Provable Security
Asymmetric Encryption

DEA - Algorithmique
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Summary

1. Introduction
2. Computational Assumptions
3. Security Proofs
4. Asymmetric Encryption
5. New Assumptions
6. An Example

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Two Keys…

Asymmetric Cryptography

Alice $\xleftrightarrow[\text{authenticity}]{\text{secrecy}}$ Bob

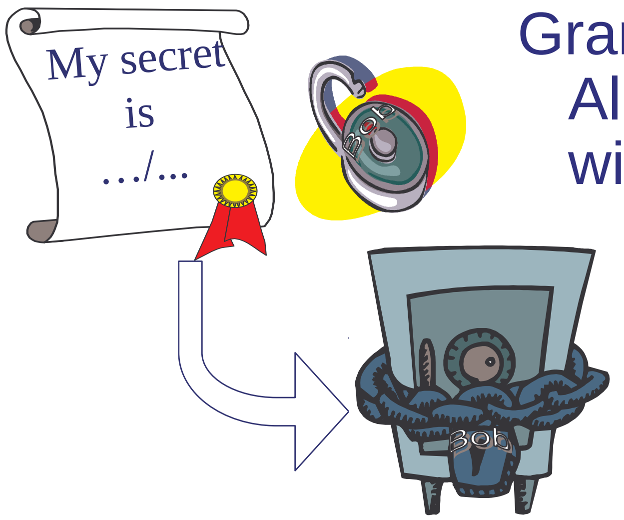
Diffie-Hellman 1976

Asymmetric Encryption:

Bob owns two "keys"

- A public key (encryption k_e)
  so that anybody can encrypt a message for him ⇒ known by everybody (included Alice)
- A private key (decryption k_d)
  to help him to decrypt ⇒ known by Bob only

Encryption / decryption attack



Granted Bob's public key,
Alice can lock the safe,
with the message inside
(*encrypt the message*)

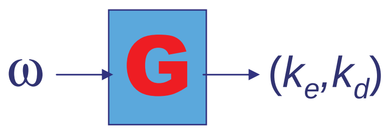
Excepted Bob,
granted his private key
(*Bob can decrypt*)

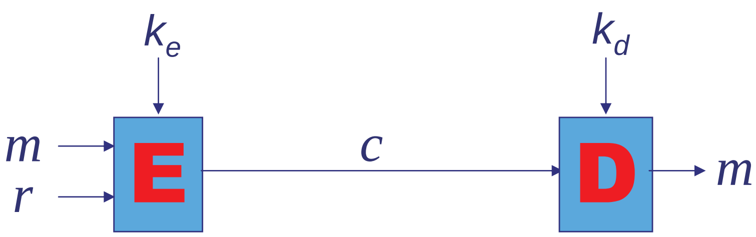
Alice sends the safe to Bob
no one can unlock it
(*impossible to break*)

Encryption Scheme

3 algorithms :

- **G** - key generation
- **E** - encryption
- **D** - decryption





Conditional Secrecy

The ciphertext comes from $c = \mathbf{E}_{k_e}(m;r)$

- The encryption key k_e is public
- A unique m satisfies the relation (with possibly several r)

At least exhaustive search on m and r can lead to m, maybe a better attack!

$\Rightarrow$ unconditional secrecy impossible

Algorithmic assumptions

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Integer Factoring and RSA

- Multiplication/Factorization :
  - $p, q \mapsto n = p.q$ easy (quadratic)
  - $n = p.q \mapsto p, q$ difficult (super-polynomial)

One-Way Function

- RSA Function, from $\mathbf{Z}_n$ in $\mathbf{Z}_n$ (with $n=pq$) for a fixed exponent e

Rivest-Shamir-Adleman 1978

  - $x \mapsto x^e \bmod n$ easy (cubic)
  - $y=x^e \bmod n \mapsto x$ difficult (without p or q)

RSA Problem

$x = y^d \bmod n$ where $d = e^{-1} \bmod \varphi(n)$ trapdoor

key

decryption

The Discrete Logarithm

- Let $\mathbf{G} = (<g>, \times)$ be any finite cyclic group
- For any $y \in \mathbf{G}$, one defines

$$\mathrm{Log}_g(y) = \min\{x \geq 0 \mid y = g^x\}$$

- One-way function
  - $x \rightarrow y = g^x$ easy (cubic)
  - $y = g^x \rightarrow x$ difficult (super-polynomial)

$$\boxed{\mathrm{Succ}_g^{\mathrm{dl}}(\mathcal{A}) = \Pr_{x \in \mathbf{Z}_q}\left[\mathcal{A}(y) = x \middle| y = g^x\right]}$$

