Unknot recognition, linear programming and the elusive polynomial time algorithm

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Outline

1. Decision problems in geometric topology

2. Complexity classes

3. Approaches for a polynomial time algorithm
   - Normal surfaces and linear programming
   - Diagram simplification
   - Integer programming over homology

4. Average and generic case complexity
What is geometric topology?

Geometric topology is essentially “rubber-sheet geometry”.

Two topological objects are considered equivalent if we can “bend or stretch” one to make the other.

Examples from 2-manifolds (2-dimensional surfaces):

- Sphere
- Klein bottle
- Torus
What is geometric topology (ctd.)

Examples from knot theory:

\[ \text{Unknot} \quad = \quad \text{Figure 8} \]

\[ \text{Trefoil} \quad = \quad \text{Trefoil} \]

Much research is driven by decision problems:

- Are the 2-manifolds \( M, N \) equivalent? \( \ldots \) Easy!
- Are the knots \( K, L \) equivalent? \( \ldots \) Difficult
- Are the 3-manifolds \( M, N \) equivalent? \( \ldots \) Very difficult
What is geometric topology (ctd.)

Examples from knot theory:

Unknot = Figure 8

Trefoil =

Much research is driven by decision problems:

- Are the 2-manifolds $M, N$ equivalent? ... Easy!
- Are the knots $K, L$ equivalent? ... Difficult
- Are the 3-manifolds $M, N$ equivalent? ... Very difficult
- Are the 4-manifolds $M, N$ equivalent? ... Undecidable!

[Markov, 1960]

We study the simplest cases: Does $M \equiv sphere$? Does $K \equiv unknot$?
2-sphere recognition

Is the 2-manifold (surface) $M$ equivalent to the 2-sphere?

**Theorem**

For every triangulation of the 2-sphere:

$$\text{vertices} - \text{edges} + \text{faces} = 2.$$

For any triangulation of any other 2-manifold:

$$\text{vertices} - \text{edges} + \text{faces} < 2.$$

**2-sphere recognition algorithm**

Triangulate $M$ and test whether $\text{vertices} - \text{edges} + \text{faces} = 2$.

Simple to implement and very fast (small polynomial time).
Unknot and 3-sphere recognition

Is the knot $K$ equivalent to the unknot?
- First algorithm based on normal surface theory [Haken, 1961]
- Later algorithm based on diagram simplification [Dynnikov, 2003]

Is the 3-manifold $M$ equivalent to the 3-sphere?
- First algorithm used almost normal surfaces [Rubinstein, 1992]
- Later algorithm based on Pachner moves [Mijatović, 2003]

Most are messy to implement.
All have at least exponential time in the worst case.
Complexity classes

Do these algorithms need to run in exponential time?

What do we know?

- Unknot recognition is in NP [Hass-Lagarias-Pippenger, 1999]
- 3-sphere recognition is in NP [Schleimer, 2004]
- Knot genus in an arbitrary 3-manifold is NP-complete [Agol-Hass-Thurston, 2002]

There are hints that unknot / 3-sphere recognition might lie in P . . .

- Unknot recognition is also in co-NP . . . [Claim by Agol]
- . . . and in AM \( \cap \) co-AM [Hara-Tani-Yamamoto, 2005]
- Bad cases are extremely rare [B., 2010]
- Several “near miss” polynomial-time algorithms, with linear programming as a key tool
Approach #1: Normal surface theory

The key idea is to look for interesting surfaces within a 3-D space.

*Haken’s unknot recognition algorithm*: Find the 2-dimensional disc that the unknot surrounds.

*Input*: A triangulation of a 3-D space (e.g., drill out the knot from $\mathbb{R}^3$)
- Glue together faces of $n$ tetrahedra ($n$ is the input size).
- Tetrahedra may be “bent” and/or self-identified.
Searching for normal surfaces

We look for embedded normal surfaces.

