Hardness of Minimum Barrier Shrinkage and Minimum Installation Path

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Abstract

In the MINIMUM INSTALLATION PATH problem, we are given a graph G with edge weights $w(\cdot)$ and two vertices s, t of G. We want to assign a non-negative power $p\colon V \to \mathbb{R}_{\geq 0}$ to the vertices of G, so that the activated edges $\{uv \in E(G) \mid p(u) + p(v) \geq w(uv)\}$ contain some s-t-path, and minimize the sum of assigned powers. In the MINIMUM BARRIER SHRINKAGE problem, we are given, in the plane, a family of disks and two points x and y. The task is to shrink the disks, each one possibly by a different amount, so that we can draw an x-y curve that is disjoint from the interior of the shrunken disks, and the sum of the decreases in the radii is minimized.

We show that the MINIMUM INSTALLATION PATH and the MINIMUM BARRIER SHRINKAGE problems (or, more precisely, the natural decision problems associated with them) are weakly NP-hard.

Keywords: installation path, activation network, barrier problem, NP-hardness

1 Introduction

Let X be a subset of the plane, let x and y be points in X, and let $\mathbb{S}$ be a family of shapes in the plane. An x-y ***curve*** is a curve in $\mathbb{R}^2$ with endpoints x and y. We say that $\mathbb{S}$ ***separates*** x and y in X if each x-y curve contained in X intersects some shape from $\mathbb{S}$. Let $D(c, r)$ denote the *open* disk centered at c with radius r.

In this work we show that the following two decision problems are weakly NP-hard. This means that in our reduction we will use numbers that are exponentially large, but have polynomial length when written in binary.

> MINIMUM BARRIER SHRINKAGE.
> Input: a family $\{D(c_i, r_i) \mid i = 1, \ldots, n\}$ of n open disks; two points $x, y \in \mathbb{R}^2$; a real number C.
> Output: Whether there exist ***shrinking values*** $\delta_1, \ldots, \delta_n \geq 0$ such that their ***cost*** $\sum_i \delta_i$ is at most C and the family of open disks $\{D(c_i, r_i - \delta_i) \mid i = 1, \ldots n\}$ does not separate x and y in $\mathbb{R}^2$.

> MINIMUM INSTALLATION PATH.
> Input: a graph $G = (V, E)$ with positive edge weights $w\colon E \to \mathbb{R}_{>0}$; two vertices s and t of G; a real number C.
> Output: Whether there exists an assignment of ***powers*** $p\colon V \to \mathbb{R}_{\geq 0}$ to the vertices such that its ***cost*** $\sum_{v \in V} p(v)$ is at most C and the ***activated edges*** $E(p) = \{uv \in E \mid p(u) + p(v) \geq w(uv)\}$ contain an s-t-path.

We next discuss the motivating and closest related work.

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Minimum barrier shrinkage Kumar, Lai and A. Arora [10] introduced the following ***barrier resilience problem*** in the plane. The input is specified by a domain $X \subseteq \mathbb{R}^2$, a family $\mathbb{D}$ of disks in $\mathbb{R}^2$, and two points x and y in X. The task is to find an x-y curve in X that intersects as few disks of $\mathbb{D}$ as possible, without counting multiplicities. An alternative statement is that we want to find a minimum cardinality subfamily $\mathbb{D}' \subseteq \mathbb{D}$ such that $\mathbb{D} \setminus \mathbb{D}'$ does not separate x and y in X. The intuition is that we have sensors detecting movements from x to y, and we want to know how many sensors can suffer a total failure and still any agent moving from x to y within X is detected by some of the remaining sensors.

Kumar, Lai and A. Arora [10] showed that the problem can be solved in polynomial time when the domain X is a vertical strip bounded between two vertical lines ℓ and ℓ', the point x lies above and the point y lies below all disks of $\mathbb{D}$. Let us call this scenario the ***rectangular scenario***. The main insight is to consider the intersection graph G defined by $\mathbb{D} \cup \{\ell, \ell'\}$ and to note that the solution is the maximum number of ℓ-ℓ' internally vertex-disjoint paths in G. Thus, the problem can be solved in polynomial time by solving maximum flow problems. The same argument works for any family of shapes $\mathbb{S}$, not just disks, as far as each shape of $\mathbb{S}$ is connected.

Despite the claim in the preliminary version [9] of [10], we do not know whether the barrier resilience problem can be solved exactly in polynomial time when the domain X is all of $\mathbb{R}^2$. In fact, we know that when $X = \mathbb{R}^2$ and the family $\mathbb{D}$ of disks is replaced by some other family $\mathbb{S}$ of shapes, the problem is NP-hard [2, 8, 14]. The difference between the strip and the whole plane is that in the former case we can use Menger's theorem to relate the number of ℓ-ℓ' paths in the intersection graph of $\mathbb{S} \cup \{\ell, \ell'\}$ to the ℓ-ℓ' vertex connectivity, but no such statement applies to cycles that "separate" x and y. The computational complexity of the barrier problem in the plane for (unit) disks and (unit) squares is a challenging open problem, and several approximation algorithms have been devised [4, 6, 8].

