Provable Security for Public-Key Schemes

II – Encryption

David Pointcheval

Ecole normale supérieure, CNRS & INRIA

IACR-SEAMS School
Cryptographie: Foundations and New Directions
November 2016 – Hanoi – Vietnam

Outline

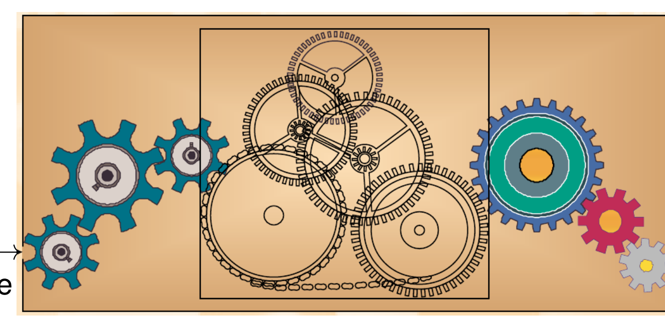
Outline

Provable Security

One can prove that:

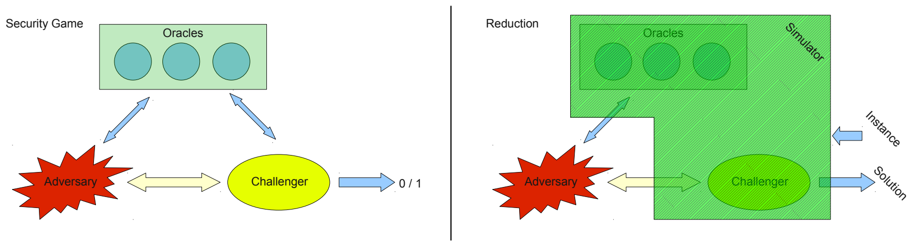
- if an adversary is able to break the cryptographic scheme
- then one can break the underlying problem (integer factoring, discrete logarithm, 3-SAT, etc)

hard instance →



→ solution

Direct Reduction



Unfortunately

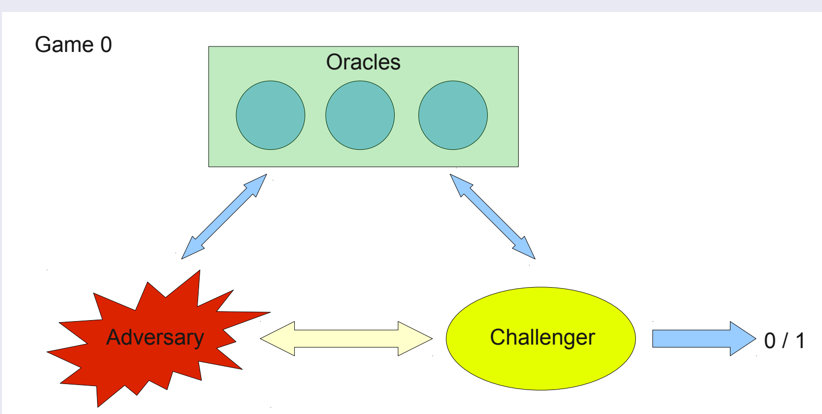
- Security may rely on several assumptions
- Proving that the view of the adversary, generated by the simulator, in the reduction is the same as in the real attack game is not easy to do in such a one big step

Outline

Sequence of Games

Real Attack Game

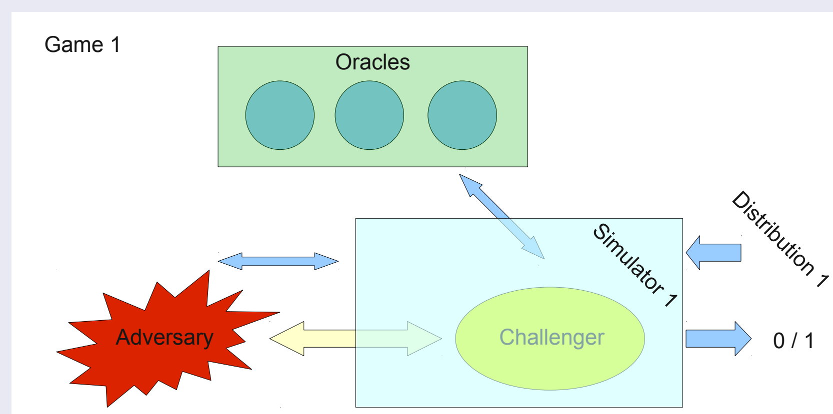
The adversary plays a game, against a challenger (security notion)



Sequence of Games

Simulation

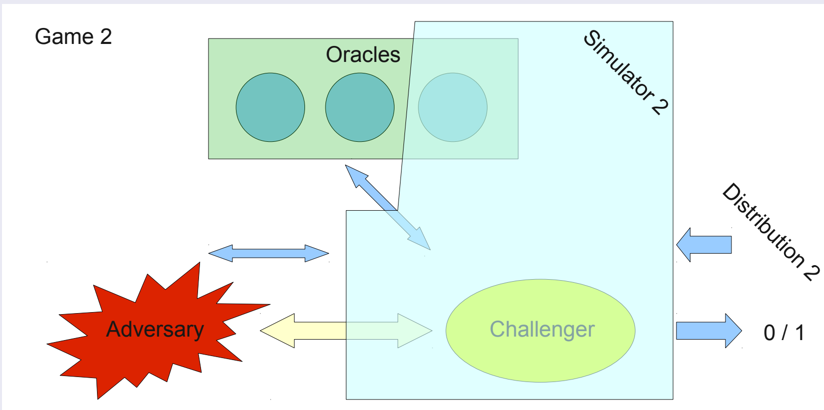
The adversary plays a game, against a sequence of simulators