These slice through tetrahedra in triangles and quadrilaterals with no self-intersections.
Normal surfaces as integer vectors

A normal surface can be described by a sequence of $7n$ integers. These count the discs of each type in each tetrahedron.

This vector uniquely identifies the normal surface.

Theorem (Haken, 1961)

A vector $\mathbf{x} \in \mathbb{Z}^{7n}$ represents an embedded normal surface if & only if:

- $\mathbf{x}$ is non-negative;
- $\mathbf{x}$ satisfies a series of linear homogeneous matching equations;
- $\mathbf{x}$ uses at most one quadrilateral type per tetrahedron (the quadrilateral constraints).
The magic

The projective solution space is a cross-section of the cone described by $\mathbf{x} \geq \mathbf{0}$ and the matching equations.

This is a rational polytope.

**Theorem (Haken, 1961; Jaco-Tollefson, 1984)**

If the knot spans a disc, then it spans a normal disc that projects to a vertex of the projective solution space.

**Unknot recognition algorithm**

Enumerate the vertices of the projective solution space. If a vertex satisfies the quadrilateral constraints, reconstruct the surface and test whether it is the disc that we are looking for.

3-sphere recognition uses similar techniques.
Can we do this in polynomial time?

We cannot enumerate all vertices in polynomial time:
Pathological cases exist with $O(17^{n/4})$ vertices that all satisfy the quadrilateral constraints. [B., 2010]

**Lemma**

For every polygonal decomposition of a disc, $\text{vertices} - \text{edges} + \text{faces} = 1$.

For any polygonal decomposition of any other bounded surface, $\text{vertices} - \text{edges} + \text{faces} \leq 0$.

**Observation:** $\text{vertices} - \text{edges} + \text{faces}$ is **linear** on the solution space!

**Corollary**

*We have the unknot if and only if* $\max(\text{vertices} - \text{edges} + \text{faces}) > 0$ *under the quadrilateral constraints.*
Linear programming and the projective solution space

This sounds like a job for linear programming!

The problem is the quadrilateral constraints, which are non-linear and have a non-convex solution set.

Workarounds:

- Run $3^n$ distinct linear programs on the $3^n$ convex pieces that make up this solution set. \[\text{[Casson, } \sim 2002\text{]}\]

  This is always slow, since all $3^n$ steps are necessary if the input knot is non-trivial.

- Add integer and binary variables to enforce the quadrilateral constraints. \[\text{[B.-Ozlen, 2011]}\]

  This is extremely fast in practice, but requires integer programming which is non-polynomial in general.
Approach #2: Diagram simplification

Try to monotonically simplify a knot diagram / triangulation into its simplest possible form.

**Grid diagrams** for knots:
- Constructed from *n* horizontal rods and *n* vertical rods.
- Vertical rods always cross above horizontal rods.

**Theorem (Dynnikov, 2003)**
Any two grid diagrams of the same knot can be related by elementary moves.
Simplifying grid diagrams

Some elementary moves reduce $n$:

Some elementary moves leave $n$ unchanged:

Theorem (Dynnikov, 2003)

For any grid diagram of the unknot, there is a non-strict monotonic sequence of simplification moves that reduces the diagram to the trivial square.

If non-strict could be made strict, this would yield a polynomial time algorithm!
3-sphere recognition by Pachner moves

Any two triangulations of the same 3-manifold can be related by Pachner moves:

2-3 / 3-2 move

1-4 / 4-1 move

The same is true if we consider only one-vertex triangulations and 2-3 / 3-2 moves.

[Pachner, 1991]

[Matveev, 2003]
Simplifying by Pachner moves

**In theory:** Triangulations might become much larger along the way.
Current best bound: $6 \cdot 10^6 n^2 2^2 \cdot 10^4 n^2$ moves \[\text{[Mijatović, 2003]}\]

**In practice:**

**Computer theorem (B., 2011)**
For all $n = 3, \ldots, 9$, any 3-sphere triangulation of size $n$
can be simplified by passing through $\leq 2$ extra tetrahedra,
and by making $\leq 3$ “composite jumps”.