The Diffie-Hellman Problems

- The Diffie-Hellman Problem (1976):
  - Given $A=g^a$ and $B=g^b$
  - Compute DH$(A,B) = C=g^{ab}$
- The **Decisional Diffie-Hellman Problem**:
  - Given A, B and C in $<g>$
  - Decide whether $C =$ DH(A,B)

$$\mathrm{Adv}_g^{\mathrm{ddh}}(\mathcal{A}) = \left| \begin{array}{l} \Pr_{a,b,c\in \mathbb{Z}_q}\left[\mathcal{A}(A,B,C)=1 \middle| A=g^a, B=g^b, C=g^c\right] \\ \quad - \Pr_{a,b\in \mathbb{Z}_q}\left[\mathcal{A}(A,B,C)=1 \middle| A=g^a, B=g^b, C=g^{ab}\right] \end{array} \right|$$

Complexity Estimates

Estimates for integer factoring Lenstra-Verheul 2000

	Modulus (bits)	Mips-Year ($\log_2$)	Operations (en $\log_2$)
	512	13	58
Mile-stone	1024	35	80
	2048	66	111
	4096	104	149
	8192	156	201

Can be used for RSA too
Lower-bounds for DL in $\mathbb{Z}_p^*$

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Algorithmic Assumptions *necessary*

- $n=pq$: **public modulus**
  e : **public exponent**
- $d=e^{-1} \bmod \varphi(n)$: **private**

RSA Encryption

$\mathbf{E}(m) = m^e \bmod n$

$\mathbf{D}(c) = c^d \bmod n$

If the RSA problem is easy,
secrecy is not satisfied:
anybody may recover m from c

Algorithmic Assumptions *sufficient?*

Security proofs give the guarantee that the assumption is ***enough*** for secrecy:

- if an adversary can break the secrecy
- one can break the assumption

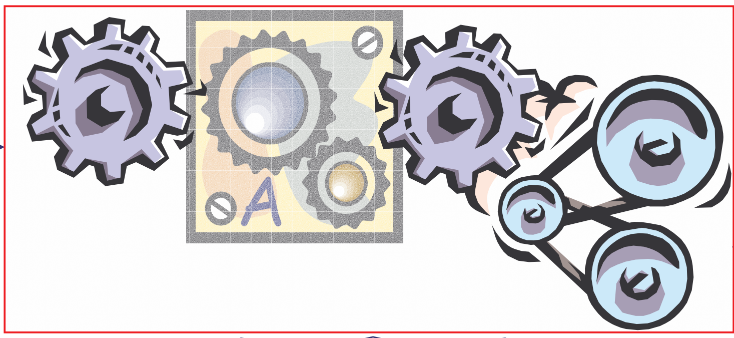
$\Rightarrow$ “reductionist” proof

Proof by Reduction

Reduction of a problem **P** to an attack *Atk*:

- Let *A* be an adversary that breaks the scheme then *A* can be used to solve **P**

Instance **I** of **P** $\rightarrow$



$\rightarrow$ Solution of **I**

P intractable $\Rightarrow$ scheme unbreakable

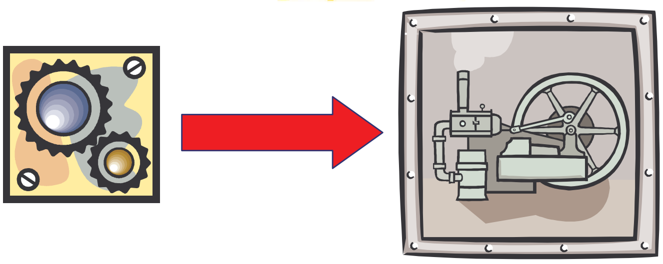
Provably Secure Scheme

To prove the security of a cryptographic scheme, one has to make precise

- the algorithmic assumptions
- the security notions to be guaranteed
- a reduction:
  an adversary can help
  to break the assumption

Practical Security

Adversary within t



Algorithm against **P** within $t' = T(t)$

- Complexity theory: T polynomial
- Exact Security: T explicit
- Practical Security: T small (linear)

