relative interior of $\Gamma_E(e)$
does not meet $\Gamma_V(v)$ (no edge passes through a vertex).

We can actually replace $\mathbb{R}^2$ above with any
topological space $Y$ and talk
about an embedding of a graph in $Y$.

When we speak of embedded graphs, we sometimes implicitly identify the
graph, its embedding, and the image of its embedding.

1.1.3 Planar graphs and the Jordan curve theorem

In the remaining part of this chapter, we only consider embeddings of
graphs into the sphere $S^2$ or the plane $\mathbb{R}^2$.

A graph is planar if it admits an embedding into the plane. By Exercise 1.2, this is equivalent to the existence of an embedding into the
sphere $S^2$.

The faces of a graph embedding are the connected components of the
complement of the image of the vertices and edges of the graph.

Here are the most-often used results in the area.

**Theorem 1.3** (Jordan curve theorem, reformulated; see [75]). Let $G$ be
a graph embedded on $S^2$ (or $\mathbb{R}^2$). Then $G$ disconnects $S^2$ if and only if it
contains a circuit.

**Theorem 1.4** (Jordan–Schönflies theorem; see [75]). Let $f : S^1 \to S^2$ be
a one-to-one continuous map. Then $S^2 \setminus f(S^1)$ is homeomorphic to two
disjoint copies of the open disk.

**Exercise 1.5.** Sketch a proof of the Jordan curve theorem in the
case where $G$ is embedded in the plane with polygonal edges.

These results are, perhaps surprisingly, difficult to prove: the difficulty
comes from the generality of the hypotheses (only continuity is required).
For example, if in the Jordan curve theorem one assumes that $G$ is em-
bedded in the plane with polygonal edges, the theorem is not hard to
prove.

A graph is **cellularly embedded** if its faces are (homeomorphic to) open
disks. Henceforth, we only consider cellular embeddings. It turns out that
a graph embedded on the sphere is cellularly embedded if and only if it is
connected.\(^3\)

1.2 Combinatorics

So far, we have considered curves and graph embeddings in the plane that
are rather general.

1.2.1 Combinatorial maps for planar graph embeddings

We now focus on the combinatorial properties of cellular graph embeddings
in the sphere. Since we are not interested in the geometric properties,
it suffices to specify how the faces are “glued together”, or alternately
the cyclic order of the edges around a vertex. Embeddings of graphs on
the plane are treated similarly: just choose a distinguished face of the
embedding into $S^2$, representing the “infinite” face of the embedding in the
plane.

An algorithmically sound way of representing combinatorially a cellular
graph embedding in $S^2$ is via **combinatorial maps**, which we now de-
scribe. The basic notion is the flag, which represents an incidence between a vertex, an edge, and a face of the embedding. Three involutions allow
move to a nearby flag, and, by iterating, to visit the whole graph em-
deeding; see Figure 1.2:

\[^3\]Although this statement should be intuitively clear, it is not so obvious to prove. It
may help to use the results of Chapter 4, especially the fact that every face of a graph
embedding is a surface with boundary.
Figure 1.2. The flags are represented as line segments parallel to the edges; there are four flags per edge. The involutions $vi$, $ei$, and $fi$ on the thick flag are also shown.

```c
int vertex_degree(Flag fl) {
    int j=0;
    Flag fl2=fl;
    do {
        ++j;
        fl2=fl2->ei()->fi();
    } while (fl2!=fl);
    return j;
}
```

```c
int face_degree(Flag fl) {
    int j=0;
    Flag fl2=fl;
    do {
        ++j;
        fl2=fl2->ei()->vi();
    } while (fl2!=fl);
    return j;
}
```

Figure 1.3. C++ code for degree computation.

- $vi$ moves to the flag with the same edge-face incidence, but with a different vertex incidence;
- $ei$ moves to the flag with the same vertex-face incidence, but with a different edge incidence;
- $fi$ moves to the flag with the same vertex-edge incidence, but with a different face incidence.

Example 1.6. Figure 1.3, left, presents code to compute the degree of a vertex, i.e., the number of vertex-edge incidences of this vertex. The function takes as input a flag incident with that vertex. Note that a loop incident with the vertex makes a contribution of two to the degree. Dually, on the right, code to compute the degree of a face (the number of edge-face incidences of this face) is shown.

Note that a flag is not necessarily uniquely defined by its triple (vertex, edge, and face), as shows the example of a graph with a single vertex and a single (loop) edge.

The complexity of a graph $G = (V, E)$ is $|V| + |E|$. The complexity of a cellular graph embedding is the total number of flags involved, which is linear in the number of edges (every edge bears four flags), and also in the number of vertices, edges, and faces. Therefore the complexity of a graph cellularly embedded in the plane and of one of its embeddings are linearly related.

The data structure indicated above allows to “navigate” throughout the data structure, but does not store vertices, edges, and faces explicitly. In many cases, however, it is necessary to have one data structure (“object”) per vertex, edge, or face. For example:

- if one has to be able to check in constant time whether an edge is a loop (incident twice to the same vertex), the data structure given above is not sufficient. On the other hand, if every flag has a pointer to the incident vertex, then testing whether an edge is a loop can be done by testing the equality of two pointers in constant time;
- in coloring problems, one need to store colors on the vertices of the graph. Such information can be stored in the data structure used for each vertex.

For such purposes, each flag can have a pointer to the underlying vertex, edge, and face (called respectively $vu$, $eu$, $fu$). Each such vertex, edge, or face contains no information on the incident elements, only information needed in the algorithms. If needed, one may similarly put some information in the vertex-edge, edge-face, vertex-face, and vertex-edge-face incidences. Maintaining such informations, however, comes with a cost, which is not always desirable. For example, assume we want to be able to remove one edge incident to two different faces in constant time. If we keep the information $fu$, this must take time proportional to the smaller
degree of the two faces (since the two faces are merged, the fu pointer has to be updated at least on one side of the edge). If we only keep vu, say, then such an update is not needed, and this edge removal can be done in constant time.

1.2.2 Duality and Euler’s formula

A dual graph of a cellular graph embedding \( G = (V, E) \) on \( S^2 \) is a graph embedding \( G^* \) defined as follows: put one vertex \( f^* \) of \( G^* \) in the interior of each face \( f \) of \( G \); for each edge \( e \) of \( G \), create an edge \( e^* \) in \( G^* \) crossing \( e \) and no other edge of \( G \) (if \( e \) separates faces \( f_1 \) and \( f_2 \), then \( e^* \) connects \( f_1^* \) and \( f_2^* \)). See Figure 1.4.

A dual graph embedding is also cellular. The combinatorial map of the dual graph is unique. Actually, with the map representation, dualizing is easy: simply replace \( fi \) with \( vi \) and vice-versa. This in particular proves that duality is an involution: \( G^{**} = G \).

Exercise 1.7 (easy). \( \star \star \star \) Every tree (acyclic connected graph) with \( v \) vertices and \( e \) edges satisfies \( v - e = 1 \).

Lemma 1.8. Let \( G = (V, E) \) be a cellular graph embedding in \( S^2 \), and let \( G^* = (F^*, E^*) \) be its dual graph. Furthermore, let \( E' \subseteq E \).

Then \( (V, E') \) is acyclic if and only if \( (F^*, (E \setminus E')^*) \) is connected. In particular, \( (V, E') \) is a spanning tree if and only if \( (F^*, (E \setminus E')^*) \) is a spanning tree.

Proof. \( (V, E') \) is acyclic if and only if \( S^2 \setminus E' \) is connected, by the Jordan curve theorem 1.3. Furthermore, \( S^2 \setminus E' \) is connected if and only if \( (F^*, (E \setminus E')^*) \) is connected: Two points \( x \) and \( x' \) in faces \( f \) and \( f' \) of \( G \) can be connected by a path avoiding \( E' \) and not entering any face other than \( f \) and \( f' \) if and only if \( f \) and \( f' \) are adjacent by some edge not in \( E' \), i.e. if and only if \( f^* \) and \( f'^* \) are adjacent in \( (F^*, (E \setminus E')^*) \). \( \square \)

Corollary 1.9 (Euler’s formula for cellular graph embeddings in \( S^2 \)). For every cellular graph embedding in \( S^2 \) with \( v \) vertices, \( e \) edges, and \( f \) faces, we have \( v - e + f = 2 \).

Hence this formula also holds for every embedding of a connected graph in the plane.

Proof. Let \( T \) be the edge set of a spanning tree of \( G \). The dual edges of its complement, \( (E \setminus T)^* \), is also a spanning tree. The number of edges of \( G \) is \( e = |T| + |(E \setminus T)^*| \), which, by Exercise 1.7, equals \( (v - 1) + (f - 1) \). \( \square \)

Exercise 1.10 (easy direction of Kuratowski’s theorem). \( \star \star \star \star \) Show that the complete graph with 5 vertices, \( K_5 \), is not planar. Indication: Use Euler’s formula and double-counting on the number of vertex-edge and edge-face incidences. Also show that the bipartite graph \( K_{3,3} \) (with 6 vertices \( v_1, v_2, v_3, w_1, w_2, w_3 \) and 9 edges, connecting every possible pair \( \{v_i, w_j\} \)) is not planar.

1.3 Notes

For more information on basic topology, see for example Armstrong [3] or Henle [44]; see also Stillwell [73]. For more informations on planar graphs, see (the next two chapters and) Mohar and Thomassen [64, Chapter 2].
There are many essentially equivalent ways of representing planar graph embeddings [27, 52]; the computational geometry library CGAL implements one of them\(^4\). We will see later that (most of) these data structures generalize to graphs embedded on surfaces. There are further generalizations to higher dimensions [6, 57, 58]; this is important especially in geometric modelling.

Eppstein provides many proofs of Euler’s formula\(^5\).

Exercise 1.10 shows that \(K_5\) and \(K_{3,3}\) are not planar. There is a converse statement: Kuratowski’s theorem asserts that a graph \(G\) is planar if and only if it does not contain \(K_5\) or \(K_{3,3}\) as a subdivision; in other words, if and only if one cannot obtain \(K_5\) or \(K_{3,3}\) from \(G\) by removing edges and isolated vertices and replacing every degree-two vertex and its two incident edges with a single edge [53, 60, 74].

Let \(G\) be a cellular embedding of a graph on \(S^2\). By overlaying \(G\) with its dual graph \(G^*\), we obtain a quadrangulation: a cellular embedding of a graph \(G^*\) where each face has degree four. See Figure 1.4. Every face of \(G^*\) is incident with four vertices: one vertex \(v_G\) of \(G\), one vertex \(v_{G^*}\) of \(G^*\), and two vertices that are the intersection of an edge of \(G\) and an edge of \(G^*\). If, within each face, we connect \(v_G\) with \(v_{G^*}\), we obtain a triangulation, called the barycentric subdivision of \(G\) (Figure 1.5). Each triangle in the barycentric subdivision corresponds to a flag; its three neighbors are the flags reachable via the operations \(v_i\), \(e_i\), and \(f_i\).

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\(^5\)http://www.ics.uci.edu/~eppstein/junkyard/euler/.
Chapter 2

Planarity testing and graph drawing

Given a graph $G$ in a “usual” form, e.g., where each vertex has a linked list of pointers to its incident edges, and each edge has two pointers to its incident vertices, how can we determine whether $G$ is planar? Section 2.1 answers this question. Then we move on by considering algorithms to draw a planar graph in the plane or in $\mathbb{R}^3$.

2.1 Planarity testing

Given a graph $G$, how hard is it to determine whether $G$ is planar?

**Theorem 2.1.** Given a graph $G$, one can, in (optimal) linear time, determine whether $G$ is planar, and if so, compute a combinatorial map of $G$ in the plane.

We shall here prove this theorem with a weaker, cubic complexity. With much care, refining these ideas indeed leads to a linear-time algorithm [48].

A graph $G$ is **biconnected** if it has at least three vertices, and removing zero or one vertex (together with their incident edges) from $G$ does not disconnect $G$. A **cutvertex** of $G$ is a vertex whose removal increases the number of connected components of $G$. A **block** of $G$ is an inclusionwise maximal subgraph of $G$ that has no cutvertex.

**Lemma 2.2.** $G$ is planar if and only if all its blocks are planar.

**Proof.** Let $C$ be the set of cutvertices of $G$, and $B$ be the set of blocks of $G$. Let $H$ be the **block graph** of $G$, whose vertex set is the disjoint union of $B$ and $C$, and such that a block $b$ and a cutvertex $c$ are adjacent if and only if $c \in b$. This is a bipartite graph which is easily seen to be a forest; it gives a “coarse” description of $G$. For each tree of this forest (corresponding to a connected component of $G$), one can traverse the tree, embedding each block in turn without interfering with the other blocks. \hfill $\Box$

**Lemma 2.3.** Given a graph $G$, we can determine all its blocks in linear time.

**Proof.** We can obviously assume that $G$ is connected, because we could apply the algorithm to each connected component of $G$ in turn. We first focus on computing the cutvertices. For this purpose, run a depth-first search on the graph $G$, starting from an arbitrary root vertex. Recall that this partitions the edges of $G$ into **link edges**, which belong to the rooted search tree $T$, and **back edges**, which connect a vertex $v$ with an ascendent of $v$ in $T$. Clearly, the root of $T$ is a cutvertex if and only if it has degree at least two in $T$; this property is trivial to test. It should be also clear that a non-root vertex $v$ is a cutvertex if and only if some subtree of $T$ rooted at some child of $v$ is incident to no (back) edge whose other endpoint is an ascendent of $v$. To test the latter property efficiently, during the depth-first search, we maintain the following information:

- the depth of each vertex in the depth-first-search tree (once it gets visited), and
- for each vertex $v$, the **lowpoint** of $v$, namely, the lowest depth of an endpoint of a back edge incident to a descendent of $v$ (possibly $v$ itself), or $\infty$ if no such back edge exists.

The above characterization indicates that (the non-root vertex) $v$ is a cutvertex if and only if the lowpoint of some child $w$ of $v$ is at least the depth of $v$. Thus, we can compute the cutvertices in linear time, provided we can compute the depths and the lowpoints in linear time. The depth is standard to maintain during a depth-first search. The lowpoint of $v$ can be computed after visiting all descendents of $v$ (i.e., just before $v$ gets popped off the depth-first-search stack), since if we know the lowpoint of the children of $v$, we can compute it for $v$ in time linear in its degree.
Lemma 2.4. Each piece has at least two attachments.

**Proof.** Assume first that $G$ is planar. In a planar embedding, each piece is bipartite; thus $G$ is planar if and only if, for every piece $P$ of $G$ with respect to $C$, the graph obtained by adding $P$ to $C$ is planar.

**Proof.** Assume first that $G$ is planar. In a planar embedding, each piece is drawn either entirely inside or outside $C$. Furthermore, two pieces $P$ and $Q$ drawn on the same side of $C$ must be non-conflicting because, in the cyclic order around $C$, edges of $P$ and of $Q$ cannot be interlaced. (Otherwise, we would essentially have, after removal, contractions, and expansions of edges if needed, four vertices $v_1, v_2, v_3, v_4$ in this order on circuit $C$, with $v_1$ connected to $v_3$ and $v_2$ connected to $v_4$ by edges inside $C$; adding a new vertex outside $C$ and connecting it to all four vertices, we would get $K_5$, which is not planar.)

Lemmas 2.2 and 2.3 imply that, for the proof of Theorem 2.1, we can without loss of generality assume that the input graph $G$ is biconnected.

Let $C$ be a circuit of $G$. We partition the edges of $G - C$ into pieces as follows (see Figure 2.1): Two edges are in the same class if there is a path in $G$ between them that does not contain any vertex of $C$. The vertices of a piece $P$ that are in $C$ are called its attachments. Since $G$ is biconnected, each piece has at least two attachments.

**Lemma 2.4.** In linear time, we can either compute a circuit of $G$ that has at least two pieces, or certify that $G$ is planar.

**Proof.** First compute any circuit $C$, using, e.g., depth-first search. Determine the pieces of $C$. If $C$ has no piece, then $G = C$, thus $G$ is planar. If $C$ has two or more pieces, then $C$ satisfies the conclusion, so we are done. So assume that $C$ has a single piece $P$. Let $v_1, \ldots, v_k$ be the attachments of $P$ on $C$, in cyclic order around $C$. Let $p$ be a path in $P$ between $v_1$ and $v_2$. Now, let $C'$ be the circuit obtained by concatenating $p$ with the subpath of $C$ with endpoints $v_1$ and $v_2$ that also contains $v_3, \ldots, v_k$ (pick either of the two subpaths if $k = 2$). One piece of $C'$ is the other subpath of $C$, and another piece of $C'$ is $P \setminus p$, unless $P = p$, in which case $G = C \cup \{p\}$ is planar.

All of this takes linear time.

If $G$ is planar then, in a planar drawing of $G$, each piece of a circuit $C$ must be entirely inside or outside $C$. We say that two pieces $P$ and $Q$ of $G$ are non-conflicting with respect to $C$ if, intuitively, in any planar drawing of $G$ (if it exists), exactly one of $P$ and $Q$ must be drawn inside $C$. More formally, $P$ and $Q$ are non-conflicting if there are two (possibly identical) vertices $u$ and $v$ of $C$, splitting $C$ into two subpaths $C_1$ and $C_2$ with endpoints $u$ and $v$, such that all attachments of $P$ are in $C_1$ and all attachments of $Q$ are in $C_2$. Otherwise, $P$ and $Q$ are in conflict. The conflict graph of $G$ with respect to $C$ is a graph with vertex set the pieces of $C$; two pieces are connected if and only if they conflict.

**Lemma 2.5.** Let $C$ be a circuit of $G$. The graph $G$ is planar if and only if the following conditions are satisfied:

i. The conflict graph of $G$ with respect to $C$ is bipartite;

ii. for every piece $P$ of $G$ with respect to $C$, the graph obtained by adding $P$ to $C$ is planar.

**Proof.** Assume first that $G$ is planar. In a planar embedding, each piece is drawn either entirely inside or outside $C$. Furthermore, two pieces $P$ and $Q$ drawn on the same side of $C$ must be non-conflicting because, in the cyclic order around $C$, edges of $P$ and of $Q$ cannot be interlaced. (Otherwise, we would essentially have, after removal, contractions, and expansions of edges if needed, four vertices $v_1, v_2, v_3, v_4$ in this order on circuit $C$, with $v_1$ connected to $v_3$ and $v_2$ connected to $v_4$ by edges inside $C$; adding a new vertex outside $C$ and connecting it to all four vertices, we would get $K_5$. This graph is not planar.)

So assume that $G$ is not planar. Then, by Theorem 2.1, there exists a circuit $C$ of $G$ such that $G$ has a piece $P$ of $C$ with at least two attachments, and $P$ is non-planar. That is, $G$ and the conflict graph of $G$ with respect to $C$ both contain subgraphs isomorphic to $K_5$, which is a contradiction.

Figure 2.1. A graph $G$ with a circuit $C$ (on the outside of the figure) and the four pieces with respect to $C$ numbered from 1 to 4. All pairs of pieces conflict except $(1, 3)$ and $(3, 4).$
which is nonplanar.) This implies that the conflict graph is bipartite. The second property is trivial.

For the opposite direction, by (i), we consider a bipartition $\mathcal{P} \cup \mathcal{Q}$ of the conflict graph. We next describe how to embed all pieces of $\mathcal{P}$ inside $C$; this concludes, since using a similar method we can embed all pieces of $\mathcal{Q}$ outside $C$. We embed each piece of $\mathcal{P}$ iteratively. Assume that we have already embedded some pieces of $\mathcal{P}$. By (ii), in order to embed the next piece, we only need to prove that all its attachments $v_1, \ldots, v_k$ (in cyclic order around $C$) are incident to a same inner face of the already constructed graph. But this is clear, since each other piece of $\mathcal{P}$ has its attachment points on a subpath of $C$ between two consecutive $v_i$’s (because pieces in $\mathcal{P}$ do not conflict).

At a high level, the algorithm first applies Lemma 2.4 to compute a circuit $C$ with at least two pieces (unless $G$ is planar, which concludes). Then it uses the characterization of Lemma 2.5: If the conflict graph of $G$ with respect to $C$ is non-bipartite, it returns that $G$ is non-planar; otherwise, it recursively checks that $C \cup P$ is planar, for each piece $P$ of $G$ (such graphs are clearly biconnected). The correctness is clear.

To get an efficient algorithm, however, we need to be slightly more specific. The algorithm takes as input a biconnected graph $G$, and a circuit $C$ of $G$ with at least two pieces.

1. Compute the pieces of $G$ with respect to $C$.
2. For each piece $P$ of $G$ that is not a path:
   (a) let $G'$ be the graph obtained by adding $P$ to $C$;
   (b) let $C'$ be the circuit of $G'$ obtained from $C$ by replacing the portion of $C$ between two consecutive attachments with a path of $P$ between them;
   (c) apply the algorithm recursively to graph $G'$ and circuit $C'$. If $G'$ is non-planar, return “non-planar”.
3. Compute the conflict graph of the pieces.
4. Return “planar” or “non-planar” according to whether the conflict graph is bipartite or not.

