Algorithmique et Programmation

Devoir n° 5

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Optimal coloring of triangulated graphs

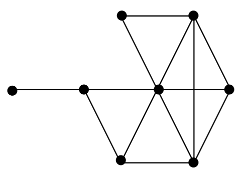
The purpose of this exercise is to study another graph search algorithm, which can recognize a class of graphs for which an optimum coloring can be found efficiently.

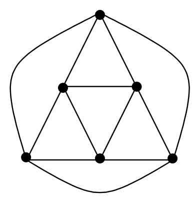
We consider the following algorithm:

Algorithm 1: Maximum cardinality search (MCS)

```
Input: Undirected G = (S, A)
begin
    for u ∈ S do
        ℓ[u] ← 0 ;
        π[u] ← NIL ;
    endfor
    F ← S;
    while F ≠ ∅ do
        Let u ∈ F such that ℓ[u] is maximal;
        for neighbor v of u in F do
            ℓ[v] ← ℓ[v] + 1 ;
            π[v] ← u;
        endfor
        F ← F \ {u};
    endw
end
```

(a) Give an order of traversal of the vertices for the graphs below using this algorithm and spanning tree built using π.





A graph is called *triangulated* if in any cycle of length greater than or equal to 4, there are two non-consecutive vertices in this cycle which are adjacent in the graph. A vertex is called *simplicial* if its neighborhood is a clique. Let $c = s_1, \ldots, s_k$ be a path in a graph G. A *chord* for this path is an edge $\{s_i, s_j\}$ with $|j - i| > 1$ (a path is *chordless* if no such edge exists on this path and a graph is triangulated if all its cycles of length greater than or equal to 4 have a chord).

(b) Are the above graphs triangulated? What are their simplicial vertices?

(c) Show that if G admits a simplicial vertex and $G_{|S\setminus\{s\}}$ is triangulated, then G is triangulated.

(d) Show that every triangulated graph admits a simplicial vertex. To do this, we can show the following steps:

- Show that if G is a clique, then all its vertices are simplicial.
- Show that if x is adjacent to all other vertices, then each vertex $y \neq x$ is simplicial in $G - \{x\}$ it is in G and vice versa.
- Otherwise, let x be a vertex of G and T the set of vertices farthest from x. Let H be a connected component of T, U all neighbors of H and Q the connected component of $G_{|S-U}$ containing x. Show that U is a clique.
- Conclude that a simplicial vertex exists in H.

(e) Show that if G is triangulated, then the last vertex s visited by the algorithm is simplicial. To do so, let $F : S \to \{1, \ldots, n\}$ be a numbering of the vertices according to the order they were processed by the MCS algorithm. Let $c = u, v_1, \ldots, v_k, w$ be a path such as $\forall i \in \{1, \ldots, k\}$, $f(u) < f(v_i)$ and $f(w) < f(v_i)$. Prove that such a path necessarily has a chord.

(f) Derive a greedy algorithm for the recognition of triangulated graphs.

A proper k-coloring c of a graph $G = (S, A)$ with k color is a mapping $c : S \to \{1, \ldots, k\}$ such that for every $(u, v) \in A$ $c(u) \neq c(v)$. A proper k-coloring is said to be optimal if there is no proper $k - 1$-coloring of the graph.

(g) Derive a greedy algorithm graphs recognition triangulated.

(h) From the above, deduce a greedy algorithm to properly color triangulated graphs. Can we establish a relationship between the minimum number of colors needed to color a triangulated graph (chromatic number) and the maximum size of a clique of the graph?