Eg : $t' = 4t$

P intractable within less than 2^{80} operations
$\Rightarrow$ scheme unbreakable
within less than 2^{78} operations

Security Notions

According to the needs, one defines

- the goals of an adversary
- the means of an adversary,
  i.e. the available information

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   - Formal Security Model
   - Examples
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Asymmetric Encryption

- Formal Security Model
- Examples

Encryption Scheme

3 algorithms :

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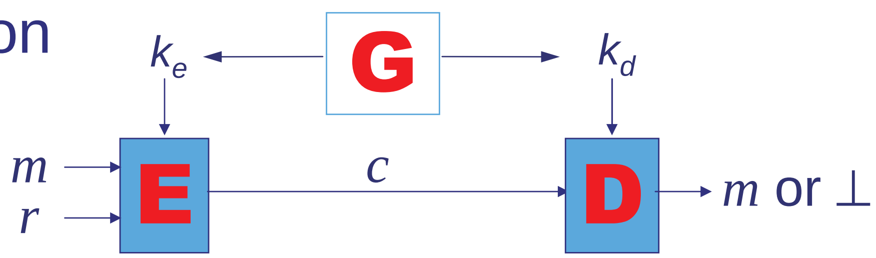


OW-Security: it is impossible to get back m just from c, k_e, **E** and **D** (without k_d)

Basic Secrecy

- One-Wayness (OW) :

  without the private key, it is computationally impossible to recover the plaintext

$$\mathrm{Succ}^{ow}(\mathcal{A}) = \Pr_{m,r}\left[\mathcal{A}(k_e, c) = m \middle| c = \mathbf{E}(m;r)\right]$$

Not enough if one already has some information about m :

- "Subject: XXXXX"
- "My answer is XXX" (XXX = Yes/No)

Strong Secrecy

- Semantic Security (IND - Indistinguishability) :

  GM 1984

  the ciphertext reveals *no more* information about the plaintext to a **polynomial adversary**

$$\mathrm{Adv}^{ind}(\mathcal{A}) =$$

$$2\Pr_{r,b}\left[\mathcal{A}_2(m_0, m_1, c, s) = b \middle| \begin{array}{c} (m_0, m_1, s) \leftarrow \mathcal{A}_1(k_e) \\ c \leftarrow \mathbf{E}(m_b, r) \end{array}\right] - 1$$

Non-Malleability

- Non-Malleability (NM):

DDN 1991

No polynomial adversary can derive, from a ciphertext $c = \mathbf{E}(m;r)$, a second one $c' = \mathbf{E}(m';r')$ so that the plaintexts m and m' are meaningfully related

non-malleability
$\Downarrow$
semantic security
$\Downarrow$
one-wayness

Basic Attacks

- Chosen-Plaintext Attacks (CPA)

In public-key cryptography setting,
the adversary can encrypt any message of his choice, granted the public key

$\Rightarrow$ the basic attack

Improved Attacks

- More information: oracle access
- Chosen-Ciphertext Attacks (CCA)

  The adversary has access to the strongest oracle: the decryption oracle

  The adversary can obtain the plaintext of any ciphertext of his choice (excepted the challenge)