The correctness follows from the proof of Lemma 2.4 and from the fact that each graph considered is biconnected.

Now, we study the complexity of Step 3:

**Lemma 2.6.** Given a circuit $C$, we can determine the conflict graph of $G$ with respect to $C$ in quadratic time.

**Proof.** Let $P$ be a piece of $C$, with attachments $v_1, \ldots, v_k$ in cyclic order around $C$. Then another piece $Q$ does not conflict with $P$ if and only if all its attachments are in some interval $[v_i, v_{i+1}]$, in cyclic order around $C$ (indices are taken modulo $k$). This suggests the following approach: Mark each vertex of $C$ according to which interval(s) $[v_i, v_{i+1}]$ it belongs to; for each piece $Q \neq P$, determine if all its attachments belong to a single interval using this marking. This takes linear time plus a time linear in the number of attachment points of all the pieces, which is also linear. Iterating for every piece $P$, we obtain the conflict graph of $G$ in quadratic time.

Since testing whether a graph is bipartite can be done in linear time, this shows that each recursive invocation of the algorithm takes quadratic time. Furthermore:

**Lemma 2.7.** The number of recursive invocations is linear in the complexity of the input graph.

**Proof.** We associate a different edge of $G$ to each invocation of the recursive algorithm. Namely, for a given invocation on graph $G$ and circuit $C$, we select a witness edge $e$ of $C$ that does not belong to the circuit of the parent invocation. That edge $e$ does not appear in the siblings’ graphs, so it will not show up as a witness edge in any sibling invocation nor in any descendant of a sibling. There remains to prove that $e$ does not appear as the witness edge of a descendent invocation. Consider a path of recursive invocations towards that descendent. If $e$ belongs to the circuit of the current invocation, it cannot be chosen as the witness of a child’s invocation, so we are safe; and if $e$ ceases to belong to the circuit of the current invocation, then by choice of the new circuit $C'$, $e$ now belongs to a piece of $C'$ that is a path, and therefore is absent from any descendent invocation.
This proves Theorem 2.1 with a weaker, cubic-time complexity... well, actually not quite: We only determined whether the input graph is planar or not; in the former case, a little bit more work is needed to actually compute a combinatorial map:

Exercise 2.8. ✩ Convince yourself that one can, also in cubic time, compute an embedding if the input graph \( G \) is indeed planar.

## 2.2 Graph drawing on a grid

Now we consider the following problem: Given a planar graph \( G \), given in the form of a combinatorial map (for example, obtained by the algorithm in the previous section), how can we build an actual embedding of \( G \) in the plane?

To be more specific, we need some definitions. A simple graph is a graph without loops or multiple edges. A planar graph is triangulated if every face of \( G \), including the outer face, has degree three. A graph embedding in the plane is straight-line if every edge is a straight-line segment (such an embedding is thus uniquely determined by the coordinates of its vertices).

We shall prove:

**Theorem 2.9.** Let \( G \) be a simple planar graph, given in the form of a combinatorial map. In \( O(n) \) time, we can compute a straight-line embedding of \( G \) where the vertices are on a regular \( O(n) \times O(n) \)-grid.

The restriction of having a simple graph is legitimate, because non-simple graphs do not have a straight-line embedding. Furthermore, we can remove all loops and multiple edges in a graph in linear time if desired:

**Lemma 2.10.** Let \( G \) be a graph (not necessarily planar) of complexity \( n \). In \( O(n) \) time, we can determine all loop edges and multiple edges of \( G \).

**Proof.** Let \( v \) be a vertex of \( G \). Mark each neighbor \( w \) of \( v \) with the list of edges with endpoints \( v \) and \( w \), by visiting each edge incident with \( v \) in turn. Any list containing more than one edge corresponds to multiple edges; if the list of \( v \) is non-empty, it corresponds to one or several loops.

\[ \rightarrow \]

**Figure 2.2.** After flipping a multiple edge (left) or a loop (right) in a planar graph, the new edge is not a loop and is not a multiple edge.

Finally, we erase the marks on the neighbors of \( v \). This operation takes a time linear in the degree of \( v \). We can iterate the process over all vertices \( v \) in turn.

Reusing the technique, we also obtain:

**Lemma 2.11.** Let \( G \) be a simple planar graph. In linear time, we can add edges to \( G \) to obtain a simple, triangulated, planar graph.

**Proof.** It is easy to add edges so that the resulting graph is connected, and then triangulated, in linear time. The only problem is that the resulting graph may be non-simple. Let \( e \) be an edge of a triangulated graph; removing \( e \) yields a degree-four face, which we can triangulate by inserting the unique edge \( e' \neq e \); we call this procedure a flip of \( e \).

For each vertex \( v \), compute all loops and multiple edges incident with \( v \), using the technique of the previous lemma. Now, we flip all loop edges incident with \( v \) (no such edge belongs to the original graph \( G \)). Furthermore, for each neighbor \( u \) of \( v \), consider the set of edges \( E_{uv} \) with both \( u \) and \( v \) as endpoints. Assume \( |E_{uv}| \geq 2 \). The original graph \( G \) has at most one edge in \( E_{uv} \); if \( G \) contains one edge of \( E_{uv} \), we let \( e \) be that edge, otherwise we let \( e \) be an arbitrary edge of \( E_{uv} \). Now flip all edges in \( E_{uv} \setminus \{e\} \).

The crucial observation is that none of these flips introduce loops or multiple edges, by planarity of the triangulated graph (Figure 2.2).

Iterating this process for each vertex \( v \) in turn, we obtain the desired linear-time algorithm. \( \square \)
The previous lemma implies that, to prove Theorem 2.9, we can assume that $G$ is triangulated. Another key ingredient for the proof of this theorem is the following inductive decomposition of a planar, simple, triangulated graph, depicted in Figure 2.3.

**Proposition 2.12.** Let $G$ be a planar, simple, triangulated graph. Let $v_1$ and $v_2$ be two vertices on its outer circuit. In linear time, we can order the vertices of $G$ as $v_1, \ldots, v_n$ such that, for each $k \geq 3$, the subgraph $G_k$ of $G$ induced by $v_1, \ldots, v_k$ satisfies:

- $G_k$ is connected;
- the boundary of $G_k$ is a circuit;
- each inner face of $G_k$ has degree three;
- $v_{k+1}$ is in the outer face of $G_k$.

The proof of this proposition rests on the following lemma.

**Lemma 2.13.** Let $G$ be a planar, simple graph; assume that the boundary of the outer face forms a circuit (without repeated vertices) $C$. Let $v_1 v_2$ be an edge on $C$. There exists a vertex $v$ on $C$, different from $v_1$ and $v_2$, that has exactly two neighbors on $C$.

**Proof.** If every vertex of $C$ has exactly two neighbors on $C$, we are done. Let the vertices of $C$ be $v_1 = w_1, \ldots, w_m = v_2$, in this order. Consider an edge connecting $w_i$ to $w_j$ where $j - i$ is minimal but at least two. Then the only neighbors of $w_{i+1}$ in $C$ are $w_i$ and $w_{i+2}$ (Figure 2.4): None of $w_{i+3}, \ldots, w_j$ can be a neighbor of $w_{i+1}$ by minimality of $j - i$, and none of the other vertices on $C$ either, by planarity.

**Proof of Proposition 2.12.** We choose $v_n, \ldots, v_3$ in this order by repeated applications of Lemma 2.13; the conditions are obviously satisfied.

To do this in linear total time, we maintain the following information on each vertex $v$ of the current graph: Whether $v$ belongs to the outer circuit and, if so, its number of neighbors on the outer circuit. We maintain a list of (pointers to) vertices on the outer circuit that have exactly two neighbors on the outer circuit; by Lemma 2.13, this list is never empty. The algorithm iteratively picks a vertex in the list, updates the data, and iterates until exactly three vertices are left.

This takes linear time, since each edge is considered only if one of the endpoints enters or leaves the circuit.

**Proof of Theorem 2.9.** The algorithm iteratively embeds the subgraph $G_k$ of $G$ induced by $v_1, \ldots, v_k$, where $k$ goes from 3 to $n$. Actually, instead of
computing $x$- and $y$-coordinates of the vertices, we compute $y$-coordinates of the vertices and $x$-spans of the edges, namely, the difference between the $x$-coordinates of their endpoints; trivially, this information is enough to recover the embedding.

Assume inductively that we already embedded $G_k$ ($k \geq 3$) on the grid in such a way that (Figure 2.5):

1. The $y$-coordinates of $v_1$ and $v_2$ are zero;
2. If $v_1 = w_1, \ldots, w_2, \ldots, w_m = v_2$ are the vertices on the outer face of $G_k$, in cyclic order, then the $x$-spans of each edge $w_iw_{i+1}$ is positive;
3. each edge $w_iw_{i+1}$, $1 \leq i \leq m$, has slope $+1$ or $-1$.

Vertex $v_{k+1}$ is incident, in $G_{k+1}$, to a contiguous set of vertices $w_p, \ldots, w_q$ on the boundary of the outer face of $G_k$. Let $P(w_p, w_q)$ be the intersection point of the line of slope $+1$ passing through $w_p$ with the line of slope $-1$ passing through $w_q$; Condition (3) implies that $P(w_p, w_q)$ has integer coordinates. Putting $v_{k+1}$ at position $P(w_p, w_q)$ almost yields a planar drawing of $G_{k+1}$, except that it may fail to see, e.g., $w_p$. To avoid this problem (Figure 2.5), we shift vertices $w_1, \ldots, w_p$ by one unit to the left, so that the slope of $w_pw_{p+1}$ becomes now smaller than $+1$; and similarly we shift $w_q, \ldots, w_m$ by one unit to the right. In our choice of representation of points with $x$-spans and $y$-coordinates, this takes constant time: Simply increase by one the $x$-span of $w_pw_{p+1}$ and of $w_{q+1}w_q$. The only problem is that the resulting drawing is inconsistent, so we need an adjustment phase to increase the $x$-spans of some internal edges. We first explain how to do this adjustment of the $x$-spans of internal edges at each step from $G_k$ to $G_{k+1}$. However, for the purposes of an efficient algorithm, it will be useful to do these adjustments at once.

We maintain a spanning tree $T^*$ of the dual of $G_k$, rooted at the outer face and oriented away from the root, as follows (Figure 2.3). Initially (say $k = 3$), there is one edge from the root outer face to the inner face, crossing edge $v_1v_2$. When we add vertex $v_{k+1}$, for each newly created internal face of the drawing, we create an edge of $T^*$ arriving to that face by crossing the unique edge incident to that face that belongs to $G_k$.

When adding edges in $G_k$ to build $G_{k+1}$, the adjustment phase consists in increasing by one the $x$-span of the edges crossed by the subpath of $T^*$ from the root to the first vertex incident to $(w_pw_{p+1})^*$, and similarly of the edges crossed by the subpath to the first vertex incident to $(w_{q+1}w_q)^*$. (Edges crossed by both subpaths have thus their $x$-span increased by two.) Combined with the initial shift of the boundary edges, this results in a shift of a “left” part of the graph to the left and of a “right” part of the graph to the right. To prove that this results in a valid embedding, it suffices to note that the following property is maintained during the algorithm: For every edge $e^*$ of $T^*$ oriented away from the root, the vertex of $G_k$ to the left of $e^*$ has smaller $x$-coordinate than the vertex of $G_k$ to the right of $e^*$. It is clear that, at the end, the vertices are on an $O(n) \times O(n)$-grid.

To implement this idea in linear time, we first compute the $x$-spans and $y$-coordinates in $G_3, \ldots, G_n$ without doing the adjustment steps; this takes $O(n)$ time. Omitting this adjustment step does not harm because, at each step, we only need to know that the $x$-spans and $y$-coordinates of the vertices on the outer face are correct. Afterwards, we need to increase the
2.3 Tutte’s barycentric embedding theorem

We give another method to build straight-line embeddings in the plane, which has other desirable properties: In particular, assuming 3-connectivity of the input graph (defined below), in the planar output embedding, all faces are convex. This leads to an interesting result regarding 3-dimensional polytopes, described in the next section.

A graph is 3-connected if it is connected and if it is still connected after removing zero, one, or two vertices and their incident edges.

**Theorem 2.14.** Let $G = (V, E)$ be a 3-connected graph without loops or multiple edges. Assume $G$ is embedded on $\mathbb{R}^2$. Let $v_1, \ldots, v_k$ be the vertices of the outer face. Assign unique positions $f(v)$ in $\mathbb{R}^2$ for each vertex $v$, such that

- the $f(v_i)$, $i = 1, \ldots, k$ are mapped to the vertices of a convex polygon (respecting the order of the vertices);
- the image of every vertex $v$ different from the $v_i$’s is a barycenter with strictly positive coefficients of the images of its neighbors in $G$.

Then drawing straight-line edges between the image points gives an embedding of $G$.

Given such a 3-connected graph, it is always possible to achieve the conditions of the theorem. For example, choose the barycentric coefficients to be all equal to one. The barycentric condition yields an affine system, which is solvable by an argument of “dominant diagonal”. Equivalently, one may view the edges as “springs” with the same rigidity, and the interior vertices as being free to move. The equilibrium of this physical system is met when the “energy” is minimized [69, p. 124].

We refer to the vertices $v_1, \ldots, v_k$ as exterior vertices, and to the other ones as interior vertices. Let $v$ be an interior vertex of $G$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be an affine function vanishing on $f(v)$. If all the neighbors of $v$ lie on $h^{-1}(0)$, we say $v$ is $h$-inactive. Otherwise, $v$ is $h$-active. In this case, $v$ has neighbors in both $h^{-1}((0, \infty))$ and $h^{-1}((-\infty, 0))$. In particular one can find a rising path from $v$ to an exterior vertex: a path whose value of $h$ strictly increases. Similarly one can find a falling path.

**Proposition 2.15.** The image of every interior vertex of $G$ is in the interior of the convex polygon $f(v_1) \ldots f(v_k)$.

**Proof.** Let $h$ be an affine form such that the polygon $f(v_1) \ldots f(v_k)$ lies in $h^{-1}((0, \infty))$. If there is a vertex (whose image is) in $h^{-1}((-\infty, 0))$, then consider the one that has minimum value of $h$. Since it is a barycenter with positive coefficients of its neighbors, all its neighbors must have the same value of $h$. By induction and connectivity of $G$, some exterior vertex must have that value of $h$, which is not possible. Therefore, each interior vertex lies in the interior or on the boundary of the polygon $f(v_1) \ldots f(v_k)$.

Let $v$ be an interior vertex; assume $v$ lies on an edge of the outer polygon, whose supporting line is $h^{-1}(0)$. Then all the neighbors of $v$ are $h$-inactive. Thus, all interior vertices that can be reached from $v$ by a path using only interior vertices lie on $h^{-1}(0)$. This contradicts the 3-connectivity of $G$, because removing the two exterior vertices on $h^{-1}(0)$ destroys the
Proposition 2.16. For any affine form \( h \), there is no \( h \)-inactive vertex.

See Figure 2.7 for an illustration of the next lemma.

Lemma 2.17 (Y–lemma). Let \( w_1, w_2, w_3 \) and \( v \) be pairwise distinct vertices of a graph \( H \). Assume, for \( i = 1, 2, 3 \), that there is a path \( P_i \) from \( w_i \) to \( v \) which avoids the \( w_j \)'s (for \( j \neq i \)). Then there exist three paths \( P_i' \), from \( w_i \) to a common vertex \( v' \), which are pairwise disjoint (except at \( v' \)).

Proof. First, using \( P_1 \) and \( P_2 \), we easily get a (simple) path \( R \) from \( w_1 \) to \( w_2 \), so that \( R \) and \( P_1 \) have the same first edge \( w_1z \). Then we consider the path \( P_3 \). If this path \( P_3 \) intersects \( R \), let \( v' \) be the first vertex of intersection on \( P_3 \). \( v' \) splits \( R \) in two parts, which we call \( P_1' \) (from \( w_1 \) to \( v' \)) and \( P_2' \) (from \( w_2 \) to \( v' \)); \( P_3' \) is the part of \( P_3 \) going from \( w_3 \) to \( v' \), with loops removed (if any). The \( P_i' \)'s satisfy the property stated in the lemma. If \( P_3 \) does not intersect \( R \), we call \( v' \) the last vertex on \( P_1 \) (when going from \( w_1 \) to \( v \)) which is also on \( R \). Such a vertex exists and is different from \( w_1 \) because \( w_1z \) is the first edge of \( R \) and \( P_1 \). Let \( P_3' \) be the path defined by \( P_3 \) followed by the part of the path \( P_1 \) which goes from \( v \) to \( v' \), with loops removed (if any). \( v' \) splits \( R \) in two parts, which we call \( P_1' \) and \( P_2' \). The paths \( P_i' \)'s satisfy the desired property.

Proof of Proposition 2.16. For the sake of a contradiction, assume \( v \) is an \( h \)-inactive vertex. We prove the existence of a subdivision of \( K_{3,3} \) in \( G \): a subgraph of \( G \) such that, after contracting edges, we get \( K_{3,3} \). Thus \( G \) cannot be planar. See Figure 2.8.

Let \( G(h) \) be the subgraph of \( G \) induced by the vertices on \( h^{-1}(0) \). Since \( G \) is 3-connected, there are, in \( G(h) \), three distinct \( h \)-active vertices \( w_1, w_2, \) and \( w_3 \) and three paths \( P_i \) connecting \( v \) with \( w_i \) such that, for any \( i \), the path \( P_i \) contains no vertex \( w_j \) for \( j \neq i \). Indeed, let \( w \) be a vertex of \( G \) so that \( h(f(w)) \neq 0 \). By connectivity of \( G \), take a path from \( v \) to \( w \) and call \( w_1 \) the first \( h \)-active vertex on this path. Do the same in \( G - \{w_1\} \), and choose \( w_2 \), by 2-connectivity. Similarly, use 3-connectivity to select \( w_3 \) in \( G - \{w_1, w_2\} \).

Applying then the Y–lemma in \( G(h) \), we get the existence of a vertex \( v' \) in \( G(h) \), together with three distinct paths (except at \( v' \)) \( P_i \) from \( w_i \) to \( v' \) in \( G(h) \). We can build rising paths \( Q_i \) from each of the \( w_i \) to a vertex \( x \) maximizing the value of \( h \). Then, the Y–lemma allows us to assume, by changing \( x \) and the \( Q_i \)'s if necessary, that these three paths are disjoint (except at \( x \)). Similarly, we can build falling paths \( R_i \) from each of the \( w_i \) to a vertex \( y \) and apply the Y–lemma. Using the paths \( P_i, Q_i, R_i \), which are all pairwise disjoint except at their endpoints, and the vertices \( x, v', y \) and \( w_1, w_2, w_3 \), we get a subdivision of the graph \( K_{3,3} \). This contradicts the planarity of \( G \).

By Proposition 2.16, the convex hull of the neighbors of an interior vertex \( v \) is a non-degenerate polygon, and \( v \) lies in its interior. We now triangulate
the faces of $G$, except the outer face: faces are added to split the faces of $G$ into triangles, without adding vertices (this is done in a purely combinatorial way). It still holds that $v$ is in the interior of the convex hull of its neighbors, which readily implies that $v$ is a barycenter with positive coefficients of them. To summarize, the hypotheses of the theorem are satisfied, but we can now even assume that every face of $G$, except maybe the outer face, is a triangle.

**Proposition 2.18.** Let $uvy$ and $uvz$ be two triangles of $G$ sharing the edge $uv$. Let $h$ be an affine form vanishing on $f(u)$ and $f(v)$. Then

$$h(f(y))h(f(z)) < 0.$$  

The proof relies on the following lemma.

**Lemma 2.19.** If $h(f(y)) > 0$, then $h(f(z)) < 0$.

**Proof.** By assumption $u$, $v$, and $y$ are $h$-active. Find strictly falling paths going from $u$ and $v$ to an exterior vertex. The falling paths may share a vertex. In any case, we get a simple circuit $C$ in $G$ using $uv$ whose set of vertices are on the strictly negative side of $h$, except $u$ and $v$. See Figure 2.9. (We may need some exterior vertices if the falling paths do not share a vertex.)

On the original (not necessarily straight-line) embedding of $G$, the circuit $C$ bounds a disk. Let $S$ be the set of vertices in the interior of this disk; $S$ contains no exterior vertices, so every vertex in $S$ is a barycenter with positive coefficients of its neighbors. Under $f$, all the vertices of $C$ are mapped to the half-space $h \leq 0$ except $u$ and $v$, which are mapped to $h = 0$; therefore, as in the proof of Proposition 2.15, all the vertices in $S$ belong to the open half-space $h < 0$. Since $h(f(y)) > 0$, the vertex $y$ cannot belong to $C$ or $S$.