Modeling the fact that sensors are less reliable further away from their placement, Cabello et al. [5] considered the problem of minimizing the total shrinkage of the disks such that there is an x-y curve disjoint from the interior of the disks. This is precisely the problem MINIMUM BARRIER SHRINKAGE. Cabello et al. also provided an FPTAS for the rectangular scenario. The algorithm uses the connection to vertex-disjoint paths.

We believe that showing NP-hardness for the problem MINIMUM BARRIER SHRINKAGE is interesting because of the computational complexity of two closely related problems, the barrier resilience problem for $X = \mathbb{R}^2$ and the minimum barrier shrinkage problem in the rectangular scenario, are unknown.

Minimum installation path There is a rich literature on so-called Activation Network problems. The task is to assign a power $p(v)$ to each vertex v of an edge-weighted graph $G = (V, E)$ so that the activated edges satisfy a certain connectivity property, such as for example spanning the whole graph. Whether an edge uv is activated depends only on $p(u)$ and $p(v)$. In the most general scenario, one only assumes an oracle telling, given $p(u)$ and $p(v)$, whether u and v are activated, together with a natural monotonicity constraint: if some choice of $p(u)$ and $p(v)$ activates uv, then increasing the powers at u and v still leaves uv activated. In many cases, the following simplifying assumption is made: the possible powers at the vertices are discretized as a finite set of values, denoted by D (the *domain*). See the survey by Nutov [12] for an overview of the area.

In this context, Panigrahi [13, Section 4.1] considered the MINIMUM ACTIVATION PATH problem: the connectivity constraint is that the activated edges must include a path between two fixed vertices s and t of G. He provided an algorithm with running time $O(\text{poly}(n, |D|))$, where n is the number of vertices of G and D is the finite domain of values for the power assignments.

Compared to the problem studied by Panigrahi, our MINIMUM INSTALLATION PATH has two differences. First, power assignments are not discretized and can be arbitrary nonnegative real numbers. Second, whether an edge uv is activated is simply determined by whether $p(u) + p(v) \geq w(uv)$. In this article, we show that the MINIMUM INSTALLATION PATH is weakly NP-hard. We also provide a simple fully polynomial time approximation scheme (FPTAS) relying on the algorithm by Panigrahi [13].

Our weak NP-hardness result of the MINIMUM INSTALLATION PATH problem is consistent with the result of Panigrahi. In our reduction, we use large *integer* weights: they have a polynomial bit length, but they are exponentially large. Taking $D = \mathbb{Z} \cap [0, \max_{uv} w(uv)]$ in the algorithm of Panigrahi, one only gets a pseudopolynomial time algorithm for such instances. This is consistent with a weakly NP-hardness proof.

In a similar vein, we remark that Alqahtani and Erlebach [1] presented algorithms parameterized by the treewidth of the graph in the case where the goal is to activate k node-disjoint st-paths, or node-disjoint paths between k pairs of terminals. See also Lando and Nutov [11] and Althaus et al. [3].

Relation between the problems We are not aware of any polynomial-time reduction from one problem to the other. Nevertheless, the NP-hardness proofs for both problems are very similar. The underlying connection between both problems is the following classical property: in a planar graph $G = (V, E)$, a set of edges $F \subset E$ is a minimum s-t cut if and only if, in the dual graph G^*, the edges $\{e^* \mid e \in F\}$ form a shortest cycle separating the face s^* from the face t^*. This relation does not directly provide a reduction even in the case of planar graphs, but does inspire the adaptation we make. Actually, our hardness proof for MINIMUM BARRIER SHRINKAGE reuses components of the hardness proof for MINIMUM INSTALLATION PATH; we reformulate some special instances of MINIMUM BARRIER SHRINKAGE in terms of graphs and then remark that each reformulated instance is equivalent to an instance of MINIMUM INSTALLATION PATH.

Organization It seems more convenient to present the NP-hardness of MINIMUM INSTALLATION PATH first. We achieve this in Section 2, together with an FPTAS for this problem. Then, in Section 3, we show that the MINIMUM BARRIER SHRINKAGE problem is NP-hard.

2 Minimum installation path

In this section we study the complexity of the MINIMUM INSTALLATION PATH problem is NP-hard. We first provide a simple FPTAS for this problem, and then prove that it is weakly NP-hard.