Sequence of Games

Simulation

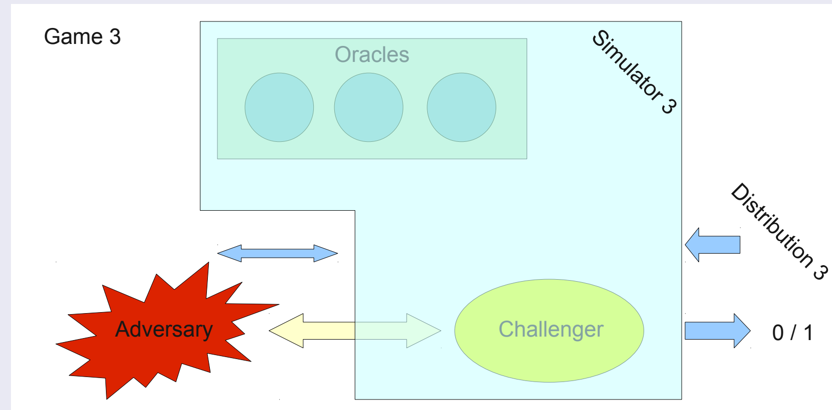
The adversary plays a game, against a sequence of simulators



Sequence of Games

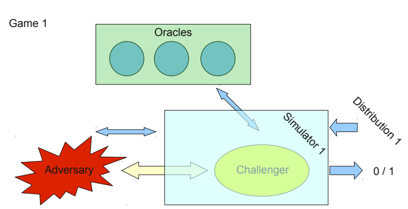
Simulation

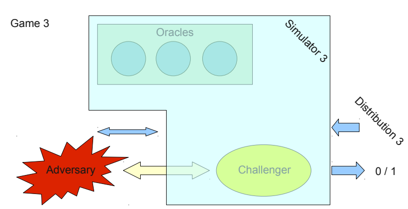
The adversary plays a game, against a sequence of simulators



Output

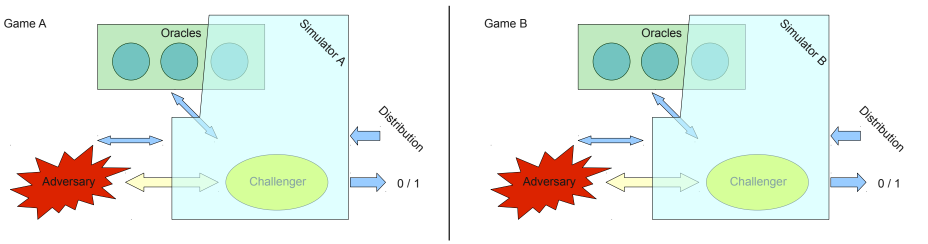
- The output of the simulator in Game 1 is related to the output of the challenger in Game 0 (adversary's winning probability)
- The output of the simulator in Game 3 is easy to evaluate (e.g. always zero, always 1, probability of one-half)
- The gaps (Game 1 $\leftrightarrow$ Game 2, Game 2 $\leftrightarrow$ Game 3, etc) are clearly identified with specific events





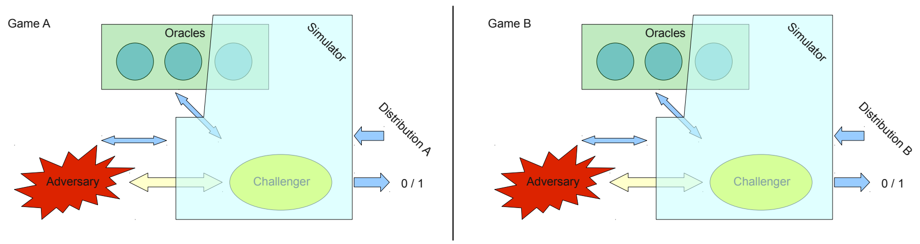
Outline

Two Simulators



- perfectly identical behaviors **[Hop-S-Perfect]**
- different behaviors, only if event **Ev** happens
  - **Ev** is negligible **[Hop-S-Negl]**
  - **Ev** is non-negligible **[Hop-S-Non-Negl]**
    and independent of the output in $\textbf{Game}_A$
    $\rightarrow$ Simulator B terminates in case of event **Ev**

Two Distributions



- perfectly identical input distributions **[Hop-D-Perfect]**
- different distributions
  - statistically close **[Hop-D-Stat]**
  - computationally close **[Hop-D-Comp]**

Two Simulations

- Identical behaviors: $\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] = 0$
- The behaviors differ only if **Ev** happens:
  - **Ev** is negligible, one can ignore it
    Shoup's Lemma: $\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] \leq \Pr[\textbf{Ev}]$

$$\begin{aligned}
&|\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B]| \\
&= \left| \begin{array}{l} \Pr[\textbf{Game}_A|\textbf{Ev}]\Pr[\textbf{Ev}] + \Pr[\textbf{Game}_A|\neg\textbf{Ev}]\Pr[\neg\textbf{Ev}] \\ - \Pr[\textbf{Game}_B|\textbf{Ev}]\Pr[\textbf{Ev}] - \Pr[\textbf{Game}_B|\neg\textbf{Ev}]\Pr[\neg\textbf{Ev}] \end{array} \right| \\
&= \left| \begin{array}{l} (\Pr[\textbf{Game}_A|\textbf{Ev}] - \Pr[\textbf{Game}_B|\textbf{Ev}]) \times \Pr[\textbf{Ev}] \\ +(\Pr[\textbf{Game}_A|\neg\textbf{Ev}] - \Pr[\textbf{Game}_B|\neg\textbf{Ev}]) \times \Pr[\neg\textbf{Ev}] \end{array} \right| \\
&\leq |1 \times \Pr[\textbf{Ev}] + 0 \times \Pr[\neg\textbf{Ev}]| \leq \Pr[\textbf{Ev}]
\end{aligned}$$