In the original embedding of $G$, the circuit $C$ uses edge $uv$, and $y$ is outside $C$. Therefore $z$ must be inside $C$ (i.e., in $S$) or on $C$. In the former case, as seen above, we get $h(f(z)) < 0$. In the latter, since $u$ and $v$ are the only vertices of $C$ on the line $h = 0$, we also get $h(f(z)) < 0$. \qed

**Proof of Proposition 2.18.** Lemma 2.19 shows that, whenever one triangle is non-degenerate, then its incident triangles are non-degenerate. Necessarily, any triangle having one exterior edge is non-degenerate (Proposition 2.15). So every triangle is non-degenerate. The result follows. \qed

We can now conclude the proof of Tutte’s theorem.

**Proof of Theorem 2.14.** As we discussed earlier, a consequence of Proposition 2.16 is that we may assume that $G$ is triangulated (except possibly for the outer face). Since the triangles are non-degenerate by Proposition 2.18, it suffices to prove that the interiors of two distinct triangles are disjoint. For the sake of a contradiction, let $a$ be a point of $\mathbb{R}^2$ in the interior of two triangles $t$ and $t'$. Shoot a ray from $a$ to the boundary of the polygon $f(v_1)f(v_2)\ldots f(v_k)$ avoiding the image of every vertex. Whenever the ray leaves $t'$, by Proposition 2.18, it enters another triangle. So we get a sequence of triangles $t' = t'_1,t'_2,\ldots,t''$ where $t''$ is the unique triangle incident to the boundary edge that is on the end of the ray. Similarly, we get a sequence of triangles $t'' = t''_1,t''_2,\ldots,t''$. Going back in both sequences from $t''$, we pass from a triangle to an unambiguously defined preceding triangle. Since we start with the same triangle, we get $t = t'$. \qed

In particular, we have:

**Corollary 2.20** (Fáry–Stein–Wagner’s theorem). Every planar graph can be drawn in the plane with straight line edges.
Furthermore, if the graph is 3-connected, then we can choose the faces to be convex, since it is the case in a Tutte embedding. This latter fact is in particular important in the next application.

2.4 Steinitz’ theorem

Every convex polytope in \( \mathbb{R}^3 \) has a set of vertices (extremal points) and edges. This is called the 1-skeleton of the polytope. In this section, we will prove the following theorem.

**Theorem 2.21** (Steinitz’ theorem (1922)). Let \( G = (V, E) \) be a graph without loops or multiple edges. Then \( G \) is a planar 3-connected graph if and only if it is the 1-skeleton of a convex polytope in \( \mathbb{R}^3 \).

The “if” part is the easiest direction: Imagine a big physical model of your convex polytope, where the faces and the interior of the polytope are transparent. If you sit outside the polytope, close enough to the center of a face, you will see no crossing between the edges. In other words, the 1-skeleton is planar. We omit the proof that the 1-skeleton of a 3-dimensional polytope is 3-connected; this can be proved directly without too much trouble, and follows from a more general theorem by Balinski (see the proof for example in Ziegler’s book [80, Sect. 3.5]).

Let \( \omega : E \to (0, \infty) \) be a function from the (undirected) edges of \( G \) to a set of strictly positive coefficients. (In the sequel, we could take \( \omega \) to be constant, equal to one.)

Let \( f : V \to \mathbb{R}^2 \) be the corresponding Tutte equilibrium given by Theorem 2.14, where every vertex \( v \) is barycenter with coefficients \( \omega_{w_1}, \ldots, \omega_{w_m} \) of its neighbors \( w_1, \ldots, w_m \). We actually assume that \( f : V \to \mathbb{R}^3 \) maps the vertices into the plane \( z = 1 \) of \( \mathbb{R}^3 \).

To every interior face \( f \) of \( G \) we associate a vector \( q_f \) in \( \mathbb{R}^3 \). We choose an arbitrary interior face \( f_0 \), for which \( q_{f_0} = 0 \). The other \( q_f \)'s are defined by the following formula: For every interior edge \( uv \) with left face \( f_1 \) and right face \( f_2 \), we define

\[
q_{f_1} = \omega_{uv}(f(u) \times f(v)) + q_{f_2}
\]  

(2.1)

where \( \times \) denotes the cross-product in \( \mathbb{R}^3 \).

**Lemma 2.22.** The vectors \( q_f \) are well-defined.

**Proof.** First note that exchanging \( u \) exchanges \( f_1 \) and \( f_2 \) in (2.1), and thus gives \( q_{f_2} = \omega_{uv}(f(v) \times f(u)) + q_{f_1} \), which rewrites \( q_{f_2} = \omega_{uv}(-f(u) \times f(v)) + q_{f_1} \); this is exactly Equation (2.1).

Let \( v \) be a vertex of \( G \), and \( w_1, \ldots, w_m \) be its neighbors. We get:

\[
\sum_{i=1}^m \omega_{vw_i}(f(v) \times f(w_i)) = f(v) \times \left( \sum_{i=1}^m \omega_{vw_i}(f(w_i) - f(v)) \right) = 0.
\]

Therefore, Equation (2.1) gives consistent vectors \( q_f \) for all faces around an interior vertex.

Now, starting from the initial face \( f_0 \), we may define the value of \( q_f \) by choosing an arbitrary sequence of faces from \( f_0 \). In other words, in the dual graph \( G^* \), every path starting at \( f_0 \) and ending at some face \( f \) gives a value of \( q_f \). (Incidentally, this shows that the function \( q \), if it exists, is unique.)

We need to check that this value does not depend on the particular path chosen. For this purpose, consider two such paths \( p \) and \( p' \) in \( G^* \). We may assume that \( p \) and \( p' \) use distinct vertex sets of \( G^* \) except at the endpoints \( f_0 \) and \( f^* \). Then by the Jordan curve theorem 1.3, \( p \) and \( p' \) enclose a set of faces of \( G^* \). The result is proven by induction on the number of faces of \( G^* \) enclosed by \( p \) and \( p' \): The case of one face is the previous paragraph. For the induction step, build one path \( p'' \) in \( G^* \) that is in-between \( p \) and \( p' \), and apply induction. See Figure 2.10.

We define a piecewise linear function \( g \) from the union of the interior faces to \( \mathbb{R} \) by setting, for every point \( x \) in a face \( f \), \( g(x) = \langle x \mid q_f \rangle \).

**Lemma 2.23.** This map \( g \) is well-defined.

**Proof.** We only need to prove that, whenever \( x \) belongs to an edge \( uv \) incident with faces \( f_1 \) and \( f_2 \), the value of \( g(x) \) is the same, whichever face \( f_1 \) or \( f_2 \) we choose for the computation; in other words, \( \langle x \mid q_{f_1} \rangle = \langle x \mid q_{f_2} \rangle \).

By linearity it suffices to prove the result for \( x = f(u) \) and \( x = f(v) \).

\[
\langle f(v) \mid q_{f_1} \rangle = \langle f(v) \mid \omega_{uv}(f(u) \times f(v)) + q_{f_2} \rangle = \langle f(v) \mid q_{f_2} \rangle.
\]
There is one subtlety, however: the vertices of the outer face are not necessarily coplanar; though, if the outer face is incident with three vertices, this condition is automatically satisfied. If \( G \) contains a triangle, we may have taken that triangle to be the outer face in the application of Tutte’s theorem. Thus, the only case that remains to be shown is when \( G \) contains no triangle.

From Exercise 3.2, we know that \( G^* \), the dual graph of \( G \), contains a triangle. Clearly, \( G^* \) is planar, and it can be shown that it is also 3-connected. We may therefore realize \( G^* \) as the 1-skeleton of a convex polytope. Now, a known construction, polarity, allows to transform a 3-polytope into another one, whose 1-skeleta are dual to each other [80, Sect. 2.3]. So \( G \) is the 1-skeleton of a convex polytope as well. \( \square \)

### 2.5 Notes

The planarity testing algorithm is taken from [23, Section 3.3]. (The algorithm to compute the blocks is due to Hopcroft and Tarjan [47]; the presentation is inspired from Wikipedia’s article “Biconnected component”.) The algorithm to draw a graph on a grid is due to de Fraysseix et al. [21], with simplifications from Castelli Aleardi et al. [11].

The proof of Tutte’s theorem we described uses arguments from Edelsbrunner and Harer [25] other sources [19,69]; the proof of Steinitz’ theorem is also taken from Richter-Gebert [69]. In addition to the original paper proving Tutte’s theorem [77], there are many other proofs [4,15,19,36,42,69,76]. The correspondence between Tutte embeddings where every vertex is barycenter of its neighbors and the height function \( g \) is the Maxwell-Cremona correspondence (see for example Hopcroft and Kahn [49]).

There are some straight-line graph embeddings that cannot be “lifted” to a convex polytope (Figure 2.11).

The fact that every planar graph without loops or multiple edges admits a straight-line embedding was shown a few decades before the discovery of the algorithms given above [34,72,79]. Actually, if \( G \) is a planar graph without loops or multiple edges with \( n \) vertices, a straight-line embedding exists where all vertices lie in the \((n-2) \times (n-2)\)-grid [35]. Many other representations exist, such as circle packing representations: the vertices are mapped to non-overlapping
Figure 2.11. An embedding that cannot be lifted to a convex polytope. Indeed, assume every interior edge is an edge on the “bottom” of the convex polytope. We can suppose, by adding a suitable affine form to all the $z_i$’s, that $z_4 = z_5 = z_6 = 0$. Then $z_1 > z_2 > z_3 > z_1$, which is impossible.

Disks in the plane, two of which are tangent if and only if an edge between the corresponding vertices exists (see Mohar and Thomassen [64, Chapter 2] for a proof and references).

In this chapter, we illustrate the general idea that algorithmic problems on graphs are easier to deal with when the graph is assumed to be planar. By Theorem 2.1, if we are given a planar graph $G$, we can compute in linear time a combinatorial map of $G$ in the plane; therefore, we can assume that in all algorithms for planar graphs, a combinatorial map of the graph is given.

### 3.1 Minimum spanning tree algorithm

Let $G = (V, E)$ be a cellular graph embedding in $S^2$, with a weight function $w : E \to \mathbb{R}$ on its edges. Let $n$ be its complexity.

**Theorem 3.1.** A minimum spanning tree of $G$ can be computed in $O(n)$ time.

We note that, by Lemma 1.8, $E' \subseteq E$ is a minimum spanning tree of $G$ if and only if $(E \setminus E')^*$ is a maximum spanning tree of $G^*$ (where the weight of a dual edge equals the weight of the corresponding primal edge).

**Exercise 3.2.** Prove that a connected planar graph has either a vertex or a face with degree at most three.

We introduce two operations to transform a cellular graph embedding in $S^2$
into another one. These operations (together with their reverses) are called
Euler operations. Let $e$ be an edge of $G$ that is incident with two dif-
ferent faces. Then removing $e$ yields a cellular graph embedding, denoted
by $G \setminus e$. The dual operation is contraction: let $e$ be an edge of $G$
that is incident with two different vertices (i.e., that is not a loop), then we
may contract $e$ by identifying its two incident vertices; the resulting graph
embedding is denoted by $G/e$. Obviously, these two operations preserve
the planarity.

Proof of Theorem 3.1. The two following dual rules allow to build induct-
ively the set of edges $T(G)$ of a minimum spanning tree of $G$:

- Let $v$ be a vertex of $G$. If all edges incident with $v$ are loops, then
  $G$ has exactly one vertex, so there is a unique, empty, spanning tree.
  Otherwise, let $e$ be a minimum-weight edge incident exactly once
  with $v$. Necessarily, edge $e$ belongs to a minimum spanning tree
  of $G$. Hence $T(G/e) \cup e$ is a minimum spanning tree of $G$;

- let $f$ be a face of $G$. If all edges incident with $f$ have $f$ on both sides,
  then $G$ has exactly one face, so $G$ is a tree, and there is a unique
  spanning tree, $G$ itself. Otherwise, let $e$ be a maximum-weight edge
  incident exactly once with $f$. Then $e$ does not belong to a minimum
  spanning tree of $G$ (because $e^*$ belongs to a maximum spanning tree
  of $G^*$). It follows that $T(G \setminus e)$ is a minimum spanning tree of $G$.

The number of iterations of this algorithm is $O(n)$. Assuming we can pick
a vertex $v$ or a face $f$ with degree $O(1)$ (whose existence is guaranteed
by Exercise 3.2) in constant amortized time, we have a linear-time algo-

Proof. Root $T$ at an arbitrary vertex $r$. Using a traversal of the tree, one
can label each vertex $v$ of $T$ with the sum of the weights of $v$ and all
other vertices and faces, possibly some of them being destroyed in the
course of the algorithm after they are put in the bucket). Initially, put
all vertices and faces in $B$. When contracting or deleting an edge $e$, only
the degrees of the vertices and faces incident with $e$ can change, so we put
them in the bucket before contracting or deleting $e$. Therefore in total
$O(n)$ vertices and faces are put into $B$.

To find a vertex or face of degree at most three in the current graph, pick
an element of $B$, check in $O(1)$ time whether it still belongs to the current
graph and, if so, whether it has degree at most three. If it is not the case,
remove it from $B$ and proceed with the next element. Since $O(n)$ elements
in total are put in $B$, also $O(n)$ elements are removed from $B$, so the total
time spent to find vertices and faces with degree at most three is $O(n)$.

3.2 Separators

Let $G$ be a graph. Here we assume that $G$ is weighted: every vertex gets
a non-negative weight, and all the weights sum up to at most one. (As an
important special case, one could choose $1/n$ for the weight of each of the
$n$ vertices of $G$. However, we will need the general case at some point.)

The weight of $G$ is the sum of the weights of its vertices. A separator
for $G$ is a set $S$ of vertices such that every connected component of $G − S$
has weight at most $1/2$.

Separators of small size, when they exist, are very useful, and often allow
for efficient divide-and-conquer strategies. The purpose of this section is
to show how to compute efficiently (optimally) small separators in planar
graphs.

Let us first focus on two special cases. The first one studies separators for
trees and is the mother of all examples for graph separators.

Proposition 3.3. Let $T$ be a weighted tree with $n$ vertices and edges. In
$O(n)$ time, one can compute a separator for $T$ made of a single vertex.

Proof. Root $T$ at an arbitrary vertex $r$. Using a traversal of the tree, one
can label each vertex $v$ of $T$ with the sum of the weights of $v$ and all
its descendents. This allows to compute in \(O(d)\) time the weight of each component of \(T - v\), for an arbitrary vertex \(v\) of degree \(d\). Now, starting at an arbitrary vertex \(v\) of the tree:

- if all components of \(T - v\) have weight at most \(1/2\), return \(v\);
- otherwise, replace \(v\) with the neighbor \(w\) of \(v\) belonging to the largest connected component of \(T - v\).

When the algorithm returns, its result is correct. Furthermore, it moves from vertex to vertex along edges, but it never visits the same vertex twice, because \(T\) is a tree and because, in the second case, the component of \(T - w\) containing \(v\) has weight at most one minus the weight of the component of \(T - v\) containing \(w\), and this is at most \(1/2\). The running time is thus proportional to the sum of the degrees of the vertices, which is \(O(n)\). \(\square\)

The second special case considers planar graphs with small radius.

**Lemma 3.4.** Let \(G\) be a weighted planar graph with \(n\) vertices and edges. Let \(U\) be a spanning tree of \(G\) rooted at some vertex \(r\), such that any vertex is at distance at most \(d\) from \(r\) in \(U\). Then a separator for \(G\) of size at most \(3d + 1\) can be computed in \(O(n)\) time.

**Proof.** Without loss of generality, we may assume that \(G\) is triangulated. Indeed, we can without harm iteratively remove edges forming faces of degree one or two and then triangulate every face of degree at least four. Define weights on the vertices of the dual graph \(G^*\) by charging the weight of each vertex of \(G\) to the dual of exactly one incident face of \(G\). Now \(G^*\) qualifies as a weighted graph.

\(T^* := G^* - U^*\) is a tree. By Proposition 3.3, we can compute a vertex \(c^*\) of \(T^*\) such that each connected component of \(T^* - c^*\) has weight at most \(1/2\).

Let \(t\) be the triangle of \(G\) containing \(c^*\). Let \(S\) be the subgraph of \(G\) that is the union of the three edges of \(t\) together with the three shortest paths from the vertices of \(t\) to the root \(r\). \(S\) has at most \(3d + 1\) vertices.

There remains to prove that each component \(C\) of \(G - S\) has weight at most \(1/2\). To see this, note that \(S\) splits the plane into several connected regions (Figure 3.1), and \(C\) belongs entirely to a single region. By the Jordan curve theorem, the vertices of \(G^*\) inside that region belong to a single component of \(T^* - c^*\) and therefore have total weight at most \(1/2\). Furthermore, when we assigned weights to \(G^*\), all the weight of \(C\) was charged into that region. In other words, \(C\) has weight at most \(1/2\). \(\square\)

Finally, here is the general result on planar graphs.

**Theorem 3.5.** Let \(G\) be a planar graph with \(n\) vertices and edges. Then one can compute in \(O(n)\) time a separator for \(G\) of size \(O(\sqrt{n})\).

One can prove that this result is optimal: Each separator of the \(\sqrt{n} \times \sqrt{n}\)-grid has size \(\Omega(\sqrt{n})\).

**Proof of Theorem 3.5.** We can obviously assume that \(G\) is connected. Let \(r\) be an arbitrary vertex of \(G\). Compute a breadth-first search tree \(T\) in \(G\) rooted at \(r\). For every vertex \(v\) of \(G\), define its level to be its distance to \(r\).

Let \(\ell_1\) be a weighted median level of a vertex in \(G\): namely, the total weight of the vertices with level lower (resp., higher) than \(\ell_1\) is at most \(1/2\). Let \(\ell_0\) be the largest level smaller than or equal to \(\ell_1\) containing at most \(\sqrt{n}\) vertices. Similarly, let \(\ell_2\) be the smallest level larger than or equal to \(\ell_1\) containing at most \(\sqrt{n}\) vertices. (It may be that these levels contain no vertices.) These computations take \(O(n)\) time.

Here are the key properties of the levels \(\ell_0\) and \(\ell_2\):
3. Efficient algorithms for planar graphs

Let \( G = (V, E) \) be a graph and \( k \geq 1 \) be an integer. A coloring of \( G \) with \( k \) colors is a map \( V \rightarrow \{1, \ldots, k\} \) such that adjacent vertices are mapped to different integers ("colors"). If a graph has a coloring with \( k \) colors, we say that it is \( k \)-colorable.

In coloring problems, we can safely ignore graphs with loops (edges incident twice to the same vertex), because such graphs are not \( k \)-colorable, for any \( k \). In this section, we implicitly only consider graphs without loops, and all subsequent graphs built in the proofs have this property.

Determining whether a graph is \( k \)-colorable is NP-hard, except for \( k = 1 \) (it is equivalent to have no edge in the graph) and \( k = 2 \) (it is equivalent to have a bipartite graph, a problem easily solvable in linear time). For planar graphs, life seems to be no easier: It is NP-hard to decide whether a planar graph is 3-colorable [40], by reduction from 3-SAT.

However, it is a remarkable fact that every planar graph is 4-colorable [2]; this was proved by Appel and Haken, heavily relying on computer assistance (up to date, no proof is known that does not involve a lot of case distinctions). We shall prove that every graph is 5-colorable, and give an algorithm to 5-color a planar graph in linear time, assuming a combinatorial map is given.

**Theorem 3.6.** Every (simple) planar graph is 5-colorable.

**Proof.** Consider a planar drawing of a graph \( G \) in the plane. We can assume without loss of generality that \( G \) is connected and has no face (including the outer face) of degree one or two. Let \( v, e, \) and \( f \) be the number of vertices, edges, and faces of \( G \). Euler’s formula \( v - e + f = 2 \) and double-counting of the edge-face incidences \( 2e \geq 3f \) implies \( e \leq 3v - 6 \) and thus the average degree of a vertex, \( 2e/v, \) is strictly less than 6. Thus, \( G \) has at least one vertex of degree at most five. This directly implies that \( G \) is 6-colorable, since if \( x \) is a vertex of degree at most five, we can assume by induction that the graph \( G - x \) obtained from \( G \) by removing \( x \) and its incident vertices is 6-colorable, and then color \( x \) with one color not used by any of its neighbors. To prove that every planar graph is 5-colorable, we only need to refine the argument slightly.

If \( G \) has a vertex incident with at most four distinct vertices, then by induction we are done. So let \( x \) be a vertex of degree exactly five, with distinct neighbors \( v_1, \ldots, v_5 \) in clockwise order around \( x \). Let a 5-coloring of \( G - x \) be given. If \( v_1, \ldots, v_5 \) do not have distinct colors, then at least one color remains to color \( x \), so we are done. So assume (up to permutation) that \( v_i \) bears color \( i \).

Let \( G' \) be the subgraph of \( G - x \) induced by the vertices colored 1 and 3.
Assume first that there is no path in $G_{13}$ connecting $v_1$ to $v_3$. We can exchange colors 1 and 3 in the component of $G_{13}$ that contains $v_1$ (this is clearly valid). Now, both $v_1$ and $v_3$ are colored 3, which frees one color for vertex $x$, and we are done.