2.1 A simple FPTAS

As a side note, we show that the main idea used by Cabello et al. [5] can be adapted to lead to a simple FPTAS for MINIMUM INSTALLATION PATH. Let us consider an instance of that problem.

Lemma 1. *In polynomial time, we can compute the smallest value λ such that setting $p(v) = \lambda$ for all vertices v of G activates at least one st-path.*

Proof. Whether one st-path is activated by the power assignment $p(v) = \lambda$ (for each vertex v) depends only on the set of activated edges. So, for some edge uv, the minimum value of λ activating at least one st-path is the minimum value of λ activating edge uv. In other words, the minimum value of λ is necessarily of the form $w(uv)/2$ for some edge uv. So, for each edge uv, we determine whether putting power $w(uv)/2$ to all vertices activates an st-path, and return the smallest value that does so. □

Lemma 2. *Let OPT be the optimum value of the* MINIMUM INSTALLATION PATH *instance. Then:*

1. *in an optimal solution, the power assigned to every vertex is at most $n\lambda$, where n is the number of vertices of the input graph G;*

2. $\lambda \le OPT$.

Proof. 1. $\mathrm{OPT} \le n\lambda$ because the definition of λ implies a feasible solution of cost $n\lambda$. This implies (1);

2. $\lambda \le \mathrm{OPT}$ because otherwise, some st-path would be activated by some powers strictly smaller than λ at each vertex, contradicting the definition of λ. □

Proposition 3. MINIMUM INSTALLATION PATH *admits an FPTAS.*

Proof. Let $\varepsilon > 0$ be given. We first compute λ using Lemma 1. Then we apply the algorithm by Panigrahi [13, Section 4.1] to the instance, in which the domain is defined by

$$D = \{k\varepsilon\lambda/n \mid k = 0,\dots,n^2/\varepsilon\}.$$

By Lemma 2(1), we obtain a feasible solution to the original problem, the cost of which is within an additive error of at most $\varepsilon\lambda/n$ per vertex from OPT, hence with an additive error of at most $\varepsilon\lambda$ overall. By Lemma 2(2), this is at most $\varepsilon\mathrm{OPT}$. Clearly the running time is polynomial in n and $1/\varepsilon$. □

We can readily extend this argument to more general activation functions. For example, assume that each edge uv is activated if and only if $\alpha(uv)p(u) + \beta(uv)p(v) \ge w(uv)$, for some positive constants $\alpha(uv)$, $\beta(uv)$, and $w(uv)$. (Our setup corresponds to $\alpha(uv) = \beta(uv) = 1$.) The same argument as above shows that this extended version of MINIMUM INSTALLATION PATH admits an FPTAS.

2.2 Greedy solution in a path

In the rest of Section 2, we focus on proving NP-hardness of MINIMUM INSTALLATION PATH. We first consider the particular case of a path.

Consider a graph G and a path $\pi = v_0,\dots,v_n$ in G. We define greedily a power assignment p^*_π on the vertices of G to activate π, in a way that power is pushed forward along π as much as possible. Formally, the ***greedy power assignment along*** π is

$$p^*_\pi(v) = \begin{cases} 0 & \text{if } v \text{ does not belong to } \pi \text{ or } v = v_0,\\ 0 & \text{if } v = v_i,\ i > 0 \text{ and } p^*_\pi(v_{i-1}) \ge w(v_{i-1}v_i),\\ w(v_{i-1}v_i) - p^*_\pi(v_{i-1}) & \text{if } v = v_i,\ i > 0 \text{ and } p^*_\pi(v_{i-1}) < w(v_{i-1}v_i). \end{cases} \tag{1}$$

For a power assignment p, let $\mathrm{cost}(p)$ denote the total cost of p, namely, the sum of the powers at the vertices. For path π, let $\mathrm{opt}(\pi)$ be the cost of the minimum cost power assignment that activates π. The following lemma tells that the greedy power assignment along π has minimum cost to activate π.

Lemma 4. *For each path* π*,* $\mathrm{cost}(p^*_\pi) = \mathrm{opt}(\pi)$.

Proof. It is clear that p^*_π activates all the edges of π. Let p be another power assignment activating all edges of π. We have to show that $\mathrm{cost}(p^*_\pi) \le \mathrm{cost}(p)$.

We can assume that $p(v) = 0$ at all vertices v outside π. Otherwise, we change p to have this property. This reassignment of power would decrease the cost and would keep activating the path π.