  - **Ev** is non-negligible and independent of the output in $\textbf{Game}_A$,
    Simulator B terminates in case of event **Ev**

Two Simulations

- Identical behaviors: $\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] = 0$
- The behaviors differ only if **Ev** happens:
  - **Ev** is negligible, one can ignore it
  - **Ev** is non-negligible and independent of the output in $\textbf{Game}_A$,
    Simulator B terminates and outputs 0, in case of event **Ev**:

$$\begin{aligned}
\Pr[\textbf{Game}_B] &= \Pr[\textbf{Game}_B|\textbf{Ev}]\Pr[\textbf{Ev}] + \Pr[\textbf{Game}_B|\neg\textbf{Ev}]\Pr[\neg\textbf{Ev}] \\
&= 0 \times \Pr[\textbf{Ev}] + \Pr[\textbf{Game}_A|\neg\textbf{Ev}] \times \Pr[\neg\textbf{Ev}] \\
&= \Pr[\textbf{Game}_A] \times \Pr[\neg\textbf{Ev}]
\end{aligned}$$

Simulator B terminates and flips a coin, in case of event **Ev**:

$$\begin{aligned}
\Pr[\textbf{Game}_B] &= \Pr[\textbf{Game}_B|\textbf{Ev}]\Pr[\textbf{Ev}] + \Pr[\textbf{Game}_B|\neg\textbf{Ev}]\Pr[\neg\textbf{Ev}] \\
&= \tfrac{1}{2} \times \Pr[\textbf{Ev}] + \Pr[\textbf{Game}_A|\neg\textbf{Ev}] \times \Pr[\neg\textbf{Ev}] \\
&= \tfrac{1}{2} + (\Pr[\textbf{Game}_A] - \tfrac{1}{2}) \times \Pr[\neg\textbf{Ev}]
\end{aligned}$$

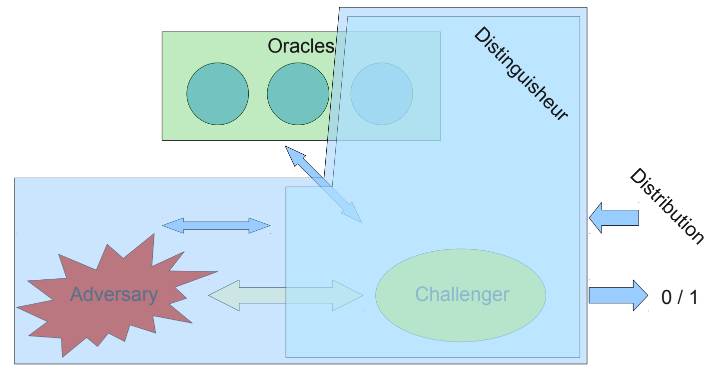
Two Simulations

- Identical behaviors: $\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] = 0$
- The behaviors differ only if **Ev** happens:
  - **Ev** is negligible, one can ignore it
  - **Ev** is non-negligible and independent of the output in $\textbf{Game}_A$, Simulator B terminates in case of event **Ev**

Event Ev

- Either **Ev** is negligible, or the output is independent of **Ev**
- For being able to terminate simulation B in case of event **Ev**, this event must be *efficiently* detectable
- For evaluating Pr[**Ev**], one re-iterates the above process, with an initial game that outputs 1 when event **Ev** happens

Two Distributions



$$\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] \leq \mathbf{Adv}(\mathcal{D}^{\text{oracles}})$$

Two Distributions

$$\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] \leq \mathbf{Adv}(\mathcal{D}^{\text{oracles}})$$

- For identical/statistically close distributions, for any oracle:

$$\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] = \textbf{Dist}(\textbf{Distrib}_A, \textbf{Distrib}_B) = \text{negl}()$$

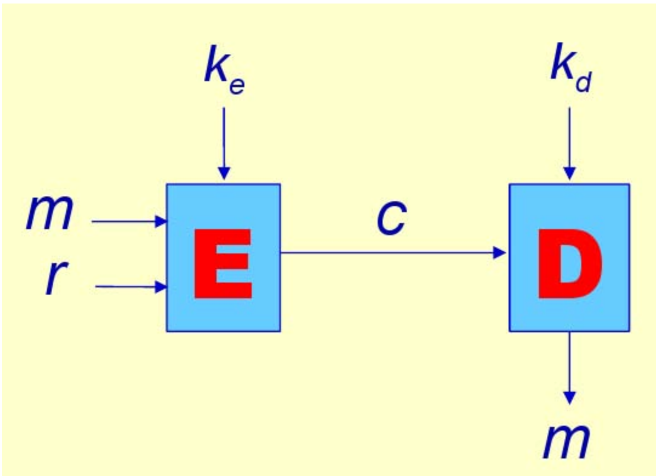
- For computationally close distributions, in general, we need to exclude additional oracle access:

$$\Pr[\textbf{Game}_A] - \Pr[\textbf{Game}_B] \leq \mathbf{Adv}^{\textbf{Distrib}}(t)$$

where t is the computational time of the distinguisheur

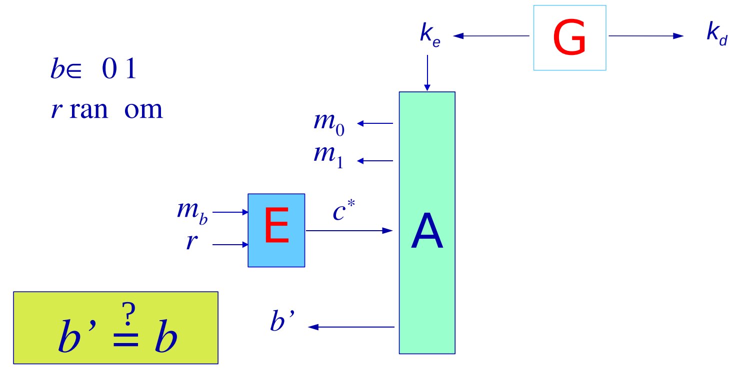
Outline

Public-Key Encryption



Goal: Privacy/Secrecy of the plaintext

IND − CPA Security Game



The adversary cannot get any information about a plaintext of a specific ciphertext (validity, partial value, etc)

Malleability

Semantic security (ciphertext indistinguishability) guarantees that no information is leaked from c about the plaintext m
But it may be possible to derive a ciphertext c'
such that the plaintext m' is related to m in a meaningful way:

- ElGamal ciphertext: $c_1 = g^r$ and $c_2 = m \times y^r$
- Malleability: $c'_1 = c_1 = g^r$ and $c'_2 = 2 \times c_2 = (2m) \times y^r$

From an encryption of m, one can build an encryption of $2m$, or a random ciphertext of m, etc

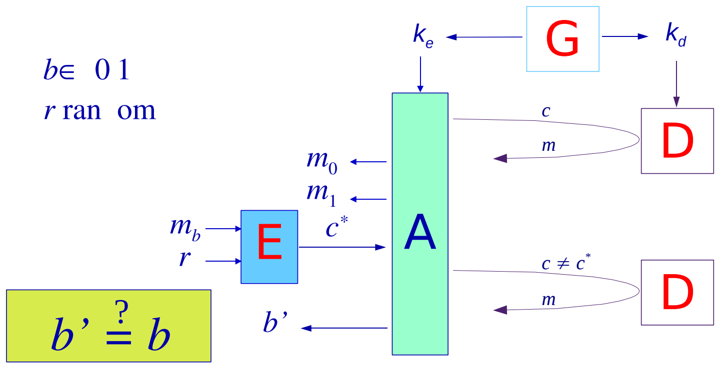
A formal security game for $\mathbf{NM} - \mathbf{CPA}$ has been defined, but we ignore it for the moment

Additional Information

More information modeled by oracle access

- reaction attacks: oracle which answers, on c, whether the ciphertext c is valid or not
- plaintext-checking attacks: oracle which answers, on a pair (m, c), whether the plaintext m is really encrypted in c or not (whether $m = \mathcal{D}_{sk}(c)$)
- chosen-ciphertext attacks ($\mathbf{CCA}$): decryption oracle (with the restriction not to use it on the challenge ciphertext) $\Longrightarrow$ the adversary can obtain the plaintext of any ciphertext of its choice (excepted the challenge)
  - non-adaptive ($\mathbf{CCA} - 1$) [Naor-Yung – STOC '90]
    only before receiving the challenge
  - adaptive ($\mathbf{CCA} - 2$) [Rackoff-Simon – Crypto '91]
    unlimited oracle access

IND – CCA Security Game



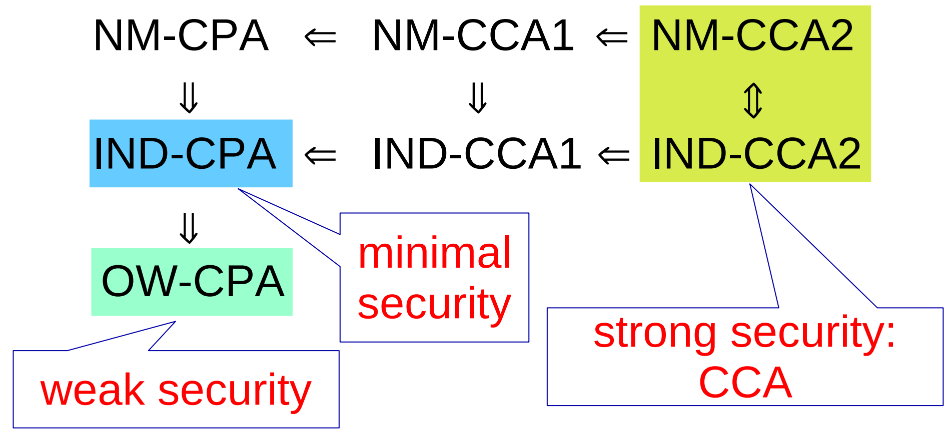
The adversary can ask any decryption of its choice:
Chosen-Ciphertext Attacks (oracle access)

$$(sk, pk) \leftarrow \mathcal{K}(); (m_0, m_1, \text{state}) \leftarrow \mathcal{A}^{\mathcal{D}}(pk);$$
$$b \stackrel{R}{\leftarrow} \{0,1\}; c = \mathcal{E}_{pk}(m_b); b' \leftarrow \mathcal{A}^{\mathcal{D}}(\text{state}, c)$$

$$\mathbf{Adv}_{\mathcal{S}}^{\text{ind-cca}}(\mathcal{A}) = \Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0] = 2 \times \Pr[b' = b] - 1$$