On the other hand, if $v_1$ and $v_3$ are connected in $G_{13}$, then we claim that $v_2$ and $v_4$ cannot be connected in $G_{21}$ (the subgraph of $G - x$ induced by the vertices colored 2 and 4), which implies that we can use the same trick with $v_2$ and $v_4$ in place of $v_1$ and $v_3$.

To prove the claim, we assume the contrary: There are two disjoint paths $p_{13}$ and $p_{24}$ in $G - x$ connecting pairs $(v_1, v_3)$ and $(v_2, v_4)$ respectively. We can modify $G - x$ to exhibit a planar graph that is $K_5$, the complete graph with five vertices (Figure 3.2), which is a contradiction (Exercise 1.10).

To do that, take the graph with vertex set \{x, v_1, v_2, v_3, v_4\} and with the following edges: $x$ is connected to all other vertices via edges drawn like in $G$; $v_1$ and $v_3$ are connected via $p_{13}$, similarly $v_2$ and $v_4$ are connected via $p_{24}$. Moreover, $(v_1, v_2)$, $(v_2, v_3)$, $(v_3, v_4)$, $(v_4, v_1)$, $(v_2, v_5)$, and $(v_3, v_5)$ can be connected together without crossing other edges because they appear in cyclic order around $x$. This is $K_5$.

We can actually obtain an efficient 5-coloring algorithm:

**Theorem 3.7.** Every (simple) planar graph, given in the form of a combinatorial map, can be 5-colored in linear time.

We will rely on the following independent proposition.

**Proposition 3.8.** Let $G$ be a planar graph with no face of degree one or two. Then $G$ has either a vertex of degree at most four, or a vertex of degree five incident to two vertices of degree at most six.

(Precisely, this should be understood as follows: among the three vertices involved in the second alternative, some of them can be the same, however the two edges involved should be distinct.)

**Proof.** It suffices to prove the result assuming $G$ is triangulated. We proceed by contradiction, assuming that such configurations cannot occur.

Let us put a charge equal to 6 in each triangle. Now each triangle $T$ sends its charge to all its incident vertices of degree five and six, in a way that the degree-5 incident vertices get all the same charge from $T$, the degree-6 incident vertices get all the same charge from $T$, and any degree-5 incident vertex gets twice the charge of a degree-6 incident vertex from $T$. (If $T$ has no vertex of degree 5 or 6, the charge remains in $T$.)

After this operation, each degree-6 vertex $v$ gets charge at least 2 from each of its incident triangles. Indeed, excepting $v$, such a triangle can be incident to at most two other degree-6 vertices, or to one degree-5 vertex and to a vertex of degree at least 7. Therefore, each degree-6 vertex gets charge at least 12. Not that the reasoning is also valid if some triangles around $v$ are identified.

Also, each degree-5 vertex $v$ gets charge at least 24. Indeed,

- If it is not adjacent to any degree-5 or degree-6 vertex, except possibly itself, it gets all the charge from its incident triangles, which is 30;
- if it is adjacent to a degree-5 vertex $w \neq v$, then it has no other vertex of degree 5 or 6 around, so it gets all the charge from the 3 triangles not incident to $w$ and half the charge from the 2 triangles also incident to $w$, and thus 24;
- if it is incident to a degree-6 vertex $w$, similarly, it gets all the charge from the 3 triangles not incident to $w$ and $2/3$ of the charge from the 2 triangles also incident to $w$, which is 26.
Let \( t \) be the number of triangles in \( G \), and \( n_i \) be the number of vertices of degree \( i \) in \( G \). The previous discussion implies \( 6t \geq 24n_5 + 12n_6 \), or equivalently \( t \geq 4n_5 + 2n_6 \) (*).

On the other hand, Euler’s formula can be rewritten in the two following ways:

- \( \sum_i (2 - i)n_i = 4 - 2t \), since \( 2e = \sum_i in_i \);
- \( \sum_i 2n_i = t + 4 \), since \( 2e = 3t \).

Eliminating \( n_7 \) in these two linear equations yields \( \sum_i (14 - 2i)n_i = t + 28 \).

Since \( n_i = 0 \) for \( i \leq 4 \), this implies \( 4n_5 + 2n_6 - 2n_8 - 4n_9 - \ldots = t + 28 \).

In particular, \( t < 4n_5 + 2n_6 \), contradicting (*).

**Proof of Theorem 3.7.** We can assume that our input planar graph \( G \) is triangulated (without loops). Indeed, we can without harm remove any edge bounding a degree-one face, and, for any set of parallel edges forming adjacent faces of degree two, we can remove all but one such edges; finally, we may triangulate the remaining edges without adding loops (as in the proof of Lemma 2.11). This can be done in linear time.

We first describe the high-level approach, without worrying about complexity. We also proceed by induction, like in the proof of Theorem 3.6.

Let \( x \) be a vertex of \( G \) obtained by Proposition 3.8. If \( x \) has at most four distinct neighbors, then we are done by applying induction to \( G - x \) (after triangulating the new face). Otherwise, \( x \) has degree five, and five distinct neighbors \( v_1, \ldots, v_5 \) in this cyclic order around \( x \), two of which, say \( v_1 \) and \( v_i \) (\( i \in \{2,3\} \)), have degree at most six.

By planarity, if \( i = 2 \), then either \( v_1 \) and \( v_3 \) are not adjacent, or \( v_2 \) and \( v_4 \) are not adjacent. If \( i = 3 \), then either \( v_1 \) and \( v_4 \) are not adjacent, or \( v_3 \) and \( v_5 \) are not adjacent. So let \( k, \ell \in \{1, \ldots, 5\} \) such that \( v_k \) and \( v_\ell \) are not adjacent and \( v_k \) has degree at most six.

Let \( H \) be the graph obtained from \( G - x \) by identifying \( v_k \) and \( v_\ell \), as in Figure 3.3 (this operation preserves planarity since \( v_k \) and \( v_\ell \) belong to the same face of \( G - x \), and no loop is created since \( v_k \) and \( v_\ell \) are not adjacent in \( G \)). We apply induction to \( H \). After \( H \) is 5-colored, this corresponds to a coloring of \( G - x \) where \( v_k \) and \( v_\ell \) have the same color, which leaves one color free to color \( x \).

![Figure 3.3. Illustration of the inductive construction in the proof of Theorem 3.7.](image)

If we omit the time taken to find vertex \( x \), then this algorithm can be implemented in linear time. Indeed, finding which vertices adjacent to \( x \) have degree at most six takes constant time. Using the fields \( uu \) described in Section 1.2.1, we can test whether the neighbors of \( x \) are distinct in constant time. We can also test whether two given vertices are adjacent in constant time if one of the vertices has bounded degree; this allows to determine \( v_k \) and \( v_\ell \) in constant time. Then identifying \( v_k \) to \( v_\ell \) takes constant time because \( v_k \) has bounded degree (this latter property is needed since updating the fields \( uu \) takes time linear in the smaller degree of \( v_k \) and \( v_\ell \)).

There remains to compute vertex \( x \) efficiently. To do this, we maintain, during the algorithm, a stack (a linked list, for example) containing the vertices of degree at most four, and the vertices of degree five adjacent to two vertices of degree at most six. At each step, a constant number of vertices can enter or leave the stack, which can be updated in constant time.

**3.4 Minimum cut algorithm**

We now give an efficient algorithm for computing minimum cuts in planar graphs.
Before that, we need to state without proof a result on shortest paths in planar graphs. Let \( G = (V, E) \) be a connected graph where each edge has a non-negative length (also called weight), and let \( s \) be a vertex of \( G \). A shortest path tree is a spanning tree of \( G \) rooted at \( s \) that contains a shortest path from \( s \) to each vertex in \( G \). Dijkstra’s algorithm (with the appropriate data structure for the priority queue, for example Fibonacci heaps) allows to compute a shortest path tree in \( O(|E| + |V| \log |V|) \) time. The following result, which is (fortunately) admitted, improves the result for planar graphs.

**Theorem 3.9.** Given a graph cellularly embedded in \( S^2 \), a shortest path tree from a given vertex can be computed in time linear in the complexity of the graph.

We shall use this result to prove the following theorem.

**Theorem 3.10.** Let \( G = (V, E) \) be a weighted planar graph of complexity \( n \), cellularly embedded in \( S^2 \). Let \( s \) and \( t \) be two vertices of \( G \). The problem of computing a minimum-weight \((s, t)\)-cut of \( G \) can be solved in \( O(n \log n) \) time.

To prove Theorem 3.10, we first dualize the problem in the following proposition, which is rather intuitive but not so easy to prove formally. Henceforth, let \( G = (V, E) \) be a weighted planar graph, and let \( F \) be the faces of \( G \).

**Proposition 3.11.** \( X \subseteq E \) is an \((s, t)\)-cut in \( G \) if and only if \( X^* \) contains the edge set of some circuit of \( G^* \) separating \( s \) and \( t \).

**Proof.** If \( X^* \) contains (the edge set of) a circuit in \( G^* \) separating \( s \) and \( t \), then any \((s, t)\)-path in \( G \) must cross an edge in \( X^* \), and thus contain an edge in \( X \), so \( X \) is an \((s, t)\)-cut.

Conversely, let \( X \) be an \((s, t)\)-cut; we will prove that \( X^* \) contains the edge set of a circuit in \( G^* \) separating \( s \) and \( t \). Without loss of generality, we may assume that \( X \) is inclusionwise minimal among all \((s, t)\)-cuts.

First, label “\( S \)” each face \( v^* \) of \( G^* \) such that there is, in \( G \), an \((s, v)\)-path avoiding \( X \). Similarly, label “\( T \)” each face \( v^* \) of \( G^* \) such that there is, in \( G \), a \((v, t)\)-path avoiding \( X \). Since \( X \) is a cut, no face of \( G^* \) is both labeled “\( S \)” and “\( T \)”. Any edge of \( G^* \) incident to faces labeled differently must be in \( X^* \). Therefore, by minimality of \( X^* \), each face of \( G^* \) is labeled either “\( S \)” or (exclusive) “\( T \)”, and \( X^* \) is the set of edges incident to faces with different labels.

Let \( S \) be the subset of the plane made of the faces of \( G^* \) labeled “\( S \)”, together with the open edges of \( G^* \) whose incident faces are both labeled “\( S \)”. Define similarly \( T \). Thus \( S \) and \( T \) are disjoint, connected subsets of the plane. Let \( f^* \) be a vertex of \( G^* \); we claim that there cannot be four faces incident to \( f^* \) that belong respectively, in cyclic order around the vertex, to \( S \), \( T \), \( S \), and \( T \). Indeed, if the opposite assertion holds, then by connectivity of \( S \), there is a closed curve in \( S \cup \{ f^* \} \) that goes through \( f^* \) and has faces of \( T \) on both sides of it, which contradicts the connectedness of \( T \) by the Jordan curve theorem.

Thus, \( X^* \) is a union of vertex-disjoint circuits in \( G^* \); let \( \gamma \) be one such circuit. Since each edge of \( \gamma \) is incident to one face labeled “\( S \)” and one face labeled “\( T \)”, \( \gamma \subseteq X^* \) separates \( s \) from \( t \). \( \square \)

We now reformulate the problem in terms of curves crossing \( G \). More precisely, we consider closed curves in general position with respect to \( G \), which do not meet any vertex of \( G \) and intersect the edges of \( G \) at finitely many points, where they cross. The length of such a closed curve is the sum of the weights of the edges of \( G \) crossed by that curve, counted with multiplicity. Computing shortest paths between two points in this setting can be done in \( O(n) \) time by applying Theorem 3.9 in the dual graph.

**Proposition 3.12.** Let \( \gamma \) be a simple closed curve in general position with respect to \( G \); assume that \( \gamma \) has minimum length among all such curves that separate \( s \) from \( t \). Then the set of edges of \( G \) crossed by \( \gamma \) is a minimum-weight \((s, t)\)-cut in \( G \).

**Proof.** The set of edges of \( G \) crossed by \( \gamma \) is an \((s, t)\)-cut, by the Jordan curve theorem. Conversely, if we have a minimum-weight \((s, t)\)-cut in \( G \), Proposition 3.11 implies that its dual contains a circuit separating \( s \) and \( t \), which corresponds to a simple closed curve \( \gamma \) separating \( s \) and \( t \) whose length is the same as the weight of the cut. \( \square \)
Now, we view $G$ as embedded on the sphere, and we remove two small disks around $s$ and $t$. We now have an embedding of $G$ on an annulus $A$, and by Proposition 3.12 it suffices to compute a shortest simple closed curve in general position with respect to $G$ that goes “around” the annulus. The general idea of the algorithm is depicted in Figure 3.4. Let $p$ be some shortest path from an arbitrary point on one boundary to an arbitrary point on the other boundary (again, where the length is measured by the sum of the weights of $G$ crossed by $p$). Let $D$ be the disk obtained by cutting the annulus along $p$; let $p'$ and $p''$ be the pieces of its boundary corresponding to $p$.

### 3.4.1 Naïve algorithm

The following lemma implies that some shortest simple closed curve separating the two boundaries of $A$ corresponds, in $D$, to a shortest path between a pair of “twin” points of $p'$ and $p''$.

**Lemma 3.13.** Some shortest closed curve separating the two boundaries of $A$ is simple and crosses $p$ exactly once.

**Proof.** Let $\gamma$ be a shortest closed curve separating the two boundaries. The image of $\gamma$ in $D$ (after cutting along $p$) must contain a simple path $q$ from $p'$ to $p''$, for otherwise $\gamma$ would not separate the boundaries of $A$.

Let $\gamma'$ be a closed curve obtained by connecting the endpoints of $q$ with a shortest path running along $p$. This closed curve is simple, separates the boundaries of $A$, crosses $p$ exactly once, and is no longer than $\gamma$, since $\gamma$ has at least the length of $q$ plus the length necessary to connect the endpoints of $q$.

This allows a naïve $O(n^2)$-time algorithm: Let $k \geq 0$ be the number of edges of $G$ crossed by $p$; let $v_0, \ldots, v_k$ be points on $p$, in this order on $p$, such that the subpath between $v_i$ and $v_{i+1}$ crosses exactly one edge of $G$. Compute all shortest paths between $v'_i$ and $v''_i$ (the twin points corresponding, in $D$, to $v_i$), and take a shortest such path. The running-time follows since $k = O(n)$ and since shortest paths can be computed in linear time in planar graphs (Theorem 3.9).

### 3.4.2 Divide-and-conquer algorithm

To beat this quadratic bound, we use a “divide-and-conquer” strategy based on the following lemma, illustrated in Figure 3.5.

**Lemma 3.14.** Let $x, y$, and $z$ be points on $p$, in this order, and $(x', x'')$, $(y', y'')$, and $(z', z'')$ be the corresponding twin points on $D$. Let $p_x$ and $p_z$ be disjoint simple shortest paths in $D$ between the corresponding twin pairs $(x', x'')$ and $(z', z'')$. Then some shortest path $p_y$ between the twin pairs $(y', y'')$ crosses neither $p_x$ nor $p_z$, and is simple.
loops in may assume that So by induction, we may assume that and the shortest closed curve separating the two boundaries of between the points If p crosses p at least twice, at points a and b, we may replace the part of p between a and b by a path running along p, removing two crossings between p and p; this does not decrease the length of p, since p is a shortest path; and this does not introduce additional crossings between p and p, since p and p are disjoint. So by induction, we may assume that p is disjoint from p. Similarly, we may assume that p is also disjoint from p. Finally, we may remove the loops in p to make it simple.

We first describe the two base cases of the recursion, which can be solved in linear time:

1. If k = O(1) (for example, if k ≤ 1), we may conclude by computing all shortest paths, in D, between each pair of twin vertices v′ and v′′, and taking the shortest of these paths;
2. similarly, if there is a face f of the graph incident with both boundaries of A, then the shortest closed curve has to go through this face; we can conclude by cutting the annulus A into a disk along a path entirely contained in f and computing a shortest path, in this disk, between the two copies of the path.

Otherwise, we consider vertex v := v[l..r] and compute a shortest path in D between the points v′ and v′′ corresponding to v on p′ and p″, respectively; this is thus a shortest closed curve γ passing through v and crossing p exactly once. Let A1 and A2 be the two annuli obtained by cutting A along γ. The previous lemma implies that it suffices to recursively compute the shortest closed curve separating the two boundaries of A1 and of A2 (using the pieces of p within A1 and A2 as new shortest paths), and to take the shortest of these closed curves. This concludes the description of the algorithm.

3.4.3 Correctness and complexity analysis

Proof of Theorem 3.10. The execution of the algorithm can be represented with a binary tree, where each node corresponds to an annulus. The root corresponds to A; internal nodes always have two children; leaves correspond to the base case of the recursion.

The algorithm terminates, since the path p crosses at most ⌈k/2⌉ edges at the i-th level in the recursion tree, and by base case (1). In fact, this proves that there are at most ⌈log k⌉ = O(log n) levels in the recursion tree. The correctness follows from Proposition 3.12 and from the above considerations. There remains to show the O(n log n) complexity.

Consider a given edge e of G. At some level r of the recursion tree, that edge is cut by some closed curves into a number of subedges e1, ..., ej (j ≥ 1), all belonging to distinct annuli at level r. However, only the subedges e1 and ej can belong to an annulus that is an internal node of the recursion tree: the other ones end in base case (2). Therefore, e occurs at most twice in total in the annuli that are internal nodes at level r, and thus at most four times in total in the annuli at level r + 1. Hence, the total number of non-boundary edges of the annuli at a given level is at most 4n.

Furthermore, every boundary edge of an annulus can be charged to an adjacent non-boundary edge of that annulus, in a way that every non-boundary edge is charged at most twice. Thus, the total number of boundary edges of the annuli at a given level is at most 8n.

Bottom line: the total number of edges of all annuli at a given level is O(n); by Euler’s formula, this is also a bound on the sum of the complexities of all annuli at a given level. Since, at each node, all the operations (cutting and shortest paths computations) take linear time in the complexity of the annulus, the overall complexity of the algorithm is proportional to the total complexity of the annuli.
complexity of the annuli appearing in the recursion tree, which is made of \(O(\log n)\) levels, each containing annuli of total complexity \(O(n)\).

3.5 Notes

The minimum spanning tree algorithm described above is based on Matsui [62] (see also Cheriton and Tarjan [14] for a more complicated, but more general, algorithm). Actually, the same technique shows that a minimum spanning tree of a graph cellularly embedded on a surface of genus \(g\) can be computed in \(O(gn)\) time. (See Chapter 4 for more on surfaces.) On arbitrary graphs, things are more complicated: there is a randomized algorithm with linear time [51], and a deterministic algorithm with almost linear time (where “almost” means up to a factor involving the inverse Ackermann function) [13].

The linear-time separator algorithm is inspired by the original paper by Lipton and Tarjan [59], but was quite simplified (in particular the weighting scheme of the dual graph in the proof of Lemma 3.4). It might be that the simplified proof is new, but it is, however, more likely that such or similar simplification was found earlier, and the author welcomes any information on this subject. The interest of separators is that they often allow for divide-and-conquer strategies, by cutting the problem into two subproblems of roughly half the size of the original problem, computing a solution in these subproblems, and using them to compute the entire solution. What is actually more powerful is a separator decomposition of the graph, where separators of the subgraphs are recursively computed; such a decomposition is also computable in linear time [41]. There are alternate proofs for the existence of separators, see Alon et al. [1] for a graph-theoretic approach and Miller et al. [63] for a geometric approach using circle packing with applications.

Regarding the section on graph coloring, Proposition 3.8 is due to Franklin [38]. The linear-time 5-coloring algorithm is a variant of an algorithm sketched by Robertson et al. [70], which seems to have a subtle flaw. In that paper, a weaker version of Proposition 3.8 is used; the algorithm still needs to identify two vertices \(v_k\) and \(v_l\), but with that weaker version, none of these vertices can be assumed to have bounded degree. Thus updating the \(vu\) fields requires linear time. Such \(vu\) fields are needed because we must be able to test whether two vertices are adjacent in constant time, assuming (only) one of these vertices has bounded degree.

The algorithm for finding a minimum cut in a planar graph was found by Reif [68]. Here we used, as a black-box, the linear-time algorithm for shortest paths in planar graphs, due to Henzinger et al. [45]; this algorithm relies on graph separators. The presentation of the min-cut algorithm above differs slightly, by using closed curves in general position with respect to \(G\); this concept will be refined when we introduce the notion of cross-metric surface in Chapter 5.1. Frederickson [39] provides a different method. The proof of Proposition 3.11 used here is a variant of the one found in Colin de Verdière and Schrijver [20, Lemma 7.2].

A shortest circuit in a graph separating two given faces translates, in the dual, to a minimum cut separating the two dual vertices. By the max-flow min-cut theorem, a maximum flow yields immediately a minimum cut, but not conversely. A very recent paper shows that both the minimum cut and maximum flow problems can be solved in \(O(n \log \log n)\) in planar graphs [50].
Chapter 4

Topology of surfaces

4.1 Definition and examples

A surface is a topological space in which each point has a neighborhood homeomorphic to the unit open disk \( \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \} \). We only consider compact surfaces in this chapter (and even later, unless specifically noted).