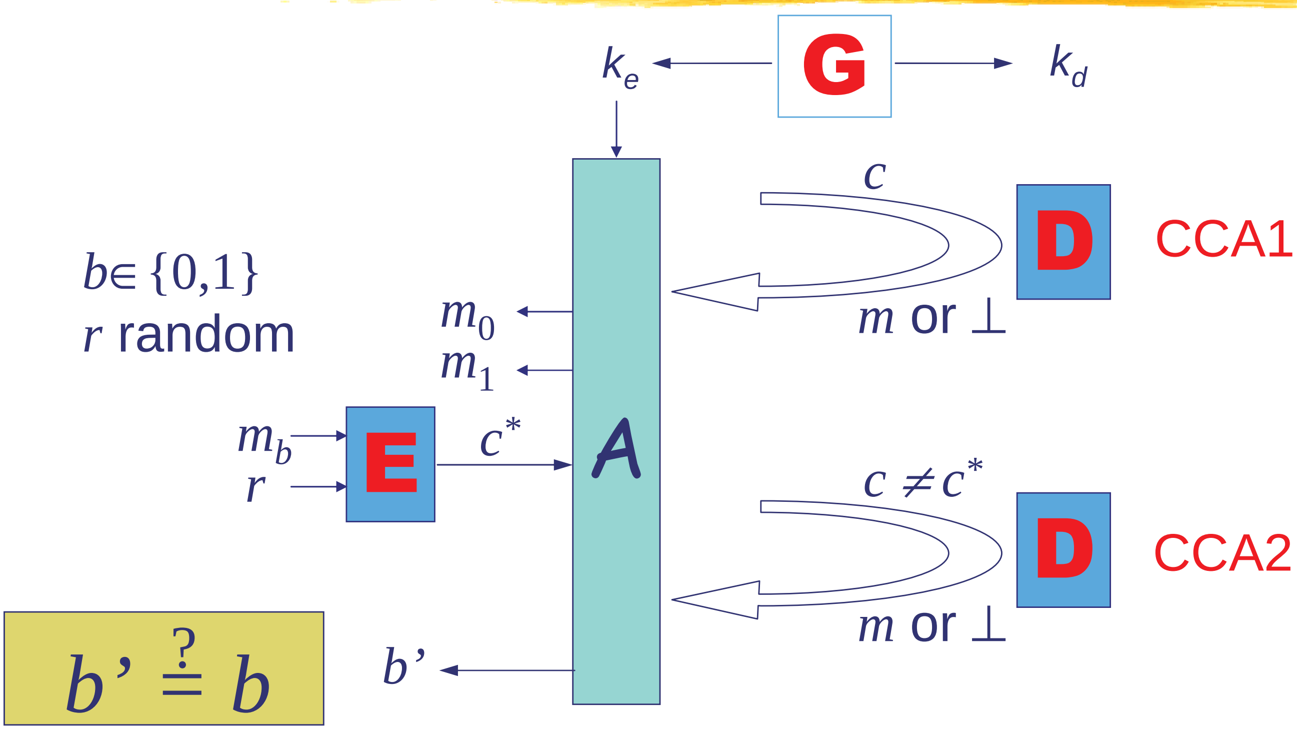
  – non-adaptive (CCA1) NY 1990

  only before receiving the challenge

  – adaptive (CCA2) RS 1991

  unlimited oracle access

IND-CCA2



Relations BDPR C-1998

Implications and separations

$$
\begin{array}{ccccc}
\text{NM-CPA} & \Leftarrow & \text{NM-CCA1} & \Leftarrow & \text{NM-CCA2} \\
\Downarrow & & \Downarrow & & \Updownarrow \\
\text{IND-CPA} & \Leftarrow & \text{IND-CCA1} & \Leftarrow & \text{IND-CCA2} \\
\Downarrow & & & & \\
\text{OW-CPA} & & & &
\end{array}
$$

minimal security

weak security

strong security: CCA

Asymmetric Encryption

- Formal Security Model
- Examples

RSA Encryption

- $n = pq$, product of large primes
- e, relatively prime to $\varphi(n) = (p\text{-}1)(q\text{-}1)$
- n, e : **public** key
- $d = e^{-1} \bmod \varphi(n)$: **private** key

$$\mathbf{E}(m) = m^e \bmod n \qquad \mathbf{D}(c) = c^d \bmod n$$

OW-CPA = RSA problem

Nothing to prove = definition

El Gamal Encryption

- $\mathbf{G} = (<g>, \times)$ group of order q
- x : **private** key
- $y=g^x$: **public** key

$$\mathbf{E}(m;a) = (g^a, y^a m) \rightarrow (c,d) \qquad \mathbf{D}(c,d) = d / c^x$$

OW-CPA = CDH Assumption

IND-CPA = DDH Assumption

To be proven to see the restrictions

El Gamal: OW-CPA

$$\mathbf{E}(m;a) = (g^a, y^a m) \rightarrow (c,d) \quad \mathbf{D}(c,d) = d / c^x$$

$$\mathrm{Succ}^{ow}(\mathcal{A}) = \Pr_{m,r}\left[\mathcal{A}(y,(c,d)) = m \,\middle|\, (c,d) = \mathbf{E}(m;a)\right]$$

$\mathcal{B}$ is given as input $\mathbf{G} = (<g>, \times)$ and (A,B)

- $y \leftarrow A$ and $c \leftarrow B$
- choose a random value d : $\mathcal{A}(y,(c,d)) \rightarrow m$
- output d/m

