TD: Linear Programming

Algorithmique et Programmation

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The goal of this exercise session is to get more familiar with linear programming as a way to solve problems, and in particular the notion of duality.

1 Primal-dual algorithm for Max-Flow

Given a directed graph $G = (V, A)$ with capacity $c(x, y) \in \mathbb{R}^+$ on each arc $(x, y) \in A$ and a source $s \in V$ and sink $t \in V$, we want to route as much flow as possible from s to t through the arcs of G without exceeding each arc capacity.

In order to simplify the coding of the problem as a linear program, we add a virtual edge from t to s with capacity $+\infty$. The problem thus becomes to find a flow that is maximal on this edge, with flow preserved at all nodes, even s and t. We note $A' = A \cup \{(t, s)\}$.

1a. Express Max-Flow as a linear program with one variable $f(x, y)$ for every $(x, y) \in A'$, one constraint for every node $x \in V$ and one constraint for every edge $(x, y) \in A$. We call this program the *primal* program.

1b. We recall that the dual of a linear program is an opposite optimization problem (maximization if the original is minimization, and vice-versa) in which each constraint of the original program becomes a variable, the coefficients of the objective function are the constant bounds on each constraint, and such that the optimal solution is equal to the optimal solution of the original problem.

Write without justification the dual linear program. We note $\pi(x)$ for $x \in V$ and $\gamma(x, y)$ for $(x, y) \in A$ the variables of that program.

1c. What is the underlying problem solved by this dual linear program?

1d. Show that if both objective functions have the same value for some feasible solutions of each problem, then those feasible solutions are optimal. We will admit without proof that the converse is true: the optimal solution of one problem is identical to that of the other.

1e. Assume f and (π, γ) are *feasible* solutions of the primal and dual programs, respectively. Relying on the inequalities established in the previous question, give a necessary and sufficient set of conditions on the values of f and (π, γ) for them to be both optimal solutions of their respective linear programs. These are known as the *complementary slackness conditions*.

1f. We consider the following scheme to build an algorithm to solve both primal and dual programs at the same time. Start with the feasible solution $f = 0$ for the primal and the non-feasible solution $(\pi, \gamma) = 0$ for the dual. At each step, improve the value of f, while improving the feasibility of (π, γ) and ensuring that the complementary slackness conditions are satisfied. We stop when the flow can no more be improved and, then, the optimality of the flow will be guaranteed by the feasibility of (π, γ) and the complementary slackness conditions.

This type of algorithm is called *primal-dual* as it builds both primal and dual solutions together using one to improve the other, the dual solution serving to prove the optimality of the primal at the end of the algorithm.

Propose such a primal-dual algorithm for Max-Flow.

1g. To what problem is the solution of the dual LP a solution? What do the optimal assignments of the variables represent?

Complementary slackness conditions can thus guide us to design an algorithm to adapt the dual variables to the improvement of the primal solution.

2 Duality of non-normal form programs

We say that a linear program is in primal *normal form* if it is a minimization problem, all variables are non-negative, and all constraints are $\geqslant$.

2a. Describe an algorithm that converts any linear program in a primal normal form with an increase at most linear in the number of variables and constraints.

2b. Give the dual of the following linear program directly without converting it into normal form. Compute the dual of the dual and check you get the original program back.

$$\begin{cases} \text{Minimize} & 3x_1 + 7x_2 - 2x_3 + 4x_4 \\ \text{such that} & 2x_1 + 3x_2 \geqslant 10 \\ & x_2 + 5x_3 + x_4 \leqslant 20 \\ & x_1 - 2x_4 = 2 \\ & x_1 \geqslant 0, x_2 \geqslant 0, x_4 \leqslant 0 \end{cases}$$

2c. Explicit the complementary slackness conditions for this problem and use them to show that the solution $x_1 = 0, x_2 = \frac{10}{3}, x_3 = \frac{53}{15}, x_4 = -1$ is optimal with value $\frac{184}{15}$. What is the corresponding dual optimal solution?

3 Unimodularity in linear programming

Consider a linear program given in matrix form, i.e.:

Minimize $c^{\mathrm{T}} x$ subject to $Ax \geqslant b$, $x \geqslant 0$

where c and x are n-dimensional vectors, b is an m-dimensional vector, and A is an $m \times n$ matrix. For simplicity, assume this linear program is feasible and all solutions are bounded.

3a. Show that any feasible solution is a point in a polytope of $\mathbb{R}^n$.

3b. Recall Cramer's rule: Let M be a $k \times k$ invertible matrix. Then the solution x of the linear equation system $Mx = \alpha$ (where α is a k-vector of constants) is such that:

$$x_i = \frac{\det M_i}{\det M}$$

where M_i is the matrix M where the i-th column has been replaced by α.

A matrix is *totally unimodular* if all its square submatrices have determinant 0, -1, or +1. Show that if A is totally unimodular and b is a vector of integers, the optimal solution to the linear program has integral coordinates.

3c. Given a graph G, formulate a linear program P with additional integral constraints (i.e., constraints forcing a variable to be an integer) for the minimum vertex cover problem.

3d. Show that minimim vertex cover can be solved in PTIME in bipartite graphs.