Examples of surfaces are the sphere, the torus, and the double torus: these are compact, connected, orientable (to be defined later) surfaces with zero, one, and two handles, respectively (see Figure 4.1). The classification of surfaces (Theorem 4.5) asserts that two compact, connected, and orientable surfaces are homeomorphic if and only if they have the same number of “handles”.

Despite the figures, note that a surface is “abstract”: the only knowledge we have of it is the neighborhoods of each point. A surface is not necessarily embedded in \( \mathbb{R}^3 \). Actually, the non-orientable surfaces cannot be embedded in \( \mathbb{R}^3 \).

4.2 Surface (de)construction

4.2.1 Surface deconstruction

A graph embedded on a surface is cellularly embedded if all its faces are topological disks. As in the case of the plane, we may consider the combinatorial map of a graph cellularly embedded on a surface; the data structures are identical. The dual graph is defined similarly.

The polygonal schema associated with a cellular graph embedding is defined as follows: assign an arbitrary orientation to each edge; for each face, record the cyclic list of edges around the face, with a bar if and only if they have the same number of “handles”. See Figure 4.2.

4.2.2 Surface construction

Conversely, the data of a polygonal schema allows to build up a surface and the cellular graph embedding. More precisely, let \( S \) be a finite set of symbols and let \( \bar{S} = \{ \bar{s} \mid s \in S \} \). Let \( R \) be a finite set of relations, each relation being a non-empty word in the alphabet \( S \cup \bar{S} \), so that for every...
4. Topology of surfaces

Figure 4.3. The “corners” incident to some vertex \( v \) can be ordered cyclically.

For each relation of size \( n \), build an \( n \)-gon; label its edges by the elements of \( R \), in order, the presence of a bar indicating the orientation of the edge (see Figure 4.2). (Polygons with one or two sides are also allowed.) Now, identify the “twin” edges of the polygons corresponding to the same symbol in \( S \), taking the orientation into account. (As a consequence, vertices get identified, too.)

Lemma 4.1. The topological space obtained by the above process is a compact surface.

Proof. Let \( X \) be the resulting topological space; \( X \) is certainly compact. We have to show that every point of \( X \) has a neighborhood homeomorphic to the unit disk. The only non-obvious case is that of a vertex \( v \) in \( X \), that is, a point corresponding to a vertex of some polygons. But it is not hard to prove that a neighborhood of \( v \) is an umbrella: the “corners” (vertices) of the polygons corresponding to \( v \) can be arranged into a cyclic order; see Figure 4.3.

We admit the following converse:

Theorem 4.2 (Kerékjártó-Radó; see Thomassen [75] or Doyle and Moran [24]). Any compact surface is homeomorphic to a surface obtained by the gluing process above.

This amounts to saying that, on any compact surface, there exists a cellular embedding of a graph. Equivalently, every surface can be triangulated.

Figure 4.4. (a) The orientations of these two faces (triangles) are compatible. (b) Two non-compatible orientations of the faces. A surface is orientable if there exist orientations of all faces that are compatible.

4.3 Classification of surfaces

4.3.1 Euler characteristic and orientability character

Let \( G \) be a graph cellularly embedded on a compact surface \( \mathcal{S} \). The Euler characteristic of \( G \) equals \( v - e + f \), where \( v \) is the number of vertices, \( e \) is the number of edges, and \( f \) is the number of faces of the graph.

Proposition 4.3. The Euler characteristic is a topological invariant: it only depends on the surface \( \mathcal{S} \), not on the cellular embedding.

Sketch of proof. The Euler characteristic is easily seen to be invariant under Euler operations. The result is then implied by the following claim: any two cellular embeddings on a given surface can be transformed one into the other via a finite sequence of Euler operations. Proving this is not very difficult but requires some work; a key property is that one can assume both embeddings to be piecewise linear with respect to a given triangulation of the surface (using for example the method by Epstein [28, Appendix]).

\( G \) is orientable if the boundary of its faces can be oriented so that each edge gets two opposite orientations by its incident faces (Figure 4.4). The orientability character is a topological invariant as well; the same proof as that of Proposition 4.3 works, but it can also be proven directly:

Exercise 4.4. \( G \) is orientable if and only if no subset of \( \mathcal{S} \) is a Möbius strip.
4.3.2 Classification theorem

**Theorem 4.5.** Every compact, connected surface $\mathcal{S}$ is homeomorphic to a surface given by the following polygonal schemata, called *canonical* (each made of a single relation):

i. $a \bar{a}$ (the sphere; Euler characteristic $2$, orientable);

ii. $a_1 b_1 \bar{a}_1 b_1 \ldots a_g b_g \bar{a}_g b_g$, for $g \geq 1$ (Euler characteristic $2 - 2g$, orientable);

iii. $a_1 a_1 \ldots a_g a_g$, for $g \geq 1$ (Euler characteristic $2 - g$, non-orientable).

Furthermore, the surfaces having these polygonal schemata are pairwise non-homeomorphic. In particular, two connected surfaces are homeomorphic if and only if they have the same Euler characteristic and the same orientability character.

In the above theorem, $g$ is called the *genus* of the surface; by convention $g = 0$ for the sphere. The orientable surface of genus $g$ is obtained from the sphere by cutting $g$ disks and attaching $g$ “handles” in place of them. Similarly, the non-orientable surface of genus $g$ is obtained from the sphere by cutting $g$ disks and attaching $g$ Möbius strips (since a Möbius strip has exactly one boundary component). See Figure 4.5. See also Figure 4.6 for a representation of a double-torus in canonical form.

**Proof.** Let $\mathcal{S}$ be a compact, connected surface, and $G$ be a graph embedded on $\mathcal{S}$ (by Theorem 4.2). By iteratively removing edges incident with different faces, we may assume that $G$ has only one face.\(^1\) By iteratively contracting edges incident with different vertices, we may assume that $G$ has only one vertex and one face\(^2\) (unless this yields a sphere, so the polygonal schema is $a \bar{a}$ — actually, we could say that the polygonal schema made of the empty relation is a degenerate polygonal schema for the sphere). The surface $\mathcal{S}$ cut along $G$ is therefore a topological disk; we use cut-and-paste operations on this polygonal schema to obtain a standard form.

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\(^1\)This amounts to removing all primal edges of a spanning tree in the dual graph.

\(^2\)This amounts to contracting the edges of a spanning tree in the primal graph.
4. Topology of surfaces

![Figure 4.6](image1)

**Figure 4.6.** (a) A canonical polygonal schema of the double torus. (b) The identification of the edges of the schema. (c) The actual graph embedded on the double torus. (d) Closeup on the order of the loops around the basepoint of the surface, as seen from below; it can be derived directly from (a).

![Figure 4.7](image2)

**Figure 4.7.** The classification of surfaces: grouping the twin edges appearing with the same orientation.

If the polygonal schema has the form $aPaQ$ (where $P$ and $Q$ are possibly empty sequences of symbols), then we can transform it into $bb\bar{P}Q$ (Figure 4.7)---$Q$ denotes the symbols of $Q$ in reverse order, inverting also the presence or absence of a bar above each letter. So inductively, we may assume that each pair of symbols appearing in the polygonal schema with the same orientation is made of two consecutive symbols. We still have one face and one vertex.

Assume some edge appears twice in the polygonal schema with opposite orientations: $aP\bar{a}Q$. Then $P$ and $Q$ must share an edge $b$, because otherwise the endpoints of $a$ would not be identified on the surface. By the preceding step, $b$ must appear in opposite orientations in $P$ and $Q$, so we may assume that the polygonal schema has the form $aPbQ\bar{a}R\bar{b}S$. Then, by further cut-and-paste operations, we may transform the polygonal schema into $d\bar{e}\bar{d}RQPS$ (Figure 4.8). We still have one face and one vertex, and can iterate the process. After this stage, the polygonal schema is the concatenation of blocks of the form $aa$ and $ab\bar{a}$.

If there are no blocks of the form $aa$, or no blocks of the form $ab\bar{a}$, then we are in form (ii) or (iii), respectively. Otherwise, one part of the boundary of the polygonal schema has the form $aab\bar{e}\bar{c}$. We may transform it to $d\bar{e}\bar{d}b\bar{c}$ (Figure 4.9), and, applying the method of Figure 4.7 to $b$, $c$, and $d$ in order, we obtain that we replaced the part of the boundary we considered into $e\bar{f}\bar{f}g\bar{g}$; the other part of the boundary is unchanged. So iterating, we may convert the polygonal schema into form (iii).

The Euler characteristics and orientability characters of the surfaces are readily computed, since the canonical polygonal schemata have exactly one
Figure 4.8. The classification of surfaces: grouping pairs of twin edges appearing with different orientations.

Figure 4.9. The classification of surfaces: transforming one form into the other.

vertex and one face. Since two distinct canonical polygonal schemata do not have the same Euler characteristic and the same orientability character, they cannot be homeomorphic, by Proposition 4.3 and Exercise 4.4.

Example 4.6.

- The orientable surface with genus 1 is a torus; the orientable surface with genus 2 is the double torus; and so on.
- The non-orientable surface with genus 1 is a projective plane; with genus 2 it is the Klein bottle.

Exercise 4.7. Identify the surfaces with the following schemata:
1. $a\overline{a}b\overline{b}$;
2. $abab$;
3. $aba\overline{b}$;
4. $a_{1}a_{2}...a_{n}\overline{a_{1}}\overline{a_{2}}...\overline{a_{n}}$;
5. $a_{1}a_{2}...a_{n-1}a_{n}\overline{a_{1}}\overline{a_{2}}...\overline{a_{n-1}}\overline{a_{n}}$.

4.4 Surfaces with boundary

A surface (possibly) with boundary $\mathcal{S}$ is a topological space in which each point has a neighborhood homeomorphic to the unit open disk $\{(x, y) \in \mathbb{R}^{2} \mid x^{2}+y^{2} < 1\}$ or to the unit half disk $\{(x, y) \in \mathbb{R}^{2} \mid x^{2}+y^{2} < 1 \text{ and } x \geq 0\}$.

The boundary of $\mathcal{S}$, denoted by $\partial \mathcal{S}$, comprise the points of this surface that have no neighborhood homeomorphic to the unit disk. The interior of $\mathcal{S}$ is the complementary part of its boundary.

A cellular embedding on a surface with boundary is defined as in the case of surfaces without boundary. In particular, since each face must be an open disk, the boundary of the surface must be the union of some edges of the graph. The classification theorem (Theorem 4.5) can be extended for surfaces with boundary: Given a surface with boundary $\mathcal{S}$, we may attach a disk to each of its boundary components, obtaining a surface without
boundary $\mathcal{J}$, and apply the previous classification theorem. Furthermore, the number of boundary components is a topological invariant.

The Euler characteristic and the orientability character of a cellular embedding on a surface with boundary $\mathcal{J}$ are defined as in the case of surfaces without boundary; they are also topological invariants. The Euler characteristic of $\mathcal{J}$ equals that of $\mathcal{J}$ minus the number of boundary components of $\mathcal{J}$. So two surfaces with boundary $\mathcal{J}$ and $\mathcal{J}'$ are homeomorphic if and only if they have the same Euler characteristic, orientability character, and number of boundary components.

If we have a graph embedding $G$ without isolated vertex on a surface $\mathcal{J}$, then cutting $\mathcal{J}$ along $G$ is a well-defined operation that yields a surface with boundary, denoted by $\mathcal{J}\setminus G$. This fact is not trivial, and follows from the fact that every graph embedding on a surface $\mathcal{J}$ can be mapped by a homeomorphism of $\mathcal{J}$ (actually, an isotopy) to a piecewise-linear embedding with respect to a fixed triangulation of $\mathcal{J}$, using, e.g., the method by Epstein [28, Appendix].

### 4.5 Notes

The classification theorem is due to Brahana, Dehn, and Heegaard; the present proof is inspired from Stillwell [73]. For another, more visual proof, see Francis and Weeks [37].

The proofs of the classification theorem usually involve two steps, the first one being topological (Theorem 4.2, Proposition 4.3, Exercise 4.4), the second one being combinatorial. In the same vein, the Hauptvermutung ("main conjecture") says that any two embeddings of a graph on a surface are subdivisions of graph embeddings that are combinatorially identical. This is true, but some higher-dimensional analogs do not hold.

Let $G$ and $M$ be simple graphs (that is, without loops or multiple edges). $M$ is a minor of $G$ if $M$ can be obtained from $G$ by iteratively contracting edges, deleting edges, and deleting isolated vertices (at each step, the graph should be made simple by removing loops and identifying multiple edges). Let $\mathcal{J}$ be a fixed surface. Clearly, if $G$ is embeddable on $\mathcal{J}$, then every minor of $G$ is also embeddable on $\mathcal{J}$. Let $\mathcal{F}$ be the set of minor-minimal graphs not embeddable on $\mathcal{J}$; thus $G$ is embeddable on $\mathcal{J}$ if and only if no graph in $\mathcal{F}$ is a minor of $G$. Kuratowski's theorem asserts that $G$ is planar if and only if it does not have $K_5$ or $K_{3,3}$ as a minor; in other words, if $\mathcal{J}$ is the sphere, the family $\mathcal{F}$ is finite. This actually holds for every surface $\mathcal{J}$; however, no algorithm is known to enumerate the family $\mathcal{F}$.

More generally, this property is implied by a deep result by Robertson and Seymour [71] (whose proof needed no less than 20 papers and several hundreds of pages): In any infinite family of graphs, at least one is a minor of another.

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3This notation is not standard (yet).
Chapter 5

Computing shortest graphs with cut loci

In this chapter, we describe algorithms to compute shortest curves and graphs that “cut” a given surface into simpler pieces.

5.1 Combinatorial and cross-metric surfaces

We aim at computing “short” graphs and curves on surfaces. For this, we need to define a metric on a surface that is both accurate in the applications and simple enough so as to be handled algorithmically. We shall introduce two ways of doing this, which are dual of each other. Depending on the context, some results and algorithms are more easily described using one setting or the other.

In this chapter, all surfaces are compact, connected, and orientable. They do not have boundaries.

5.1.1 More types of curves

We already defined paths on surfaces; we need to introduce more types of curves.

An arc on a surface with boundary \( S \) is a path \( p : [0, 1] \to S \) such that \( p(t) \) belongs to \( \partial S \) if and only if \( t \in \{0, 1\} \). A loop \( \ell \) is a path with the same endpoints; \( \ell(0) = \ell(1) \) is called the basepoint of the loop. A path is simple if it is one-to-one. A loop is simple if its restriction to \( [0, 1) \) is one-to-one (of course, due to the identified endpoints, it cannot be one-to-one on \( [0, 1] \)).

The concatenation of \( p \) and \( q \), denoted by \( p \cdot q \), is the path defined by:

- \( (p \cdot q)(t) = p(2t) \), if \( 0 \leq t \leq 1/2 \);
- \( (p \cdot q)(t) = q(2t - 1) \), if \( 1/2 \leq t \leq 1 \).

A reparameterization of a path \( p \) is a path of the form \( p \circ \varphi \), where \( \varphi : [0, 1] \to [0, 1] \) is bijective and increasing. If the paths are considered up to reparameterization, the concatenation is associative. The inverse of a path \( p \), denoted by \( \overline{p} \), is the map \( t \mapsto p(1 - t) \).

5.1.2 Combinatorial surfaces

A combinatorial surface \((S, M)\) is the data of a surface \( S \) (possibly with boundary), together with a cellular embedding \( M \) of a weighted graph. The weights must be non-negative. In this model, the only allowed curves are walks in \( M \); the length of a curve is the sum of the weights of the edges traversed by the curve, counted with multiplicity.

5.1.3 Cross-metric surfaces

We will, however, use a dual formulation of this model, which allows to define crossings between curves: this turns out to be helpful both for stating the results and as intermediate steps. A cross-metric surface \((\mathcal{S}, M^*)\) is a surface \( \mathcal{S} \) together with a cellular embedding of a weighted graph \( M^* \). If \( \mathcal{S} \) has a boundary, we require in particular that each boundary of \( \mathcal{S} \) be the union of some edges in \( M^* \), with infinite crossing weight. We consider only regular paths on \( \mathcal{S} \), which intersect the edges of \( M^* \) only transversely and away from the vertices. The length \( \text{length}(\gamma) \) of a regular curve \( \gamma \) is defined to be the sum of the weights of the dual edges that \( \gamma \) crosses, counted with multiplicity. The length of a regular arc is defined similarly, excluding the endpoints of the arc (which belong to an edge of \( M^* \) with infinite crossing weight). To emphasize this usage, we sometimes refer to the weight of a dual edge as its crossing weight.
To any combinatorial surface \((\mathcal{S}, M)\) without boundary, we associate by duality a cross-metric surface \((\mathcal{S}, M^*)\), where \(M^*\) is (as notation suggests) the dual graph of \(M\). To any curve on a combinatorial surface, traversing edges \(e_1, \ldots, e_p\), we can associate a curve in the corresponding cross-metric surface, crossing edges \(e_1^*, \ldots, e_p^*\), and conversely. This transformation preserves the lengths of the curves. So far, the notions of combinatorial and of cross-metric surfaces (without boundary) are thus essentially the same, up to duality. We can easily construct shortest paths on a cross-metric surface by restating the usual algorithms (for example, Dijkstra’s algorithm) on \(M\) in terms of the dual graph \(M^*\).

### 5.1.4 Curves on cross-metric surfaces, algorithmically

We can represent an arbitrary set of possibly (self-)intersecting curves on a cross-metric surface \((\mathcal{S}, M^*)\) by maintaining the arrangement of \(M^*\) and of the curves, i.e., the combinatorial embedding associated with the union of the curves (assuming this union forms a cellular embedding, which will always be the case). Contrary to combinatorial surfaces, this data structure also encodes the crossings between curves. The initial arrangement is just the graph \(M^*\), without any additional curve. We embed each new curve regularly: every crossing point of the new curve and the existing arrangement, and every self-crossing of the new curve, creates a vertex of degree four.

Whenever we split an edge \(e^*\) of \(M^*\) to insert a new curve, we give both sub-edges the same crossing weight as \(e^*\). Each segment of the curve between two intersection points becomes a new edge, which is, unless noted otherwise, assigned weight zero. However, it is sometimes desirable to assign a non-zero weight to the edges of a curve. For example, the cross-metric surface \(\mathcal{S}\backslash \alpha\) obtained by cutting \(\mathcal{S}\) along an embedded curve \(\alpha\) can be represented simply by assigning infinite crossing weights to the edges that comprise \(\alpha\), indicating that these edges cannot be crossed by other curves.

### 5.1.5 Complexity

The complexity of a combinatorial surface \((\mathcal{S}, M)\) is the total number of vertices, edges, and faces of \(M\); similarly, the complexity of a cross-metric surface \((\mathcal{S}, M^*)\) is the total number of vertices, edges, and faces of \(M^*\). The complexity of a set of curves is the number of times it crosses edges of \(M^*\).

### 5.2 Cut loci

Let us fix the notations for the remaining part of this chapter. Unless otherwise noted, \((\mathcal{S}, M^*)\) is a cross-metric surface (connected, compact, orientable, without boundary) of genus \(g\) and complexity \(n\). Furthermore, \(b\) is a point inside a face of \(M^*\) and is the basepoint of all loops considered in this chapter (we omit the precision that the basepoint is \(b\) in the sequel).

Let \(T\) be the shortest path tree from \(b\) to a point in each face of \(M^*\).\(^1\) The cut locus \(C\) of \((\mathcal{S}, M^*)\) with respect to \(b\) is the subgraph of \(M^*\) obtained by removing all edges of \(M^*\) crossed by \(T\). It is therefore a graph embedded on \(\mathcal{S}\). See Figure 5.1.

**Lemma 5.1.** \(\mathcal{S}\backslash C\) is a disk.

**Proof.** At some stage of the growth of the shortest path tree \(T\), consider the union of all open faces of \(M^*\) visited by \(T\), and of all edges of \(M^*\) crossed by \(T\). This is an open disk; at the end, it contains all faces of \(M^*\), and its complement is \(C\). In particular, \(\mathcal{S}\backslash C\) is a disk. \(\square\)

Intuitively, we are inflating a disk around \(b\) progressively, without allowing self-intersections, until it occupies the whole surface; the cut locus \(C\) is the set of points of the surface where the boundary of the disk touches itself.

\(^1\)Strictly speaking, the shortest path tree is not always unique: there may be several shortest paths between two given points. However, uniqueness holds for generic choices of the weights; in other words, it can be enforced using an arbitrarily small perturbation of the lengths. By a slight abuse of language, we will therefore use the article “the” in such cases, since it does not harm (and may help the reader) to think that uniqueness holds. Nevertheless, no algorithm or result presented here requires uniqueness of shortest paths.
Dijkstra’s algorithm implies that we can compute $C$ in $O(n \log n)$ time.