Relations

[Bellare-Desai-Pointcheval-Rogaway – Crypto '98]



Outline

Cramer-Shoup Encryption Scheme

[Cramer-Shoup – Crypto '98]

Key Generation

- $\mathbb{G} = (\langle g \rangle, \times)$ group of order q
- $sk = (x_1, x_2, y_1, y_2, z)$, where $x_1, x_2, y_1, y_2, z \stackrel{R}{\leftarrow} \mathbb{Z}_q$
- $pk = (g_1, g_2, \mathcal{H}, c, d, h)$, where
  - g_1, g_2 are independent elements in $\mathbb{G}$
  - $\mathcal{H}$ a hash function (second-preimage resistant)
  - $c = g_1^{x_1} g_2^{x_2}$, $d = g_1^{y_1} g_2^{y_2}$, and $h = g_1^z$

Encryption

$$u_1 = g_1^r,\ u_2 = g_2^r,\ e = m \times h^r,\ v = c^r d^{r\alpha} \text{ where } \alpha = \mathcal{H}(u_1, u_2, e)$$

Cramer-Shoup Encryption Scheme vs. ElGamal

$$u_1 = g_1^r,\ u_2 = g_2^r,\ e = m \times h^r,\ v = c^r d^{r\alpha} \text{ where } \alpha = \mathcal{H}(u_1, u_2, e)$$

(u_1, e) is an ElGamal ciphertext, with public key $h = g_1^z$

Decryption

- since $h = g_1^z$, $h^r = u_1^z$, thus $m = e/u_1^z$
- since $c = g_1^{x_1} g_2^{x_2}$ and $d = g_1^{y_1} g_2^{y_2}$

$$c^r = g_1^{rx_1} g_2^{rx_2} = u_1^{x_1} u_2^{x_2} \quad d^r = u_1^{y_1} u_2^{y_2}$$

One thus first checks whether

$$v = u_1^{x_1+\alpha y_1} u_2^{x_2+\alpha y_2} \text{ where } \alpha = \mathcal{H}(u_1, u_2, e)$$

Security of the Cramer-Shoup Encryption Scheme

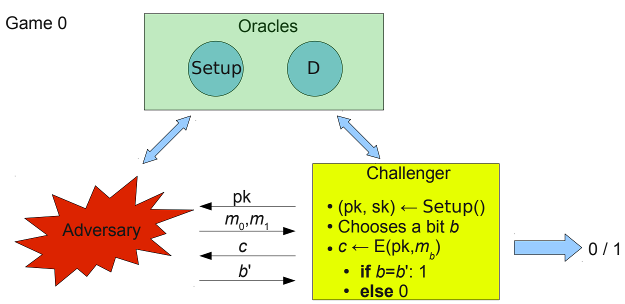
Theorem

The Cramer-Shoup encryption scheme achieves $\mathbf{IND}-\mathbf{CCA}$ *security, under the* ***DDH*** *assumption, and the second-preimage resistance of* $\mathcal{H}$*:*

$$\mathbf{Adv}_{\mathcal{CS}}^{\mathsf{ind-cca}}(t) \le 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t) + \mathbf{Succ}^{\mathcal{H}}(t) + 3q_D/q$$

Let us prove this theorem, with a sequence of games, in which $\mathcal{A}$ is an $\mathbf{IND}-\mathbf{CCA}$ adversary against the Cramer-Shoup encryption scheme

Real Attack Game



Key Generation Oracle

$x_1, x_2, y_1, y_2, z \overset{R}{\leftarrow} \mathbb{Z}_q$, $g_1, g_2 \overset{R}{\leftarrow} \mathbb{G}$: $sk = (x_1, x_2, y_1, y_2, z)$
$c = g_1^{x_1} g_2^{x_2}$, $d = g_1^{y_1} g_2^{y_2}$, and $h = g_1^z$: $pk = (g_1, g_2, \mathcal{H}, c, d, h)$

Decryption Oracle

If $v = u_1^{x_1+\alpha y_1} u_2^{x_2+\alpha y_2}$ where $\alpha = \mathcal{H}(u_1, u_2, e)$: $m = e/u_1^z$

Proof: Invalid ciphertexts

- **Game**$_0$: use of the oracles $\mathcal{K}$, $\mathcal{D}$
- **Game**$_1$: we abort (with a random output b') in case of bad (invalid) accepted ciphertext, where invalid ciphertext means $\log_{g_1} u_1 \neq \log_{g_2} u_2$

Event F

$\mathcal{A}$ submits a bad accepted ciphertext
(note: this is not computationally detectable)

The advantage in **Game**$_1$ is: $\Pr_1[b' = b|\mathbf{F}] = 1/2$

$$\Pr_{\mathbf{Game}_0}[\mathbf{F}] = \Pr_{\mathbf{Game}_1}[\mathbf{F}] \quad \Pr_{\mathbf{Game}_1}[b' = b|\neg\mathbf{F}] = \Pr_{\mathbf{Game}_0}[b' = b|\neg\mathbf{F}]$$

$\Longrightarrow$ **Hop-S-Negl**: $\mathbf{Adv}_{\mathbf{Game}_1} \ge \mathbf{Adv}_{\mathbf{Game}_0} - \Pr[\mathbf{F}]$