This was shown by enumerating and analysing all 149,676,922
distinct 3-manifold triangulations of size $n \leq 9$. 
Does this help?

If we could turn these experimental bounds into theoretical bounds...

3-sphere recognition algorithm

Try all possible sequences of \( \leq B \) moves, where \( B \) is our theoretical bound.

If this simplifies the triangulation, repeat. If not, “read off” whether we have a 3-sphere.

If \( B \) grows slower than \( O(n/ \log n) \), this yields a sub-exponential time algorithm.

If \( B \) grows like \( O(1) \), this yields a polynomial time algorithm!
Approach #3: Integer programming over homology

New problem: **Least area surface** bounded by a knot

Consider a *discrete* version:

- **triangulate** the space so the knot follows edges;
- find a least area surface built from **faces** of the triangulation.
Finding the least area surface

Theorem (Dunfield-Hirani, 2010)
In this discrete setting, the least area surface can be found in polynomial time.

Basic idea:
- Describe a surface as a sum of faces (triangles).
  If our triangulation has $n$ faces, this gives an integer vector in $\mathbb{Z}^n$.
- Express “the triangles form a surface” using linear constraints:
  Each triangle going into an edge must meet some triangle going out of an edge.
- Express area as a linear functional on $\mathbb{Z}^n$.
- Minimise this linear functional using integer programming.
Achieving polynomial time

**Theorem (Dey-Hirani-Krishnamoorthy, 2010)**

In this setting, the constraint matrix for the integer program is **totally unimodular**.

This means that we can relax the integer program to a **linear program**, which can be solved in **polynomial time**.

Unfortunately:

**Theorem (Hass-Snoeyink-Thurston, 2003)**

Even if the knot spans a disc, the **least area** surface might **not** be a disc.

If only we could express \( \text{vertices} - \text{edges} + \text{faces} \) as a linear functional on triangles, we could recognise the unknot in polynomial time!
Average and generic case complexity

If at first you fail . . .

- **Average case complexity**: average time over all possible inputs.
- **Generic complexity**: ignore a few bad cases, where
  \[ \Pr(\text{bad}) \to 0 \text{ as } n \to \infty. \]

Exhaustive analysis of all 1,537,582,427 closed 1-vertex triangulations with \( n \leq 10 \) suggests:

\[ \Pr(\text{bad}) \in O\left(\frac{1}{n^c}\right) \quad \text{for all } c > 0, \]

where “bad” means “does not simplify immediately”. That is:

**Experimental observation**

**Generic triangulations** simplify immediately!
Triangulations that do NOT simplify immediately

1, 537, 582, 427 closed 1-vertex triangulations, log-log scale:
Triangulations that do NOT simplify immediately

1,537,582,427 closed 1-vertex triangulations, linear-log scale:
Aggressive simplification

In practice, simplification is an extremely effective heuristic in 3-sphere recognition and related problems.

To find a $k$-move simplification requires $O(n^k)$ steps.

Observation

Suppose we allow $O(n^k)$ time to simplify from $t \to t - 1$ tetrahedra.

As $t$ drops, the number of moves can grow:

$$n^k = t^{(k \cdot \log n / \log t)}$$
Aggressive simplification (ctd.)

Observation (previous slide)
As $t$ drops, the number of moves can **grow**:

$$n^k = t^{(k \cdot \log n / \log t)}$$

That is, we can become **more aggressive** in our simplification as the triangulation shrinks.

$$\rightarrow$$ The **difficult small cases** become simpler!

Under the right “uniformity assumptions”:

Conjecture

Generic **3-sphere** triangulations can be simplified to the trivial case $n = 2$ in **polynomial time**.
Want to know more?

Normal surface algorithms

Simplification-based algorithms

Least area surface