**Exercise 5.2** (Complexity of the reduced cut locus). Let $C'$ be the graph obtained from the cut locus $C$ by repeatedly removing every degree-one vertex, together with its incident edge, and replacing every degree-two vertex $v$ and its incident edges with an edge connecting the two neighbors of $v$. Prove that $C'$ has complexity $O(g)$.

Given an edge $e \in C$, the loop $\sigma(e)$ is defined as a loop with basepoint $b$ that follows the shortest path tree to go from its root $b$ to a face incident with $e$, crosses $e$, and goes back from the other face incident with $e$ to the root. This can be done so that all the loops $\sigma(e)$ are simple and disjoint (except, of course, at their basepoint $b$ — we shall omit this triviality in the sequel). See Figure 5.2.

Define the weight of an edge $e$ of $C$ to be the length of the corresponding loop $\sigma(e)$ (this is not the same as the crossing weight, defined for every edge of $M^*$!); these weights can be computed with no time overhead during the cut locus computation.

**Figure 5.1.** The cut locus $C$ of a double torus (in bold lines) and the remaining edges of $M^*$ (in thin lines).

**Figure 5.2.** The loops $\sigma(e)$, for three edges $e \in C$.

### 5.3 Shortest non-contractible loop

A (possibly non-simple) loop is **contractible** if it can be continuously deformed into a point.

**Exercise 5.3.** Prove that, on a disk or a sphere, every loop is contractible.

**Lemma 5.4.** A simple loop is contractible if and only if it bounds a disk.

**Proof.** If a loop bounds a disk, it is certainly contractible. The proof of the converse is more difficult, and we admit it.

Our goal now is to give an algorithm to compute the shortest non-contractible loop.

### 5.3.1 3-path condition

A set $L$ of loops satisfies the **3-path condition** if, for any point $a \neq b$ and any three paths $p$, $q$, and $r$ from $b$ to $a$, if $p \cdot \bar{q}$ and $q \cdot \bar{r}$ belong to $L$, then $p \cdot \bar{r}$ belongs to $L$.

**Lemma 5.5.** The set of contractible loops satisfies the 3-path condition.
Lemma 5.6. Let $L$ be a set of loops satisfying the 3-path condition. Some shortest loop not in $L$ crosses the cut locus $C$ at most once.

Proof. See Figure 5.3 for an illustration of the proof. Let $\ell$ be a shortest loop not in $L$; without loss of generality, we can choose $\ell$ such that it crosses $C$ as few times as possible. Assume, for the sake of a contradiction, that $\ell$ crosses $C$ at least twice; let $a$ be a point on $\ell$ not on $M^*$ between its first and last crossing with $C$. This point $a$ splits $\ell$ into two paths $p$ and $q$, both from $b$ to $a$, and we have $\ell = p \cdot \bar{q}$. Furthermore, let $r$ be the shortest path from $b$ to $a$; this path does not cross $C$.

The 3-path condition applied to $p$, $q$, and $r$ implies that $p \cdot \bar{r}$ or $q \cdot \bar{r}$ does not belong to $L$. Both paths are no longer than $\ell = p \cdot \bar{q}$ and cross $C$ fewer times than $\ell$, implying the desired contradiction.

5.3.2 Structural lemmas

Lemma 5.7. Some shortest non-contractible loop has the form $\sigma(e)$.

Proof. Let $\ell$ be a shortest non-contractible loop. By Lemmas 5.5 and 5.6, some shortest non-contractible loop crosses the cut locus at most once. On the other hand, every non-contractible loop has to cross $C$ at least once (since $\mathcal{J}\setminus C$ is a disk). Hence some shortest non-contractible loop crosses the cut locus exactly once, at some edge $e$. This loop deforms continuously to $\sigma(e)$, which cannot be longer. The result follows.

Lemma 5.8. Let $e$ be an edge of $C$. Then $\sigma(e)$ is contractible if and only if some component of $C - e$ is a tree.

Proof. Assume first that one component of $C - e$ is a tree. One can then move $\sigma(e)$ continuously over the tree to make it disjoint from $C$; the resulting loop is contractible.

Conversely, if $\sigma(e)$ is contractible, it bounds a disk $D$ by Lemma 5.4. We want to prove that the part of $C$ inside $D$ is a tree. But if it is not the case, this part contains a circuit, which further bounds a disk $D' \subset D$, and therefore $C$ cuts $\mathcal{J}$ into at least two pieces, one of which is $D'$; this is impossible (Lemma 5.1).

5.3.3 Algorithm

Theorem 5.9. Finding a shortest non-contractible loop can be done in $O(n \log n)$ time. The loop computed is simple.

Proof of Theorem 5.9. We first compute the cut locus $C$, and assign to every edge $e$ of $C$ a weight that is the length of $\sigma(e)$, in $O(n \log n)$ time. We show how to eliminate the edges $e$ such that at least one component of $C - e$ is a tree. This concludes, since it then suffices to select the minimum-weight remaining edge of $C$ (by Lemmas 5.7 and 5.8).

This graph pruning can be done in $O(n)$ time: put all edges incident with a degree-one vertex in a list. Then, while the list is non-empty, remove an edge $e$ from it; remove it from $C$ (unless it was already removed); if one or both of its endpoints have now degree one in $C$, put the corresponding edge(s) in the list. Clearly, this removes only edges $e$ such that no component of $C - e$ is a tree. All them must eventually be removed, because a tree has a degree-one vertex (a leaf).

Corollary 5.10. Finding a shortest non-contractible loop without specified basepoint can be done in $O(n^2 \log n)$ time.
\textbf{Proof.} For every face of $M^*$, run the algorithm in Theorem 5.9 with the basepoint in that face, and return the shortest loop. \qed

\section{5. Shortest non-separating loop}

\subsection{5.4.1 Types of simple loops}

A simple loop $\ell$ is \textit{separating} if $\mathcal{S} \backslash \ell$ is not connected. A simple contractible loop bounds a disk, hence is separating; the converse is false. So there are (essentially) three kinds of simple loops: contractible, separating but not contractible, and non-separating. These three types are illustrated in Figure 5.2.

\textbf{Exercise 5.11.} \bigstar

1. Give an algorithm that determines whether a given simple loop is separating.

2. Give an algorithm that determines whether a given simple loop is contractible. Indication: use Lemma 5.4.

Our present goal is to compute the shortest non-separating (simple) loop. We need first to define the notion of homology boundary, which generalizes the notion of separating loop to possibly non-simple loops. To anticipate, we introduce a bit more technicalities than those needed for this sole purpose.

\subsection{5.4.2 Preliminaries on homology}

We introduce \textit{1-dimensional homology for graphs embedded on surfaces, over $\mathbb{Z}/2\mathbb{Z}$}.\footnote{This would not be needed if we introduced \textit{singular homology}, but it seems preferable.}

To simplify matters, we assume here (and in Section 5.5) that all curves considered are drawn on a very dense graph $G = (V,E)$ embedded on $\mathcal{S}$, transversely to $M^*$.\footnote{This would not be needed if we introduced \textit{singular homology}, but it seems preferable.} We consider \textit{chains}: subsets of $E$. It is a natural $\mathbb{Z}/2\mathbb{Z}$-vector space: the addition of two subsets of $E$ is the symmetric difference, multiplication by 0 gives the empty subset of $E$, and multiplication by 1 is the identity.

A chain $E' \subseteq E$ is a \textit{homology cycle} if every vertex of $V$ is incident with an even number of edges of $E'$. A chain $E' \subseteq E$ is a \textit{homology boundary} if the faces of $G$ can be colored black and white so that $E'$ is the set of edges of $E$ with exactly one black and one white incident face. Equivalently, if we consider the “dual” graph of $(V,E')$, which has one vertex inside each face of $(V,E')$ and one edge crossing each edge of $(V,E')$, then $E'$ is a homology boundary if and only if this dual graph is bipartite.

\textbf{Exercise 5.12.} \bigstar\bigstar\bigstar

1. Prove that the set of homology cycles (resp. homology boundaries) forms a vector space, and that every homology boundary is a homology cycle.

2. Assume $\mathcal{S}$ is a sphere. Prove that every homology cycle is a homology boundary.

\textbf{Lemma 5.13.} A simple loop $\ell$ in $G$ disconnects $\mathcal{S}$ if and only if its edge set forms a homology boundary.

\textbf{Proof.} Let $E'$ be the set of edges of $\ell$. Either the graph $(V,E')$ has one face, in which case the only boundary is the empty set, or it has two faces, in which case coloring one face in black and the other one in white yields a non-zero boundary formed by the edge set of $\ell$. \qed

So the notion of homology boundary extends the notion of being separating.

As shown in Exercise 5.12, the set of homology boundaries, $B$, is included in the set of homology cycles, $Z$. The reverse inclusion does not hold in general. Homology measures the “difference” between $Z$ and $B$; formally, it is able to avoid doing so. The assumption above is actually not needed: we only require $G$ to be dense enough so that the loops $\sigma(e)$ are disjoint walks on $G$ and so that $G$ contains some shortest non-separating loop (or some shortest system of loops, in Section 5.5). The existence of such a graph $G$ is clear, and it is never used in the algorithms, only in proofs, so its complexity does not matter.
is $\mathbb{Z}/B$, the $\mathbb{Z}/2\mathbb{Z}$-vector space that is the quotient of the two $\mathbb{Z}/2\mathbb{Z}$-vector spaces $\mathbb{Z}$ and $B$.

Given a loop $\ell$ in $G$, its mod 2 image is the set of edges of $G$ used an odd number of times by $\ell$. (We sometimes identify a loop with its mod 2 image.)

### 5.4.3 Algorithm

We prove here:

**Theorem 5.14.** Finding a shortest loop that is not a homology boundary can be done in $O(n \log n)$ time. The loop computed is simple, and is (therefore) also a shortest simple non-separating loop.

**Corollary 5.15.** Finding a shortest loop without specified basepoint that is not a homology boundary (or a shortest simple non-separating closed curve) can be done in $O(n^2 \log n)$ time.

**Lemma 5.16.** A subset $A$ of the edges of $C$ disconnects $C$ if and only if the set of loops $\sigma(A)$ disconnects $\mathcal{S}$.

**Proof.** We may certainly assume $A \neq \emptyset$. Let $D$ be the disk $\mathcal{S} \setminus C$; the basepoint $b$ belongs to the interior of $D$. Each loop $\sigma(e)$ in $\sigma(A)$ corresponds, in $D$, to two paths from $b$ to the boundary of $D$, connecting twins of $e$. See Figure 5.4.

Therefore, if we let $\tau(e)$ be the intersection of $e$ with $\sigma(e)$, any path in $\mathcal{S} \setminus \sigma(A)$ continuously retracts to a path in $C \setminus \tau(A)$, without moving the endpoints if they already belong to $C$. This implies that $\mathcal{S} \setminus \sigma(A)$ is connected if and only if $C \setminus \tau(A)$ is connected; this is in turn equivalent to having $C - A$ connected. \qed

**Proof of Theorem 5.14.** The general strategy is very similar to the proof of Theorem 5.9. The set of all loops in $G$ whose mod 2 images are homology boundaries satisfies the 3-path condition. Hence, by Lemma 5.6, some shortest loop in $G$ whose mod 2 image is not a homology boundary crosses the cut locus at most once, hence exactly once, at some edge $e$, by

![Figure 5.4. A view of the disk $\mathcal{S} \setminus C$, whose polygonal schema is $uvw\bar{u}\bar{v}$. The loops $\sigma(u)$ and $\sigma(v)$ are cut into two paths connecting the basepoint to twin points.](image)

**Exercise 5.12.** A slight extension of that exercise implies that $\sigma(e)$ is in the same homology class, and it is no longer. Hence some shortest loop whose mod 2 image is not a homology boundary has the form $\sigma(e)$.

In particular, it is simple, and is therefore a non-separating loop (Lemma 5.13). It must be a shortest non-separating loop in $G$ because every separating loop is a homology boundary. It is therefore a shortest non-separating loop, because we can (retroactively) assume that $G$ contains some shortest non-separating loop.

By Lemma 5.16, we are thus looking for a minimum-weight edge $e$ of $C$ such that $C - e$ is connected; such edges are called non-bridge edges.

**Lemma 5.17.** Let $G$ be a graph of complexity $n$. One can in $O(n)$ time determine all the bridge edges of $G$. \qed

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Proof. The proof is in the same spirit as that of Lemma 2.3. Run a depth-first search on the graph \( G \), starting from an arbitrary root vertex. Recall that this partitions the edges of \( G \) into link edges, which belong to the rooted search tree \( T \), and back edges, which connect a vertex \( v \) with an ascendent of \( v \) in \( T \). Clearly, no back edge is a bridge. A link edge \( e \) with endpoints \( u \) and \( v \), where \( u \) is visited before \( v \), is a non-bridge edge if and only if there exists a back edge from a descendent of \( v \) (maybe \( v \) itself) to an ascendent of \( u \) (maybe \( u \) itself). The algorithm will consider each back edge \((uv)\) in turn and mark as non-bridge the edges on the path from \( u \) to \( v \) in \( T \); the remaining edges are exactly the bridge edges.

To achieve this in linear time, take all back edges \((x_1,y_1),\ldots,(x_k,y_k)\) (where \( y_k \) is an ascendent of \( x_k \)), ordered such that \( y_1,\ldots,y_k \) are discovered in this order during the depth-first search (such an ordering can easily be found in \( O(n) \) time). Starting from \( x_1 \), and walking towards the root of \( T \), mark every edge as being a non-bridge edge until reaching \( y_1 \). Start from \( x_2 \), and walk towards the root of the tree, marking every edge as non-bridge, until either reaching \( y_2 \) or reaching an edge \( e \) that is already marked as non-bridge. If the latter possibility occurs, \( y_1 \) must be an ancestor of \( y_2 \) in \( T \) by the choice of the ordering, so all edges between \( e \) and \( y_2 \) must be already marked. Continue similarly with the other back edges. This process clearly takes linear time in total.

\[Q.E.D.\]

### 5.5 Shortest system of loops

In this section, we describe an algorithm to compute a shortest topological decomposition of the surface. Namely, a system of loops \( L \) is a set of simple loops meeting pairwise only at their common basepoint, such that \( \mathcal{S} \setminus L \) is a disk (refer to Figure 4.6(c) for an example). We give an algorithm to compute the shortest system of loops of a given surface.

#### 5.5.1 Algorithm

Define a homology basis of loops to be a set of loops whose homology classes (of their mod 2 images) form a basis of the homology vector space.

There exist homology bases of loops:

**Exercise 5.18.** Prove that every homology cycle is the mod 2 image of a loop.

Recall that a system of loops \( L \) is a set of simple loops meeting pairwise only at their common basepoint, such that \( \mathcal{S} \setminus L \) is a disk. Denote by \([\ell]\) the homology class of a loop \( \ell \), and by \([L]\) the set of homology classes of a set of loops \( L \).

**Lemma 5.19.** Some shortest homology basis is made of loops of the form \( \sigma(e) \). In particular, the loops in that basis are simple and disjoint.

**Proof.** Let \( \ell \) be a loop in the shortest homology basis. Let \( e_1,\ldots,e_k \) be the edges of the cut locus crossed by \( \ell \). Then it is not too hard, using Exercise 5.12(2), to prove that \([\ell] = [\sigma(e_1)] + \ldots + [\sigma(e_k)]\).

In particular, \( \ell \) crosses at least one edge of the cut locus. Furthermore, since \([\ell]\) is linearly independent from the homology classes of the other loops in the basis, one of the \([\sigma(e_i)]\) must be linearly independent from the homology classes of the other loops in the basis. Replacing \( \ell \) with \( \sigma(e_i) \) still yields a homology basis, which is no longer than the original one because \( \sigma(e_i) \) is a shortest loop with basepoint \( b \) among the loops that cross \( e_i \), and \( \ell \) indeed crosses \( e_i \). Iterating, we obtain that some shortest homology basis is made of loops of the form \( \sigma(e) \).

**Exercise 5.20.** Let \( L \) be a set of simple, disjoint loops in \( G \). Prove that \( L \) disconnects \( \mathcal{S} \) if and only if the homology classes of the loops in \( L \) are linearly dependent.

**Theorem 5.21.** We can compute a shortest homology basis of loops in \( O(gn + n \log n) \) time. Furthermore, there are \( 2g \) loops, each of the form \( \sigma(e) \).

**Proof.** By Lemma 5.19, computing a shortest homology basis of loops boils down to computing a shortest inclusionwise maximal set of loops \( \sigma(e_1),\ldots,\sigma(e_k) \) with linearly independent homology classes, or, equivalently, that does not disconnect \( \mathcal{S} \) (Exercise 5.20). This is equivalent to computing an inclusionwise maximal set \( S \) of edges of \( \mathcal{C} \) such that \( C - S \)
is connected, with minimal sum of weights (Lemma 5.16). This precisely means computing the complement of a maximum-weight spanning tree of $C$.

Recall that $C$ is cellularly embedded on $\mathcal{S}$ with one face (Lemma 5.1). Therefore, by Euler’s formula, the number of vertices, $v$, and edges, $e$, of $C$ satisfy $v - e = 2 - 2g = 1 - 2g$. A spanning tree always contains $v - 1$ edges (Lemma 1.7), so the complement of a spanning tree of $C$ has exactly $2g$ edges; we conclude that there are $2g$ loops in $L$.

Computing the cut locus $C$ takes $O(n \log n)$ time. A maximum spanning tree can be computed in $O(n \log n)$ time using any textbook algorithm. The actual loops may each have $O(n)$ size, and there are $2g$ of these.

**Proposition 5.22.** The shortest homology basis of loops $L$ computed in Theorem 5.21 is actually a shortest system of loops.

**Proof.** Every system of loops is made of $2g$ loops by Euler’s formula. The homology classes of a system of loops are linearly independent (Exercise 5.20), and there are $2g$ of these, so they form a basis. So any system of loops is a homology basis. It therefore suffices to prove that $L$ is a system of loops.

$L$ is a set of $2g$ simple, disjoint loops that does not disconnect $\mathcal{S}$. Cutting along it yields a (connected) surface of Euler characteristic 1 (because cutting along the first loop keeps the Euler characteristic unchanged and cutting along each subsequent loop increases it by one), hence a disk.

**5.6 Extensions**

**5.6.1 Shortest loops on surfaces with boundary**

Let $(\mathcal{S}, M^*)$ be a cross-metric surface with genus $g$ and $b \geq 1$ boundary components, and complexity $n$. We briefly indicate how the results of Sections 5.3 and 5.4 generalize to surfaces with boundary.

Let $\mathcal{F}$ be the surface $\mathcal{S}$ where a handle is attached to each boundary component. Thus $\mathcal{F}$ is a surface without boundary. The graph $M^*$ is not cellularly embedded in $\mathcal{F}$, but we can make it cellular by adding two edges per attached handle. We assign infinite weights to these new edges. Let $M^*$ be the resulting graph.

A loop in $\mathcal{S}$ is contractible if and only if it is contractible in $\mathcal{F}$. Therefore, to compute the shortest non-contractible loop in $(\mathcal{F}, M^*)$ with a given basepoint, it suffices to compute the shortest non-contractible loop in $(\mathcal{S}, M^*)$ with that basepoint. Similarly, a loop in the interior of $\mathcal{S}$ is separating if and only if it is separating in $\mathcal{F}$. So, to compute the shortest non-separating loop in $(\mathcal{S}, M^*)$, it suffices to compute the shortest non-separating loop in $(\mathcal{F}, M^*)$.

There is no direct analog of the notion of system of loops for surfaces with boundary, because no set of disjoint simple loops in the interior of the surface can cut it into a disk. We shall see a replacement of this notion in Section 5.6.3.

**5.6.2 Shortest paths relatively to a set of points**

We now extend the result of Section 5.5 to the case where there are “more than one basepoint”. Specifically, let $(\mathcal{S}, M^*)$ be a cross-metric surface without boundary, with a finite set $P$ of $k$ points, each in a different face of $M^*$. Let us call a $P$-path any path with endpoints in $P$. A **system of $P$-paths** is a set of simple $P$-paths that are pairwise disjoint, except possibly at their endpoints, cutting $\mathcal{S}$ into a topological disk, and meeting every point in $P$; equivalently, it is a graph embedded on $\mathcal{S}$ with vertex set exactly $P$ whose removal leaves a disk. Our goal is to compute a shortest system of $P$-paths.

Let $F$ be a shortest path forest in $M^*$, growing simultaneously from each point of $P$, and connecting every face of $M^*$. The cut locus $C$ of $(\mathcal{S}, M^*)$ with respect to $P$ is defined as the subgraph of $M^*$ obtained by removing all edges of $M^*$ crossed by $F$. It cuts $\mathcal{S}$ into $k$ topological disks, each containing exactly one point in $P$. Given an edge $e$ of the cut locus, the

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3Here, instead of attaching a handle to every boundary component of $\mathcal{S}$, attaching a disk would also work.
arc \( \sigma(e) \) is the concatenation of two shortest paths following the shortest path forest starting on both sides of \( e \) until each of them reaches a root.