Details: Shoup's Lemma

$$\begin{aligned}
\mathbf{Adv}_{\mathbf{Game}_1} &= 2 \times \Pr_{\mathbf{Game}_1}[b' = b] - 1 \\
&= 2 \times \Pr_{\mathbf{Game}_1}[b' = b | \neg\mathbf{F}] \Pr_{\mathbf{Game}_1}[\neg\mathbf{F}] \\
&\quad + 2 \times \Pr_{\mathbf{Game}_1}[b' = b | \mathbf{F}] \Pr_{\mathbf{Game}_1}[\mathbf{F}] - 1 \\
&= 2 \times \Pr_{\mathbf{Game}_0}[b' = b | \neg\mathbf{F}] \Pr_{\mathbf{Game}_0}[\neg\mathbf{F}] + \Pr_{\mathbf{Game}_0}[\mathbf{F}] - 1 \\
&= 2 \times \Pr_{\mathbf{Game}_0}[b' = b] - 2 \times \Pr_{\mathbf{Game}_0}[b' = b | \mathbf{F}] \Pr_{\mathbf{Game}_0}[\mathbf{F}] \\
&\quad + \Pr_{\mathbf{Game}_0}[\mathbf{F}] - 1 \\
&= \mathbf{Adv}_{\mathbf{Game}_0} - \Pr_{\mathbf{Game}_0}[\mathbf{F}](2 \times \Pr_{\mathbf{Game}_0}[b' = b | \mathbf{F}] - 1) \\
&\geq \mathbf{Adv}_{\mathbf{Game}_0} - \Pr_{\mathbf{Game}_0}[\mathbf{F}]
\end{aligned}$$

Details: Bad Accept

In order to evaluate $\Pr[\mathbf{F}]$, we study the probability that

- $r_1 = \log_{g_1} u_1 \neq \log_{g_2} u_2 = r_2$,
- whereas $v = u_1^{x_1+\alpha y_1} u_2^{x_2+\alpha y_2}$

The adversary just knows the public key:

$$c = g_1^{x_1} g_2^{x_2} \qquad d = g_1^{y_1} g_2^{y_2}$$

Let us move to the exponents, in basis g_1, with $g_2 = g_1^s$:

$$\begin{aligned}
\log c &= x_1 + s x_2 \\
\log d &= y_1 + s y_2 \\
\log v &= r_1(x_1 + \alpha y_1) + s r_2(x_2 + \alpha y_2)
\end{aligned}$$

The system is under-defined: for any v, there are (x_1, x_2, y_1, y_2) that satisfy the system $\Longrightarrow v$ is unpredictable

$\Longrightarrow \Pr[\mathbf{F}] \leq q_D/q \qquad \Longrightarrow \mathbf{Adv}_{\mathbf{Game}_1} \geq \mathbf{Adv}_{\mathbf{Game}_0} - q_D/q$

Proof: Simulations

- **Game**$_2$: we use the simulations

Key Generation Simulation

$x_1, x_2, y_1, y_2, z_1, z_2 \stackrel{R}{\leftarrow} \mathbb{Z}_q$, $g_1, g_2 \stackrel{R}{\leftarrow} \mathbb{G}$: $sk = (x_1, x_2, y_1, y_2, z_1, z_2)$

$g_2 = g_1^s$

$c = g_1^{x_1} g_2^{x_2}$, $d = g_1^{y_1} g_2^{y_2}$, and $h = g_1^{z_1} g_2^{z_2}$: $pk = (g_1, g_2, \mathcal{H}, c, d, h)$

$z = z_1 + s z_2$

Distribution of the public key: Identical

Decryption Simulation

If $v = u_1^{x_1+\alpha y_1} u_2^{x_2+\alpha y_2}$ where $\alpha = \mathcal{H}(u_1, u_2, e)$: $m = e / u_1^{z_1} u_2^{z_2}$

Under the assumption of $\neg\mathbf{F}$, perfect simulation

$\Longrightarrow$ **Hop-S-Perfect**: $\mathbf{Adv}_{\mathbf{Game}_2} = \mathbf{Adv}_{\mathbf{Game}_1}$

Proof: Computable Adversary

- **Game**$_3$: we do no longer exclude bad accepted ciphertexts
  $\Longrightarrow$ **Hop-S-Negl**:
  $\mathbf{Adv}_{\mathbf{Game}_3} \geq \mathbf{Adv}_{\mathbf{Game}_2} - \Pr[\mathbf{F}] \geq \mathbf{Adv}_{\mathbf{Game}_2} - q_D/q$

  This is technical: to make the simulator/adversary computable

Proof: DDH Assumption

- **Game$_4$**: we modify the generation of the challenge ciphertext:

Original Challenge

Random choice: $b \overset{R}{\leftarrow} \{0,1\}, r \overset{R}{\leftarrow} \mathbb{Z}_q$ $\quad [\alpha = \mathcal{H}(u_1, u_2, e)]$

$$u_1 = g_1^r,\ u_2 = g_2^r,\ e = m_b \times h^r,\ v = c^r d^{r\alpha}$$

New Challenge 1

Given $(U = g_1^r, V = g_2^r)$ from outside, and random choice $b \overset{R}{\leftarrow} \{0,1\}$

$$u_1 = U,\ u_2 = V,\ e = m_b \times U^{z_1} V^{z_2},\ v = U^{x_1+\alpha y_1} V^{x_2+\alpha y_2}$$