If m is correct, $\mathrm{DH}(A,B)=d/m$

$\mathrm{Succ}^{cdh}(\mathcal{B}) = \mathrm{Succ}^{ow}(\mathcal{A})$

El Gamal: IND-CPA

$$\mathrm{Adv}^{ind}(\mathcal{A}) = 2\Pr_{a,b}\left[\mathcal{A}_2(m_0,m_1,(c,d),s) = b \,\middle|\, \begin{array}{c}(m_0,m_1,s) \leftarrow \mathcal{A}_1(y)\\ (c,d) \leftarrow \mathbf{E}(m_b;a)\end{array}\right] - 1$$

$\mathcal{B}$ is given as input $\mathbf{G} = (<g>, \times)$ and (A, B, C)

- $y \leftarrow A$ and $c \leftarrow B$: $\mathcal{A}_1(y) \rightarrow (m_0, m_1)$
- $b \in \{0,1\}$ and $d \leftarrow C\, m_b$: $\mathcal{A}_2(c,d) \rightarrow b'$
- output $\beta = (b=b')$

Let us assume that $m_0, m_1 \in \mathbf{G}$:

- If $C=\mathrm{DH}(A,B)$, $\Pr[b=b'] = \Pr[\mathcal{A}(c,d) = b]$
- If $C\neq\mathrm{DH}(A,B)$, $\Pr[b=b'] = 1/2$

El Gamal: IND-CPA (Cnt'd)

If the messages are encoded into **G**:

- If C=DH(A,B), $\Pr[b=b'] = \Pr[\mathcal{A}(c,d) = b]$
- If C≠DH(A,B), $\Pr[b=b'] = 1/2$

$$\begin{aligned}\mathrm{Adv}^{\mathrm{ddh}}(\mathcal{B}) &= \Pr[\beta = 1 \mid C = \mathrm{CDH}(A,B)] - \Pr[\beta = 1 \mid C \neq \mathrm{CDH}(A,B)] \\ &= \Pr[b' = b\ ] - \frac{1}{2} = \frac{1}{2}\mathrm{Adv}^{ind}(\mathcal{A})\end{aligned}$$

$$\begin{aligned}\mathrm{Adv}(\mathcal{D}) &= 2\Pr[b' = b] - 1 \\ &= \Pr[b' = b \mid b = 1] + \Pr[b' = b \mid b = 0] - 1 \\ &= \Pr[b' = b \mid b = 1] - \Pr[b' \neq b \mid b = 0] \\ &= \Pr[b' = 1 \mid b = 1] - \Pr[b' = 1 \mid b = 0]\end{aligned}$$

Thus,

$\mathrm{Adv}^{ind}(t) \leq 2\ \mathrm{Adv}^{\mathrm{ddh}}(t')$

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Strong Security Notions

It is very difficult to reach CCA security

Maybe possible, but with inefficient schemes

Inefficient schemes are unuseful in practice:

Everybody wants security,
but only if it is transparent

Ideal Models

$\Rightarrow$ one makes some ideal assumptions:

- ideal random hash function:
  random oracle model
- ideal symmetric encryption:
  ideal cipher model
- ideal group:
  generic model (generic adversaries)

The Random Oracle Model

- Introduced by Bellare-Rogaway ACM-CCS '93
- The most admitted model
- It consists in considering some functions as perfectly random functions, or replacing them by random oracles:
  - each new query is returned a random answer
  - a same query asked twice receives twice the same answer

Modeling a Random Oracle

A usual way to model a random oracle H is to maintain a list Λ_H which contains all the query-answers (x,ρ):

- Λ_H is initially set to an empty list
- A query x to H is answered the following way
  - if for some ρ, $(x,\rho) \in \Lambda_H$, ρ is returned
  - otherwise,

    s a random ρ is drawn from the appropriate range
    s (x,ρ) is appended to Λ_H
    s ρ is returned

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Generic Construction Bellare-Rogaway '93