The \( 3 \)-path condition generalizes as follows to \( P \)-paths: a set \( S \) of \( P \)-paths satisfies this condition if and only if, for every point \( a \) and any three paths \( p, q, \) and \( r \) from points in \( P \) to \( a \), if \( p \cdot q \) and \( q \cdot r \) belong to \( S \), then so does \( p \cdot r \).

Homology is defined almost as in Section 5.4.2: we assume \( \sigma \) satisfies this condition if and only if, for every point \( p \), \( q \), and \( r \) from points in \( P \) to \( a \), if \( p \cdot q \) and \( q \cdot r \) belong to \( S \), then so does \( p \cdot r \).

Then as in Section 5.4.2, we have:

**Theorem 5.23.** Let \( (\mathcal{S}, M^*) \) be a cross-metric surface with genus \( g \) and without boundary; let \( n \) be its complexity. Let \( P \) be a set of \( k \) points on \( \mathcal{S} \). Finding a shortest homology basis of \( P \)-paths can be done in \( O(n \log n + (g+k)n) \) time. The basis computed is actually a shortest system of \( P \)-paths. The number of \( P \)-paths in every system of \( P \)-paths, and every homology basis of \( P \)-paths, is \( 2g + k - 1 \).

### 5.6.3 Shortest arcs on a surface with boundary

We come back to the case of surfaces with boundary: Assume \( (\mathcal{S}, M^*) \) has \( b \geq 1 \) boundary components. A **system of arcs** on \( \mathcal{S} \) is a set of disjoint simple arcs cutting \( \mathcal{S} \) into a disk (this is an appropriate analog of a system of loops for surfaces with boundary).

Let \( \mathcal{S} \) be the cross-metric surface without boundary obtained by attaching disks to each boundary component of \( \mathcal{S} \). Let \( M^* \) be the graph \( M^* \) where all edges on the boundary of \( \mathcal{S} \) are assigned a fixed large enough crossing weight \( W \); thus \((\mathcal{S}, M^*)\) is a cross-metric surface without boundary. Furthermore, let \( P \) be a set of points, one inside every disk glued to the boundaries of \( \mathcal{S} \).

Call a \( P \)-path on \( \mathcal{S} \) **admissible** if it intersects the boundary of \( \mathcal{S} \) in exactly two points. Admissible \( P \)-paths in \( \mathcal{S} \) precisely correspond to arcs in \( \mathcal{S} \); this correspondence preserves the lengths, up to a shift of \( 2W \). Furthermore, a system of admissible \( P \)-paths in \( \mathcal{S} \) corresponds to a system of arcs in \( \mathcal{S} \). Since the algorithm of Theorem 5.23 only computes admissible \( P \)-paths (provided \( W \) is chosen large enough), the above considerations yield:

**Theorem 5.24.** Let \( (\mathcal{S}, M^*) \) be a cross-metric surface with genus \( g \) and \( b \geq 1 \) boundary components, and complexity \( n \). Finding a system of arcs can be done in \( O(n \log n + (g+b)n) \) time. Every system of arcs is made of \( 2g + b - 1 \) arcs.

### 5.7 Notes

#### 5.7.1 Discrete vs. continuous setting

Most of the combinatorial and cross-metric surface model is taken from Colin de Verdière and Erickson [18]. Several tools of this section were described in a combinatorial setting for simplicity of exposition, but they have well-studied continuous counterparts.

In general, the cut locus of a point \( x \) in a metric space \( S \) is the set of points in \( S \) for which there exist at least two distinct shortest paths to \( x \). It is closely related to the notion of the **medial axis** of a compact set \( K \subset \mathcal{S} \): it is the set of points of \( S \setminus K \) whose distance to \( K \) is realized by at least two points of \( K \). If \( K \) is finite, the medial axis contains in particular the Voronoi diagram of \( K \).

The main topological property of a cut locus we have used (in Lemmas 5.8 and 5.16) can be stated as follows for a surface with boundary: for any subset \( A \) of the edges of \( C, \mathcal{S} \setminus \sigma(A) \) deformation retracts to \( C \setminus A \). In particular, they have the same number of connected components, and one of the components of \( \mathcal{S} \setminus \sigma(A) \) is a disk if and only if the corresponding component of \( C \setminus A \) (is connected and) contains no non-contractible loop, i.e., is a tree.

As mentioned earlier, homology can be defined in a continuous setting (**singular homology**), which vastly generalizes the ad-hoc route we took. Let \( S \) be any topological space. Let \( \Delta_n \) be the \( n \)-dimensional simplex. The set of \( n \)-chains \( C_n \)
Chapter 6

Homotopy and universal cover

In this chapter, we introduce two important tools related to surfaces. The notion of homotopy captures the intuitive notion of deformation. The universal cover of a surface provides a way to compute paths restricted to a given homotopy class, i.e., up to deformation.

In this section, \( \mathcal{S} \) is a compact, connected, orientable surface, although the definition would apply to almost arbitrary topological spaces.

6.1 Homotopy

6.1.1 Definition

Two paths \( p \) and \( q \) on \( \mathcal{S} \), having both \( u \) and \( v \) as endpoints, are homotopic if there exists a continuous family of paths whose endpoints are \( u \) and \( v \) between \( p \) and \( q \). More formally, a homotopy between \( p \) and \( q \) is a continuous map \( h : [0,1] \times [0,1] \to \mathcal{S} \) such that \( h(0, \cdot) = p \), \( h(1, \cdot) = q \), \( h(\cdot,0) = u \), and \( h(\cdot,1) = v \). This definition applies in particular to the case of loops \( (u=v) \).

6.1.2 Fundamental group

Let \( b \) be a point of \( \mathcal{S} \). The relation “is homotopic to” partitions the set of loops with basepoint \( b \) into homotopy classes. Let us denote by \([\ell]\) the homotopy class of a loop \( \ell \). The set of homotopy classes can be
equipped with the law "·" defined by $[[\ell]] \cdot [[\ell']] = [[\ell \cdot \ell']]$, and, with this law, the set of homotopy classes of loops with basepoint $b$ is a group, called the fundamental group of $(\mathcal{S}, b)$ and denoted by $\pi_1(\mathcal{S}, b)$ or more concisely $\pi_1(\mathcal{S})$, whose unit element is the class of contractible loops.

In particular, the fundamental group of the disk or the sphere is trivial: two paths having the same endpoints are homotopic. The fundamental group of the annulus is $\mathbb{Z}$ (the homotopy class of a loop is the same as the signed number of times it "winds around the hole"), and the fundamental group of the torus is $\mathbb{Z}^2$.

### 6.2 Universal cover

Informally, the universal cover $\tilde{\mathcal{S}}$ of a surface $\mathcal{S}$ is a surface $\tilde{\mathcal{S}}$ which "locally looks like $\mathcal{S}$", but is "much larger than $\mathcal{S}$": it is not compact (except in trivial cases) and every point in $\mathcal{S}$ generally corresponds ("lifts") to infinitely many points in $\tilde{\mathcal{S}}$; two paths are homotopic in $\mathcal{S}$ if and only if these paths can be lifted to paths which have the same endpoints in $\tilde{\mathcal{S}}$. The universal cover is thus a tool to compute homotopy.

#### 6.2.1 Examples

Let $\mathcal{S}$ be the annulus depicted on Figure 6.1(a). If this annulus is cut along the dashed line segment, we obtain a rectangle; if we glue together infinitely many copies of this rectangle, we obtain an "infinite strip", depicted on Figure 6.1(b), which will be denoted by $\tilde{\mathcal{S}}$. There is a natural "projection" $\pi$ from $\tilde{\mathcal{S}}$ onto $\mathcal{S}$, such that a path in $\mathcal{S}$ can be lifted to a path (in fact, infinitely many paths) in $\tilde{\mathcal{S}}$. We see that two paths $p$ and $p'$ are homotopic in $\mathcal{S}$ if two lifts of $p$ and $p'$ starting at the same point of $\tilde{\mathcal{S}}$ have the same targets. The two loops represented on the figure are not homotopic, because one of them is contractible (its lifts in $\tilde{\mathcal{S}}$ are loops), and the other one is non-contractible (its lifts are not closed).

The same kind of figure can be drawn for the torus (Figure 6.2(a)). If this torus $\mathcal{S}$ is viewed as a polygonal schema in canonical form (Figure 6.2(b)), a square whose opposite sides will be identified to obtain $\mathcal{S}$, its universal cover consists of infinitely many copies of this copy organized in a grid-like fashion: hence, it is the plane (Figure 6.2(c)).

#### 6.2.2 Definition and properties

Precisely, a universal cover of a connected surface $\mathcal{S}$ is the data of a pair $(\tilde{\mathcal{S}}, \pi)$, where:

- $\tilde{\mathcal{S}}$ is a (possibly non-compact) surface which is simply connected,
i.e., every loop in \( \tilde{\mathcal{S}} \) is contractible;
- \( \pi \) is a continuous map from \( \tilde{\mathcal{S}} \) onto \( \mathcal{S} \), called projection, which is a local homeomorphism: any point \( x \) of \( \mathcal{S} \) has an open, connected neighborhood \( U \) such that \( \pi^{-1}(U) \) is a disjoint union of open sets \((U_i)_{i \in I}\) and \( \pi|_{U_i} : U_i \to U \) is a homeomorphism.

Every connected surface (possibly with boundary) has a universal cover (we will provide constructions in the following sections). On the other hand, two universal covers are isomorphic (that is, there is a homeomorphism between them that “projects” to the identity map). This allows to speak without ambiguity of the universal cover of a surface \( \mathcal{S} \).

A lift of a path \( p \) is a path \( \tilde{p} \) in \( \tilde{\mathcal{S}} \) such that \( \pi \circ \tilde{p} = p \).

The main properties of \((\tilde{\mathcal{S}}, \pi)\) that we will use are:
- the lift property: let \( p \) be a path in \( \mathcal{S} \) whose source is \( y \); let \( x \in \pi^{-1}(y) \). Then there exists a unique path \( \tilde{p} \) in \( \tilde{\mathcal{S}} \), whose source is \( x \), such that \( \pi \circ \tilde{p} = p \);
- the homotopy property: two paths \( p_1 \) and \( p_2 \) with the same endpoints are homotopic in \( \mathcal{S} \) if and only if they have two lifts \( \tilde{p}_1 \) and \( \tilde{p}_2 \) sharing the same endpoints in \( \tilde{\mathcal{S}} \);
- the intersection property: a path \( p \) in \( \mathcal{S} \) self-intersects if and only if either a lift of \( p \) self-intersects, or two lifts of \( p \) intersect.

### 6.2.3 A construction of the universal cover

Let \( \mathcal{S} \) be a surface with boundary with a cellular graph embedding \( G \) on it (for example, \( \mathcal{S} \) is a cross-metric surface). For convenience, we sketch the construction of \( \tilde{\mathcal{S}} \) in the case where all vertices of \( G \) belong to the boundary of \( \mathcal{S} \). This latter restriction is not severe, as every surface with boundary has a cellular embedding where all vertices are on the boundary. However, the construction below does not apply to surfaces without boundary (you will have to wait until the next chapter).

The algorithm maintains a portion \( P \) of the universal cover of \( \mathcal{S} \) built so far, which is a topological disk. \( P \) is made of copies of faces of \( G \) (in other words, the image, by \( \pi \), of each point of \( P \) is known). The edges of \( P \) can be of two types: there are active edges, beyond which the construction of the universal cover needs to be proceeded, and inactive edges, which are already incident to two copies of faces of \( G \) in \( P \) (or to one copy, if they project to the boundary of \( \mathcal{S} \)). Initially, \( P \) consists of a copy of one single face of \( G \), and all the edges of this copy are active, except the edges which project to the boundary of \( \mathcal{S} \). The following process is iterated:

- let \( p \) be a copy of a face of \( G \) in \( P \) with an active edge \( a \);
- let \( p' \) be a copy of the face of \( G \) adjacent to \( \pi(p) \) through \( \pi(a) \);
- we glue \( p' \) to \( p \) via the edge \( a \);
- the edges of \( p' \) which do not project to the boundary of the surface are made active, except \( a \), which is made inactive.

Figure 6.3 presents an example of this construction. This process is, of course, infinite (except in the case of the sphere or the disk). It is clear that the space built by the algorithm is simply connected, because the dual graph of its vertex-edge graph is a tree.

![Figure 6.3: The construction of the universal cover of a surface with boundary](image)
Figure 6.2 shows a construction of the universal cover of the torus. In the next chapter, we will see how the universal cover of higher-genus surfaces without boundary looks like.

6.3 Notes

Homotopy is a very natural and “geometric” notion (compared with homology, for example). However, homology has more algebraic structure and is therefore more tractable. Homotopy problems are generally hard: determining whether a loop is contractible is undecidable in innocent-looking spaces such as two-dimensional simplicial complexes and four-dimensional manifolds. The case of manifolds of dimension three is related to the Poincaré conjecture, solved only recently [65,66]. Massey [61, Chapter 5] contains details on the construction of the universal cover of more general topological spaces. The construction of the universal cover for surfaces with boundary is also described by Hershberger and Snoeyink [46].

Chapter 7

Tightening paths on surfaces

In this chapter, we consider the following problem. Let \((\mathcal{S}, M)\) be a combinatorial surface (compact, connected, orientable, possibly with boundary). Let \(p\) be a path on \((\mathcal{S}, M)\) represented as a walk in the graph \(M\). How efficiently can we compute the shortest path homotopic to \(p\)?

We say that a path is tight if it is as short as possible among all homotopic paths. We are therefore looking for a tight path homotopic to \(p\). The problem is formulated entirely in terms of combinatorial surfaces; however, the algorithm crucially relies on cross-metric surfaces and its ability to encode crossings between curves.

Of course, the answer is obvious in cases where the surface is topologically trivial: in the sphere or the disk, any two paths with the same endpoints are homotopic, so the answer is just any shortest path between the endpoints of the input path. We henceforth assume that we are not in such cases.

The problem can be reformulated as computing a shortest path in \(\tilde{\mathcal{S}}\) between the endpoints of a lift \(\tilde{p}\) of \(p\). One difficulty is that \(\tilde{\mathcal{S}}\) is infinite (except precisely for the sphere and the disk). The general approach to solve this problem is as follows:

1. compute a lift \(\tilde{p}\) of the input path;
2. build a suitable part \(P\) of the universal cover of \(\mathcal{S}\), containing the endpoints of \(\tilde{p}\);
3. compute the shortest path \(\tilde{p}'\) in \(P\) between the endpoints of \(\tilde{p}\).

Of course, \(P\) has to be “large enough” to contain the shortest path in \(\tilde{\mathcal{S}}\) between the endpoints of \(\tilde{p}\), but “small enough” to allow a good complexity.
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7. Tightening paths on surfaces

This is done with the help of a preprocessing step: initially, we compute a certain set of curves on the surface that splits it into disks, such that \( \tilde{p} \) can cross a lift of a curve only if \( \tilde{p} \) does. For this purpose, we will see that the decomposition needs to consist of tight curves. It turns out that tightening paths on surfaces is much easier for surfaces with boundary, because there exists a shortest system of arcs (Section 5.6), so we first focus on this case.

A cycle on a surface \( \mathcal{S} \) is a continuous map from the unit circle \( \mathbb{S}^1 \) to \( \mathcal{S} \). It differs from the concept of loop in the sense that a loop is a closed curve with a distinguished basepoint; in contrast, the term cycle is used when no point is to be distinguished.

7.1 Surfaces with boundary

In this section, \( (\mathcal{S},M) \) is assumed to have at least one boundary component. We will prove:

**Theorem 7.1.** Assume \( \mathcal{S} \) has genus \( g \), \( b \geq 1 \) boundary components, and complexity \( n \). Let \( p \) be a path in \( (\mathcal{S},M) \) of complexity \( k \).

After a preprocessing step on \( (\mathcal{S},M) \) in \( O((g+b)n + n \log n) \) time, we can compute a shortest path homotopic to \( p \) in \( O((g+b)nk) \) time.

We can define a cross-metric surface with boundary, \( (\mathcal{S},M^*) \), as follows (Figure 7.1). Glue a disk to each boundary component of \( \mathcal{S} \), obtaining a surface without boundary \( \bar{\mathcal{S}} \). The graph \( M \) is cellularly embedded on \( \bar{\mathcal{S}} \); let \( M^* \) be the dual of the primal graph \( M \) on \( (\bar{\mathcal{S}},M) \); remove the disks we glued to the boundary components, and the corresponding vertices and pieces of edges of \( M^* \); furthermore, add the mandatory boundary edges of infinite weight. To tighten a path on \( (\mathcal{S},M) \), it suffices to tighten it on \( (\mathcal{S},M^*) \), because transforming the resulting path to a walk on \( M \) is trivial.

The universal cover \( \tilde{\mathcal{S}} \) is naturally a cross-metric surface: simply lift the graph \( M^* \) in \( \tilde{\mathcal{S}} \), with the same weights.

**Figure 7.1.** Primal (solid) and dual (dashed) graphs on a combinatorial annulus.

7.1.1 Tightness of the shortest system of arcs

Let \( A \) be the shortest system of arcs of \( (\mathcal{S},M^*) \). Let \( D = \mathcal{S} \setminus A \) be the disk obtained by cutting \( \mathcal{S} \) along \( A \). The construction of Section 6.2.3 applies: we may build any finite portion of the universal cover of \( \mathcal{S} \) by gluing copies of \( D \) together (forgetting about the internal vertices of \( D \), and then later re-creating the internal structure of \( D \)).

**Proposition 7.2.** Every arc in \( A \) is tight.

We give two different proofs of this result. The first one is short and natural; the second one, while more pedestrian, makes use of interesting techniques.

7.1.1.1 Homotopic arcs are homologous

Proposition 7.2 follows directly from the following important lemma, since the shortest system of arcs is made of shortest loops within their respective homology classes:
Lemma 7.3. Two homotopic arcs on a surface are homologous.

Proof. As usual when using homology, we assume the arcs, \( a \) and \( b \), are walks in some refined graph \( G = (V, E) \). Lifting them to the universal cover, we obtain lifts \( \tilde{a} \) and \( \tilde{b} \) with the same endpoints. We may as well lift \( G \), obtaining an infinite graph \( \tilde{G} = (\tilde{V}, \tilde{E}) \). Let \( \tilde{E}^c \subseteq \tilde{E} \) be the mod 2 image of \( \tilde{a} \) and \( \tilde{b} \).

By similar arguments as in Exercise 5.12(2), one can label the faces of the graph \( \tilde{G} \) with 0s and 1s so that an edge in \( \tilde{E} \) belongs to \( \tilde{E}^c \) if and only if it is incident to faces with different labels.

In particular all faces of \( \tilde{G} \) corresponding to a given face of \( (\tilde{V}, \tilde{E}^c) \) have the same label. We may actually assume, up to exchanging the 0s and the 1s, that every infinite face of \( (\tilde{V}, \tilde{E}^c) \) is labeled 0, because every boundary component of \( \mathcal{S} \) contains an even number of points of \( a \) and \( b \). In particular, finitely many faces of \( \tilde{G} \) are labeled 1.

Now, project this labeling onto \( \mathcal{S} \): every face of \( G \) gets as label the mod 2 sum of the labels of its lifts. (The sum is finite, by the preceding paragraph.) \( E^c \) is precisely the set of edges incident with faces with different labels, and is the mod 2 image of \( a \) and \( b \); this proves that the two arcs are homologous. \( \square \)

As a side remark, the same arguments show that two homotopic loops or cycles are homologous.

7.1.1.2 Crossing words

Let \( A \) be a family of arcs on \((\mathcal{S}, M^*)\), and \( c \) be a path in \((\mathcal{S}, M^*)\). Let \( X \) be the set of letters of the form \( a \) or \( \bar{a} \), where \( a \in A \). Walk along \( c \) and, each time we encounter a crossing with an arc \( a \in A \), write the letter \( a \) or \( \bar{a} \), according to the orientation of the crossing. The resulting word is called the crossing word of \( c \) with \( A \), denoted by \( A/c \). A word is parenthesized if it reduces to the empty word by successive removals of consecutive pairs of letters of the form \( a\bar{a} \) or \( \bar{a}a \).

![Figure 7.2. A step of the proof of Lemma 7.4.](image)

Lemma 7.4. Assume every arc \( a \in A \) separates the surface into two connected components, and that the arcs in \( A \) are pairwise disjoint. Let \( \ell \) be a loop. Then the crossing word \( A/\ell \) is parenthesized.

Proof. We prove the result by induction on the number of crossings between \( \ell \) and the arcs in \( A \). The result is trivial if \( \ell \) does not cross any arc in \( A \). Hence, let us assume that there is at least one crossing between \( \ell \) and an arc \( a \in A \).