With $(U = g_1^r, V = g_2^r)$: $U^{z_1}V^{z_2} = h^r$ and $U^{x_1+\alpha y_1}V^{x_2+\alpha y_2} = c^r d^{r\alpha}$
$\Longrightarrow$ **Hop-S-Perfect**: $\mathbf{Adv}_{\mathbf{Game}_4} = \mathbf{Adv}_{\mathbf{Game}_3}$

Proof: DDH Assumption

- **Game$_5$**: we modify the generation of the challenge ciphertext:

Previous Challenge 1

Given $(U = g_1^r, V = g_2^r)$ from outside, and random choice $b \overset{R}{\leftarrow} \{0,1\}$

$$u_1 = U,\ u_2 = V,\ e = m_b \times U^{z_1} V^{z_2},\ v = U^{x_1+\alpha y_1} V^{x_2+\alpha y_2}$$

New Challenge 2

Given $(U = g_1^{r_1}, V = g_2^{r_2})$ from outside, and random choice $b \overset{R}{\leftarrow} \{0,1\}$

$$u_1 = U,\ u_2 = V,\ e = m_b \times U^{z_1} V^{z_2},\ v = U^{x_1+\alpha y_1} V^{x_2+\alpha y_2}$$

The input changes from $(U = g_1^r, V = g_2^r)$ to $(U = g_1^{r_1}, V = g_2^{r_2})$:
$\Longrightarrow$ **Hop-D-Comp**: $\mathbf{Adv}_{\mathbf{Game}_5} \geq \mathbf{Adv}_{\mathbf{Game}_4} - 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$

Proof: DDH Assumption

The input from outside changes from $(U = g_1^r, V = g_2^r)$ (a CDH tuple) to $(U = g_1^{r_1}, V = g_2^{r_2})$ (a random tuple):

$$\Pr_{\mathbf{Game}_4}[b' = b] - \Pr_{\mathbf{Game}_5}[b' = b] \leq \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$$

$\Longrightarrow$ **Hop-D-Comp**: $\mathbf{Adv}_{\mathbf{Game}_5} \geq \mathbf{Adv}_{\mathbf{Game}_4} - 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$
(Since $\mathbf{Adv} = 2 \times \Pr[b' = b] - 1$)

Proof: Collision

- **Game$_6$**: we abort (with a random output b') in case of second pre-image with a decryption query

Event $\mathbf{F}_H$

$\mathcal{A}$ submits a ciphertext with the same α as the challenge ciphertext, but a different initial triple: $(u_1, u_2, e) \neq (u_1^*, u_2^*, e^*)$, but $\alpha = \alpha^*$, were "$*$" are for all the elements related to the challenge ciphertext

Second pre-image of $\mathcal{H}$: $\quad \Longrightarrow \Pr[\mathbf{F}_H] \leq \mathbf{Succ}^{\mathcal{H}}(t)$

The advantage in **Game$_6$** is: $\Pr_{\mathbf{Game}_6}[b' = b | \mathbf{F}_H] = 1/2$

$$\Pr_{\mathbf{Game}_5}[\mathbf{F}_H] = \Pr_{\mathbf{Game}_6}[\mathbf{F}_H] \quad \Pr_{\mathbf{Game}_6}[b' = b | \neg\mathbf{F}_H] = \Pr_{\mathbf{Game}_5}[b' = b | \neg\mathbf{F}_H]$$

$\Longrightarrow$ **Hop-S-Negl**: $\mathbf{Adv}_{\mathbf{Game}_6} \geq \mathbf{Adv}_{\mathbf{Game}_5} - \Pr[\mathbf{F}_H]$

$$\mathbf{Adv}_{\mathbf{Game}_6} \geq \mathbf{Adv}_{\mathbf{Game}_5} - \mathbf{Succ}^{\mathcal{H}}(t)$$

Proof: Invalid ciphertexts

- **Game**$_7$: we abort (with a random output b') in case of bad accepted ciphertext, we do as in **Game**$_1$

Event F′

$\mathcal{A}$ submits a bad accepted ciphertext
(note: this is not computationally detectable)

The advantage in **Game**$_7$ is: $\Pr_{\mathbf{Game}_7}[b' = b|\mathbf{F}'] = 1/2$

$$\Pr_{\mathbf{Game}_6}[\mathbf{F}'] = \Pr_{\mathbf{Game}_7}[\mathbf{F}'] \qquad \Pr_{\mathbf{Game}_7}[b' = b|\neg\mathbf{F}'] = \Pr_{\mathbf{Game}_6}[b' = b|\neg\mathbf{F}']$$

$\Longrightarrow$ **Hop-S-Negl**: $\mathbf{Adv}_{\mathbf{Game}_7} \geq \mathbf{Adv}_{\mathbf{Game}_6} - \Pr[\mathbf{F}']$

Details: Bad Accept

In order to evaluate $\Pr[\mathbf{F}']$, we study the probability that

- $r_1 = \log_{g_1} u_1 \neq \log_{g_2} u_2 = r_2$,
- whereas $v = {u_1}^{x_1+\alpha y_1} {u_2}^{x_2+\alpha y_2}$

Let us use "*" for all the elements related to the challenge ciphertext
Three cases may appear:

- Case 1: $(u_1, u_2, e) = (u_1^*, u_2^*, e^*)$, then necessarily

$$v \neq v^* = U^{x_1+\alpha^* y_1} V^{x_2+\alpha^* y_2} = {u_1^*}^{x_1+\alpha^* y_1} {u_2^*}^{x_2+\alpha^* y_2}$$