Let f be a trapdoor one-way permutation

then (with $G \rightarrow \{0,1\}^\ell$ and $H \rightarrow \{0,1\}^k$)

$$\mathbf{E}(m;r) = f(r) \,\|\, m \oplus G(r) \,\|\, H(m,r)$$

$$\mathbf{D}(a,b,c): \quad r = f^{-1}(a)$$

$$m = b \oplus G(r)$$

$$c = H(m,r)\ ?$$

IND-CCA2: Security Proof

Adversary $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2)$

- $\mathcal{A}_1(f) \rightarrow (m_0,m_1)$
- One randomly chooses $\beta \in \{0,1\}$ and r,

  and computes $C = \mathbf{E}(m_\beta,r) = (a,b,c)$:

  $$a = f(r),\ b = m_\beta \oplus G(r),\ c = H(m_\beta,r)$$
- $\mathcal{A}_2(C) \rightarrow \beta'$

with permanent access to

- the decryption oracle $\mathbf{D}$ — $q_{\mathbf{D}}$ queries
- the random oracles G and H — q_G, q_H queries

IND-CCA2: Security Proof (2)

Adversary $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2)$ - Simulator $\mathcal{B}$

- $\mathcal{B}(f, y=f(x))$: runs $\mathcal{A}_1(f) \rightarrow (m_0,m_1)$
- randomly chooses $b \in \{0,1\}^\ell$ and $c \in \{0,1\}^k$

  and outputs $C = \mathbf{E}(m_\beta,r) = (y,b,c)$

  this implicitly defines:

  $$r = f^{-1}(y) = x,\ G(r) = m_\beta \oplus b,\ H(m_\beta,r) = c$$
- $\mathcal{A}_2(C) \rightarrow \beta'$

IND-CCA2: Simulation (3)

$\mathcal{B}$ has to answer oracle queries:

- Random oracles G and H

  a new query is answered by

  a new random value in the proper range

  Problem if $G(r)$ (**AskG**) or $H(m_\beta,r)$ (**AskH**)

- Decryption oracle on $C' = (a',b',c')$

  one looks up for $c' = H(m',r')$

  and checks whether $C' = \mathbf{E}(m',r')$

  Problem if $H(m',r')$ not asked: rejection of a valid ciphertext (**BadD**), but with probability 2^{-k}

IND-CCA2: Simulation (4)

Without **AskG**, **AskH** or **BadD:** perfect simulation

New event **ASK**: $G(r)$ or $H(*,r)$

$$\begin{aligned}\Pr_0[\beta' = \beta] &\leq \Pr_1[\beta' = \beta] \\ &\quad + \Pr_1[\mathbf{AskG} \vee \mathbf{AskH}] + \Pr_1[\mathbf{BadD}] \\ &\leq \Pr_1[\beta' = \beta] + \Pr_1[\mathbf{ASK}] + q_{\mathbf{D}}\, 2^{-k}\end{aligned}$$

IND-CCA2: Extraction (5)

Without **ASK** adversary *A* has no information

$$\begin{aligned}\Pr_1[\beta' = \beta] &= \Pr_1[\beta' = \beta \mid \mathbf{ASK}]\, \Pr_1[\mathbf{ASK}] \\ &\quad + \Pr_1[\beta' = \beta \mid \neg\mathbf{ASK}]\, \Pr_1[\neg\mathbf{ASK}] \\ &\leq \Pr_1[\mathbf{ASK}] + \tfrac{1}{2}\end{aligned}$$

$$\begin{aligned}\mathrm{Succ}^{ow}(t') &\geq \Pr_1[\mathbf{ASK}] \geq \Pr_1[\beta' = \beta] - \tfrac{1}{2} \\ &\geq \Pr_0[\beta' = \beta] - \tfrac{1}{2} - \Pr_1[\mathbf{ASK}] - q_{\mathbf{D}}\, 2^{-k}\end{aligned}$$

$$\begin{aligned}\tfrac{1}{2}\,\mathrm{Adv}^{cca}(t) &\leq \Pr_0[\beta' = \beta] - \tfrac{1}{2} \\ &\leq 2\, \mathrm{Succ}^{ow}(t') + q_{\mathbf{D}}\, 2^{-k}\end{aligned}$$

where $t' = t + (q_G + q_H)\, T_f$

IND-CCA2: Result (6)

$$\boxed{\mathrm{Adv}^{cca}(t) \leq 4\mathrm{Succ}_f^{ow}(t + (q_G + q_H)T_f) + \frac{2q_{\mathbf{D}}}{2^k}}$$

If the parameters are properly chosen so that *f* is indeed hard to invert, the encryption scheme is semantically secure against any CCA-adversary, in the random oracle model