As \( \ell \) is a loop, it must cross \( a \) at least once with the opposite orientation. Consider now \( \ell \) as a cycle \( \gamma \) (i.e., let us forget the basepoint of \( \ell \)). The two crossings split the cycle \( \gamma \) into two paths \( c_1 \) and \( c_2 \) (Figure 7.2). For \( k = 1, 2 \), it is possible to extend \( c_k \) to a closed path \( c'_k \), so that \( A/c'_k = A/c_k \), by adding a piece of a path which goes along a part of \( a \). By the induction hypothesis, \( A/c_1 \) and \( A/c_2 \) are parenthesized. In addition, \( A/c \) equals, up to cyclic permutation,

\[ a \cdot (A/c_1) \cdot \bar{a} \cdot (A/c_2) \text{ or } \bar{a} \cdot (A/c_1) \cdot a \cdot (A/c_2). \]

Therefore, \( A/c \) is also parenthesized. \( \square \)

Proof of Proposition 7.2. Let \( a \) be a slightly translated copy of an arc in \( A \), disjoint from all arcs in \( A \); let \( c \) be a shortest arc homotopic to \( a \). These arcs \( a \) and \( c \) lift, in \( \mathcal{S} \), to two arcs \( \tilde{a} \) and \( \tilde{c} \) with the same endpoints. We will move \( \tilde{c} \) into \( \tilde{a} \) without increasing its length, which will prove the result. Let \( \tilde{A} \) be the set of all lifts of the arcs in \( A \).
homotopy class. This removes two crossings; we conclude by induction.

Therefore, the crossing word \( w := A/(\tilde{c}\cdot \tilde{a}) \) is parenthesized by Lemma 7.4, and it equals \( \tilde{A}/\tilde{c} \) since \( a \) does not cross \( A \). If \( w \) is the empty word, then \( a \) and \( c \) both live in \( D \). We may therefore assume \( c \) is simple (removing its loops cannot make it longer). Then, if we replace \( a \) by \( c \), we also get a system of arcs, so \( c \) cannot be shorter than \( a \). This indeed means that \( a \) is tight.

If \( w \) is non-empty, then \( \tilde{c} \) crosses consecutively the same lift \( \tilde{d} \) of an arc in \( A \), with opposite orientations (Figure 7.3); let \( \tilde{c}_1 \) and \( \tilde{d}_1 \) be the corresponding subpaths of these lifts between the crossing points. Since \( \tilde{c} \) is a shortest path, we may assume that \( \tilde{c}_1 \) is simple. Replacing, in \( \tilde{d} \), the subpath \( \tilde{d}_1 \) by \( \tilde{c}_1 \) would give a system of arcs, so \( \tilde{c}_1 \) cannot be shorter than \( \tilde{d}_1 \). So we may replace, in \( c \), its subpath \( c_1 \) by \( d_1 \) without changing its length or its homotopy class. This removes two crossings; we conclude by induction. \( \square \)

**Exercise 7.5.** Prove Lemma 5.4 in the case of cross-metric surfaces with boundary, using Lemma 7.4 and techniques similar as those in the previous proof.

### 7.1.2 Algorithm

We now describe an algorithm to tighten a path \( p \) in the combinatorial surface \( (\mathcal{S}, M) \). We start by computing the shortest system of arcs \( A \) on \( (\mathcal{S}, M^+) \).

Any finite portion of the universal cover of \( \mathcal{S} \) can be built using \( A \), as in Section 6.2.3. In particular, we may compute a lift \( \tilde{p} \) of the input path \( p \), together with the corresponding part \( P \) of the universal cover traversed by \( \tilde{p} \); start with a copy of the disk \( D := \mathcal{S}\setminus A \); follow \( p \) in \( D \) until it exits the currently built portion \( P \) of the universal cover, and add to this portion a copy of \( D \) necessary to prolongate the lift \( \tilde{p} \) of \( p \); iterate.

Let \( \tilde{q} \) be the shortest path in \( P \) with the same endpoints as \( \tilde{p} \). The algorithm returns \( q \), the projection of \( \tilde{q} \) onto \( \mathcal{S} \).

To prove the correctness of the algorithm, it suffices to prove that \( \tilde{q} \) is a shortest path in the universal cover \( \tilde{\mathcal{S}} \), not only in \( P \). Let \( \tilde{r} \) be the shortest path in \( \tilde{\mathcal{S}} \) with the same endpoints as \( \tilde{p} \); we will move \( \tilde{r} \) inside \( P \) without increasing its length, which concludes.

Since \( \tilde{p} \) and \( \tilde{r} \) have the same endpoints, they cross a lift \( \tilde{a} \) of an arc in \( A \) an odd number of times if and only if their endpoints are in distinct components of \( \tilde{\mathcal{S}}\setminus \tilde{a} \). On the other hand, without loss of generality, we may assume that \( \tilde{r} \) crosses every lift of \( A \) at most once: for if it crosses a given lift \( \tilde{a} \) twice, then we may shortcut \( \tilde{r} \) as in Figure 7.3; this does not make it longer, since \( \tilde{a} \) is a shortest path. So \( \tilde{r} \) crosses only lifts of \( A \) crossed by \( \tilde{p} \), and therefore remains in \( P \).

### 7.1.3 Complexity analysis and refinement

It remains to bound the complexity. For this purpose, we need to bound the number of times the input path \( p \) crosses the shortest system of arcs. This is most efficiently done using the following modification of the algorithm: Instead of considering the cross-metric surface \( (\mathcal{S}, M^+) \), we consider the cross-metric surface \( (\mathcal{S}, M^+) \), where \( M^+ \) is a weighted graph obtained by overlaying \( M^+ \) with its primal graph \( M \) (Figure 7.1).

The vertices of \( M^+ \) are either vertices of \( M \), vertices of \( M^+ \), or intersections between an edge \( e \) of \( M \) and its dual edge \( e^* \) of \( M^+ \). Each edge of \( M \) and dual edge in \( M^+ \) is partitioned into two edges in \( M^+ \). Finally, each face of \( M^+ \) is a quadrilateral. The complexity of \( M^+ \) is asymptotically the same as \( M^+ \).

To treat \( (\mathcal{S}, M^+) \) as a cross-metric surface, we assign a crossing weight to each edge \( e^+ \) of \( M^+ \) as follows. If \( e^+ \) is on the boundary of \( \mathcal{S} \), it has
infinite crossing weight. Otherwise, if \( e^+ \) is contained in an edge of \( M^* \), it has the same crossing weight as that dual edge; otherwise, its crossing weight is made infinitesimally small.\(^1\)

We actually compute the shortest system of arcs \( A \) in this cross-metric surface \((\mathcal{S}, M^+)\). By construction, every arc in \( A \) is of the form \( \sigma(e) \), and therefore crosses a given edge of \( M^+ \) at most twice. Note that the input path \( p \) is a walk in the graph \( M \), so every edge of \( p \) crosses every arc of \( A \) at most four times. So there are \( O((g+b)k) \) crossings between \( p \) and \( A \).

Each of the \( O(g+b) \) arcs of \( A \) has complexity \( O(n) \), so the disk \( D \) has complexity \( O((g+b)n) \). Furthermore, the preceding paragraph implies that \( \tilde{p} \) visits \( O((g+b)k) \) copies of \( D \). The portion \( P \) of the universal cover we thus need to explore has complexity \( O((g+b)^2nk) \). The algorithm computes a shortest path in this space, and this takes linear time since \( P \) is planar. This completes the proof of Theorem 7.1, up to an \( O(g+b) \) factor in the complexity, which is removed in the following exercise.

**Exercise 7.6.** ♣ Let \((\mathcal{S}, M)\) be a combinatorial surface (compact, connected, orientable), with genus \( g \) and \( b \geq 1 \) boundary components, and complexity \( n \). Let \((\mathcal{S}, M^*)\) be the associated cross-metric surface.

1. Prove that one can, in \( O((g+b)n+n \log n) \), compute a tight \textit{triangulated system of arcs} \( A \): a set of pairwise disjoint simple arcs such that every face of \( A \) is incident with exactly three arcs in \( A \). Indication: extend the shortest system of arcs.

2. Show how to reduce the time complexity of Theorem 7.1 from \( O((g+b)^2nk) \) to \( O((g+b)nk) \).

### 7.2 Surfaces without boundary

In this section, we explain how to tighten paths on surfaces without boundary. This is much more involved and technical, so we mostly sketch the main ideas, referring to the research paper [18] for details. We assume that \( \mathcal{S} \) has genus at least two and no boundary; the case \( g = 1 \) is indeed exceptional (but the result is similar).

**Theorem 7.7.** Assume \( \mathcal{S} \) is a surface with genus \( g \geq 2 \), no boundary, and complexity \( n \). Let \( p \) be a path in \((\mathcal{S}, M)\) of complexity \( k \). After a preprocessing step on \((\mathcal{S}, M)\) in \( O(gn+n \log n) \) time, we can compute a shortest path homotopic to \( p \) in \( O(gnk) \) time.

The main tool is an \textit{octagonal decomposition}: an arrangement of simple cycles in which every vertex has degree four and every face has eight sides. See Figure 7.4. If we lift the cycles of an octagonal decomposition to the universal cover of the surface, we obtain a tiling of the unit open disk with octagons, where each vertex has degree four; see Figure 7.5.

**Exercise 7.8.** ♣ Let \( \mathcal{O} \) be an octagonal decomposition of a surface \( \mathcal{S} \) with genus \( g \geq 1 \). Compute the number of octagons of \( \mathcal{O} \). Deduce that no octagonal decomposition exists for the torus. What kind of decomposition could be used instead?

The \textit{multiplicity} of a set of curves on a cross-metric surface \((\mathcal{S}, M^*)\) is the maximal number of times the union of the curves crosses a given edge of \( M^* \).

**Proposition 7.9.** Let \((\mathcal{S}, M^*)\) be a cross-metric surface with complexity \( n \), genus \( g \geq 2 \), and no boundary. In \( O(n^2 \log n) \) time, we can construct a tight octagonal decomposition of \((\mathcal{S}, M^*)\) in which each cycle has multiplicity \( O(1) \).

\(^1\)Equivalently, we can take the crossing weight of any edge in \( M^* \) to be a vector \((\ell, 0)\) for some non-negative real number \( \ell \), and the crossing weight of any edge in \( M \) to be \((0,1)\). Crossing weights are now vectors, which are added normally and compared lexicographically.
We will admit this result. The main steps of the construction are shown in Figure 7.6; it uses shortest non-contractible or non-separating arcs or cycles on the surface (Chapter 5), ultimately relying on Dijkstra’s algorithm and on the min-cut algorithm for planar graphs (Theorem 3.10). A tight pants decomposition (a decomposition of the surface into spheres with three boundary components) is built, and then cycles are computed within neighboring pairs of pants. The main difficulties are (1) to prove that every output cycle is tight, which essentially holds by uncrossing arguments similar to those of the previous section, and (2) to bound the multiplicity of each cycle.

### 7.2.1 The universal cover

Let $O$ be a tight octagonal decomposition of a surface $S$ without boundary. We call any lift of a cycle in $O$ to the universal cover $\tilde{S}$ a line; see Figure 7.5. The set of lines is denoted by $\tilde{O}$.

**Lemma 7.10** (Dehn [22]). Let $S$ be the non-empty union of finitely many octagons in the four-valent octagon tiling of the unit disk. Some octagon in $S$ has at least five consecutive sides on the boundary of $S$. 

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**Figure 7.5.** Universal cover of an octagonal decomposition.

**Figure 7.6.** The construction of the octagonal decomposition.
Sketch of proof. See Stillwell [73, p. 188]. Intuitively, take one octagon in $S$ as far as possible from the center of the disk in Figure 7.5. It has at least five octagons adjacent to that octagon that are even further from the center, hence not in $S$.

A trivial but important corollary is:

**Corollary 7.11.** Let $S$ be the non-empty union of finitely many octagons in the four-valent octagon tiling of the unit disk. Then at least five distinct lines contain edges on the boundary of $S$. In particular, two lines in the tiling cross at most once.

The perimeter of a set of octagons is the number of edges on its boundary.

**Lemma 7.12.** Any union of $N$ octagons, $1 \leq N < \infty$, in the four-valent octagon tiling of the unit disk has perimeter at least $2N + 6$.

**Proof.** Removing an octagon with at least five consecutive sides on the boundary of the union (Lemma 7.10) reduces the perimeter by at least two. The base case is a single octagon.

A path $p$ wraps around a cycle $\gamma$ if $p(t) = \gamma((ut + v) \mod 1)$ for some real numbers $u$ and $v$. A cycle $\delta$ wraps around $\gamma$ if $\delta(t) = \gamma(m \cdot t)$, for some integer $m$; one also says that $\delta$ is the $m$th power of $\gamma$. Observe that a subpath of a cycle wraps around it, but that the converse is not true (a path wrapping around the cycle “several times” is not a subpath). We will need the following result, which we admit [18]:

**Proposition 7.13.** Any path or cycle on $\mathcal{S}$ that wraps around a tight cycle is tight.

**Lemma 7.14.** Let $\tilde{p}$ be a path in $\tilde{\mathcal{S}}$, with endpoints $x$ and $y$; let $L$ be the set of lines in $\tilde{\mathcal{O}}$ crossed an odd number of times by $\tilde{p}$. Let $\tilde{p}'$ be a shortest path with endpoints $x$ and $y$, where the lines in $\tilde{\mathcal{O}}$ are assigned infinitesimal crossing weight. Then $\tilde{p}'$ crosses exactly once each line in $L$ and no other line.

**Proof.** Since each line $\ell$ in $\tilde{\mathcal{O}}$ separates $\tilde{\mathcal{T}}$, a path connecting $x$ and $y$ crosses $\ell$ an odd number of times if and only if $\ell$ separates $x$ and $y$. Hence it suffices to prove that $\tilde{p}'$ crosses each line at most once.

Assume that $\tilde{p}'$ crosses some line $\ell$ at least twice, at points $u$ and $v$. Since $\ell$ is a lift of a tight cycle, every subpath of $\ell$ is a shortest path, by Proposition 7.13. Thus, we can remove the two crossings from $\tilde{p}'$ by replacing the subpath from $u$ to $v$ with the shortest path in $\ell$. Since any pair of lines in $\tilde{\mathcal{O}}$ intersect at most once (Corollary 7.11), this exchange results in a path with fewer line crossings (and possibly shorter length), which is impossible.

### 7.2.2 Building the relevant region

Consider an arbitrary path $p$ on a surface $\mathcal{S}$. Let $\tilde{p}$ be a lift of $p$ to the universal cover $\tilde{\mathcal{S}}$, and let $\tilde{p}'$ be a shortest path in $\tilde{\mathcal{S}}$ between the endpoints of $\tilde{p}$. Projecting $\tilde{p}'$ back down to $\mathcal{S}$ gives us a shortest path homotopic to $p$. The algorithm exploits this characterization by constructing a subset of $\tilde{\mathcal{S}}$ of small complexity that contains both $\tilde{p}$ and some shortest path $\tilde{p}'$.

Let $\mathcal{O}$ be the tight octagonal decomposition of $\mathcal{S}$ (Proposition 7.9). Consider a path $p$ in $\mathcal{S}$, and let $\tilde{p}$ be a lift of $p$ to the universal cover $\tilde{\mathcal{S}}$. For any line $\ell$ in $\tilde{\mathcal{O}}$, let $\ell^+$ denote the component of $\tilde{\mathcal{S}} \setminus \ell$ that contains the starting point $\tilde{p}(0)$.

Let $\ell_1, \ell_2, \ldots, \ell_z$ be the sequence of lines in $\tilde{\mathcal{O}}$ crossed by $\tilde{p}$, in order of their first crossing. Let $\mathcal{L}_0 = \emptyset$, and for any integer $i$ between 1 and $z$, let $\mathcal{L}_i = \mathcal{L}_{i-1} \cup \{\ell_i\}$. For each $i$, let $\tilde{\mathcal{S}}_i$ be the subset of $\tilde{\mathcal{S}}$ reachable from $\tilde{p}(0)$ by crossing only (a subset of) lines in $\mathcal{L}_i$, in any order. Combinatorially, the region $\tilde{\mathcal{S}}_i$ is a ‘convex polygon’ formed by intersecting the ‘half-disk’ $\ell_i^+$ for all lines $\ell$ not in the set $\mathcal{L}_i$. By Lemma 7.14, some shortest path $\tilde{p}'$ between the endpoints of $\tilde{p}$ crosses only a subset of the lines that $\tilde{p}$ crosses; so $\tilde{p}'$ is contained in $\tilde{\mathcal{S}}_z$. For this reason, $\tilde{\mathcal{S}}_z$ is called the relevant region of $\tilde{\mathcal{S}}$ (with respect to $\tilde{p}$).

**Lemma 7.15.** For any line $\ell$ and any $i \geq 0$, $\ell \cap \tilde{\mathcal{S}}_i$ is either empty or connected.
Lemma 7.16. Let $\ell_i$ be in the set $T$. To prove the lemma, it suffices to show that none of the lines bounding $\ell_i$ intersect an inner line $\ell' \neq \ell$ of $O'_j$. Thus the lines $\ell$, $\ell'$, and $\ell_i$ would pairwise intersect, and these three lines would bound a disk in the tiling $\tilde{O}$, contradicting Lemma 7.10.

Hence, since every line in $L$ has a point in $\ell_1^+$, no outer line can be in $L$. Only the first and last inner lines contribute a side to the boundary of $\tilde{F}$. Neither of these two lines is in $L$, for otherwise one of the starred octagons in Figure 7.7 would also belong to $\tilde{F}_{i-1}$.

**Lemma 7.17.** $\tilde{F}_z$ contains at most $7z + 1$ octagons.

**Proof.** Let $v$ be a vertex on the boundary of $\tilde{F}_z$. Depending on whether zero or one line incident to $v$ belongs to $\{\ell_1, \ldots, \ell_z\}$, either one or two octagons incident to $v$ belong to $\tilde{F}_z$. In the former case, we say that $v$ is an extremal boundary vertex, and in the latter case, we say that $v$ is a flat boundary vertex. If there is no flat boundary vertex, then there is exactly one octagon, and the lemma holds.

Every flat boundary vertex is the intersection of some line $\ell_i$ with the boundary of $\tilde{F}_z$. There are at most $2z$ such vertices by Lemma 7.15. Between two consecutive flat boundary vertices, there are trivially at most 6 extremal boundary vertices, all on the boundary of the same octagon. Thus, the perimeter of $\tilde{F}_z$ is at most $14z$. The lemma now follows directly from Lemma 7.12.

Constructing the relevant region $\tilde{F}_z$ is now straightforward. $\tilde{F}_0$ is a copy of the octagon containing $p(0)$, the starting point of $p$. To compute $\tilde{F}_i$, we follow $\tilde{p}$ until it exits the previous region $\tilde{F}_{i-1}$. At the exit point, the path is crossing $\ell_i$, and five outer lines.

To complete $\tilde{F}_i$, we append the octagons $O'_1, \ldots, O'_u$. 

7.2.3 Conclusion

We can conclude the proof of Theorem 7.7 along the same lines as in the case with boundary. We consider a path $p$, represented as a walk in $M$, of complexity $k$. As above, we actually compute the tight octagonal
decomposition $O$ in the refined graph $M^+$. In particular, $p$ has $O(gk)$ crossings with $O$.

We build the relevant region of the universal cover corresponding to a lift $\tilde{p}$ of $p$, as described in Lemma 7.16. Since every octagon has complexity $O(n)$, Lemma 7.17 shows that this phase takes time $O(gnk)$, and this is the complexity of the relevant region.

Now let $L$ be the set of lines crossed an odd number of times by $\tilde{p}$. By Lemma 7.14, there is a shortest path $\tilde{q}$ with the same endpoints as $\tilde{p}$ that crosses each line in $L$ exactly once and no other line; in particular $\tilde{q}$ remains in the relevant region, and we can compute it in time linear in the complexity of the region, i.e., in $O(gnk)$ time.

**Exercise 7.18.** ★ Prove Lemma 5.4 in the case of cross-metric surfaces without boundary, using Lemma 7.4 and the techniques used above.

### 7.3 Notes

The material of this chapter is largely taken from Colin de Verdière and Erickson [18], where also algorithms to compute tight cycles (shortest cycles up to deformation with no fixed basepoint) are presented.

The computation of a tight octagonal decomposition can be speeded up: the $O(n^2 \log n)$ bottleneck comes from the computation of the initial cycle; but a tight non-separating cycle can actually be computed in $O(n \log n)$ time [8], improving the overall construction to $O(gn \log n)$.

The construction of the universal cover of a surface without boundary equipped with a regular tiling is closely related to regular tessellations in the hyperbolic disk, as in Figure 7.5.

Deciding whether two paths on a surface can be done in linear time [33, 56]. This problem has been considered much earlier (from a more combinatorial perspective) by Dehn [22] (see Stillwell [73, p. 188]); his algorithm is one of the starting points of combinatorial group theory.

In the specific case of the plane, several algorithms exist to test contractibility or homotopy [9] and to compute shortest homotopic paths [5, 26].

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