  Then, the ciphertext is rejected $\Longrightarrow \Pr[\mathbf{F}'_1] = 0$
- Case 2: $(u_1, u_2, e) \neq (u_1^*, u_2^*, e^*)$, but $\alpha = \alpha^*$:
  From the previous game, Aborts $\Longrightarrow \Pr[\mathbf{F}'_2] = 0$
- Case 3: $(u_1, u_2, e) \neq (u_1^*, u_2^*, e^*)$, and $\alpha \neq \alpha^*$

Details: Bad Accept (Case 3)

The adversary knows the public key, and the (invalid) challenge ciphertext:

$$c = g_1^{x_1} g_2^{x_2} \qquad d = g_1^{y_1} g_2^{y_2}$$

$$v^* = U^{x_1+\alpha^* y_1} V^{x_2+\alpha^* y_2} = g_1^{r_1^*(x_1+\alpha^* y_1)} g_2^{r_2^*(x_2+\alpha^* y_2)}$$

Let us move to the exponents, in basis g_1, with $g_2 = g_1^s$:

$$\begin{aligned} \log c &= x_1 + s x_2 \\ \log d &= y_1 + s y_2 \\ \log v^* &= r_1^*(x_1 + \alpha^* y_1) + s r_2^*(x_2 + \alpha^* y_2) \\ \log v &= r_1(x_1 + \alpha y_1) + s r_2(x_2 + \alpha y_2) \end{aligned}$$

Details: Bad Accept (Case 3)

The determinant of the system is

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & s & 0 & 0 \\ 0 & 0 & 1 & s \\ r_1^* & s r_2^* & r_1^* \alpha^* & s r_2^* \alpha^* \\ r_1 & s r_2 & r_1 \alpha & s r_2 \alpha \end{vmatrix} \\ &= s^2 \times ((r_2 - r_1) \times (r_2^* - r_1^*) \times \alpha^* - (r_2^* - r_1^*) \times (r_2 - r_1) \times \alpha) \\ &= s^2 \times (r_2 - r_1) \times (r_2^* - r_1^*) \times (\alpha^* - \alpha) \\ &\neq 0 \end{aligned}$$

The system is under-defined:
for any v, there are (x_1, x_2, y_1, y_2) that satisfy the system
$\Longrightarrow v$ is unpredictable $\Longrightarrow \Pr[\mathbf{F}'_3] \leq q_D/q$
$\Longrightarrow \mathbf{Adv}_{\mathbf{Game}_7} \geq \mathbf{Adv}_{\mathbf{Game}_6} - q_D/q$

Proof: Analysis of the Final Game

In the final **Game**$_7$:

- only valid ciphertexts are decrypted
- the challenge ciphertext contains

$$e = m_b \times U^{z_1} V^{z_2}$$

- the public key contains

$$h = g_1^{z_1} g_2^{z_2}$$

Again, the system is under-defined:
for any m_b, there are (z_1, z_2) that satisfy the system
$\Longrightarrow m_b$ is unpredictable $\qquad \Longrightarrow b$ is unpredictable
$\Longrightarrow \mathbf{Adv}_{\mathsf{Game}_7} = 0$

Conclusion

$$\begin{array}{lcl}
\mathbf{Adv}_{\mathsf{Game}_7} & = & 0 \\
\mathbf{Adv}_{\mathsf{Game}_7} & \geq & \mathbf{Adv}_{\mathsf{Game}_6} - q_D/q \\
\mathbf{Adv}_{\mathsf{Game}_6} & \geq & \mathbf{Adv}_{\mathsf{Game}_5} - \mathbf{Succ}^{\mathcal{H}}(t) \\
\mathbf{Adv}_{\mathsf{Game}_5} & \geq & \mathbf{Adv}_{\mathsf{Game}_4} - 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathsf{ddh}}(t) \\
\mathbf{Adv}_{\mathsf{Game}_4} & = & \mathbf{Adv}_{\mathsf{Game}_3} \\
\mathbf{Adv}_{\mathsf{Game}_3} & \geq & \mathbf{Adv}_{\mathsf{Game}_2} - q_D/q \\
\mathbf{Adv}_{\mathsf{Game}_2} & = & \mathbf{Adv}_{\mathsf{Game}_1} \\
\mathbf{Adv}_{\mathsf{Game}_1} & \geq & \mathbf{Adv}_{\mathsf{Game}_0} - q_D/q \\
\mathbf{Adv}_{\mathsf{Game}_0} & = & \mathbf{Adv}_{\mathcal{CS}}^{\mathsf{ind-cca}}(\mathcal{A})
\end{array}$$

$$\mathbf{Adv}_{\mathcal{CS}}^{\mathsf{ind-cca}}(\mathcal{A}) \leq 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathsf{ddh}}(t) + \mathbf{Succ}^{\mathcal{H}}(t) + 3q_D/q$$

Outline

Conclusion

Game-based Methodology: the story of OAEP [Bellare-Rogaway EC '94]

- Reduction proven indistinguishable for an IND-CCA adversary (actually IND-CCA1, and not IND-CCA2) but widely believed for IND-CCA2, without any further analysis of the reduction
  The direct-reduction methodology
- [Shoup - Crypto '01]
  Shoup showed the gap for IND-CCA2, under the OWP
  Granted his new game-based methodology
- [Fujisaki-Okamoto-Pointcheval-Stern – Crypto '01]
  FOPS proved the security for IND-CCA2, under the PD-OWP
  Using the game-based methodology