The strategy is to gradually transform p into p^*_π while keeping all edges of π activated and without increasing the value of $\mathrm{cost}(p)$. The property $\mathrm{cost}(p^*_\pi) \le \mathrm{cost}(p)$ is trivially correct if $p = p^*_\pi$. So assume $p \ne p^*_\pi$ and let i be the smallest integer such that $p(v_i) \ne p^*_\pi(v_i)$. Because all edges are activated, and by construction of p^*_π, we must have $p(v_i) > p^*_\pi(v_i)$. Let $\Delta = p(v_i) - p^*_\pi(v_i) > 0$. There are two cases:

- Assume $i \leq n-1$. Update p by decreasing $p(v_i)$ by Δ and increasing $p(v_{i+1})$ by Δ. Since each edge of π is activated by p^*_π and by p before this transformation, each edge of π is still activated by the new p. Moreover, $\text{cost}(p)$ is unchanged.
- Assume $i = n$. Update p by decreasing $p(v_n)$ by Δ. Again, each edge of π is still activated. The cost has decreased by Δ.

This transformation does not increase the value of $\text{cost}(p)$. Moreover, the new power assignment coincides with p^*_π on vertices $v_0, \ldots, v_i$. Thus, after a finite number of steps, $p = p^*_\pi$. This proves the lemma. □

For the path $\pi = v_0, \ldots, v_n$, let $\varphi(\pi) = p^*_\pi(v_n) \geq 0$. That is, $\varphi(\pi)$ is the power assignment given by the greedy power assignment along π to the final vertex. Since $p^*_\pi(v_n)$ depends on $p^*_\pi(v_{n-1})$, we have the following.

Lemma 5. *Let π be the path $v_0, \ldots, v_n$ and let π' be the path $v_0, \ldots, v_n, u$. (Thus, π' extends π by an additional edge $v_n u$.) Then $\varphi(\pi') = \max\{0, w(v_n u) - \varphi(\pi)\}$ and $\text{opt}(\pi') = \text{opt}(\pi) + \varphi(\pi')$.*

Proof. From the definition of the greedy power assignment along π and π', the power assignments p^*_π and $p^*_{\pi'}$ differ only at vertex u. We have:

$$\begin{aligned} \varphi(\pi') = p^*_{\pi'}(u) &= \max\{0, w(v_n u) - p^*_{\pi'}(v_n)\} = \max\{0, w(v_n u) - p^*_\pi(v_n)\} \\ &= \max\{0, w(v_n u) - \varphi(\pi))\}. \end{aligned}$$

This proves the claim for $\varphi(\pi')$. Because of Lemma 4 for π and π' we also get

$$\begin{aligned} \text{opt}(\pi') &= \text{cost}(p^*_{\pi'}) = \text{cost}(p^*_\pi) + p^*_{\pi'}(u) - p^*_\pi(u) \\ &= \text{opt}(\pi) + \varphi(\pi') - 0. \end{aligned}$$

□

A consequence of Lemma 4 is the following integrality property.

Lemma 6. *Assume that the weight function $w\colon E(G) \to \mathbb{R}_{>0}$ takes only integer values, and that C is also an integer. Then, for any $\alpha \in [0, 1)$,* Minimum Installation Path*(G, w, s, t, C) has a positive answer if and only if* Minimum Installation Path*$(G, w, s, t, C + \alpha)$ has a positive answer.*

Proof. Assume that Minimum Installation Path$(G, w, s, t, C + \alpha)$ is has a positive answer. Consider a power assignment p corresponding to a feasible solution of minimum cost (at most $C + \alpha$); let π be an s-t path activated by p. Because of Lemma 4 we have $\text{cost}(p) = \text{opt}(\pi) = \text{cost}(p^*_\pi)$. From the inductive definition (1) of p^*_π, we see that p^*_π assigns integral powers to all vertices, and thus $\text{cost}(p^*_\pi) = \sum_v p^*_\pi(v)$ is an integer, which is at most C. So Minimum Installation Path(G, w, s, t, C) has a positive answer. □

2.3 The reduction

Now we provide the reduction. The reduction is inspired by the reduction used to show that the restricted shortest path problem is NP-hard; this seems to be folklore and attributed to Megiddo by Garey and Johnson [7, Problem ND30]. We use the notation $[n] = \{1, \ldots, n\}$ and reduce from the following problem.

> Subset Sum
> Input: a sequence $a_1, \ldots, a_n$ of positive integers and a positive integer b.
> Question: is there a set of indices $I \subseteq [n]$ such that $\sum_{i \in I} a_i = b$?

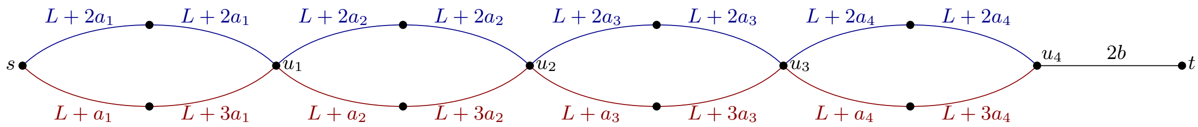

Figure 1: The graph G when $n = 4$.

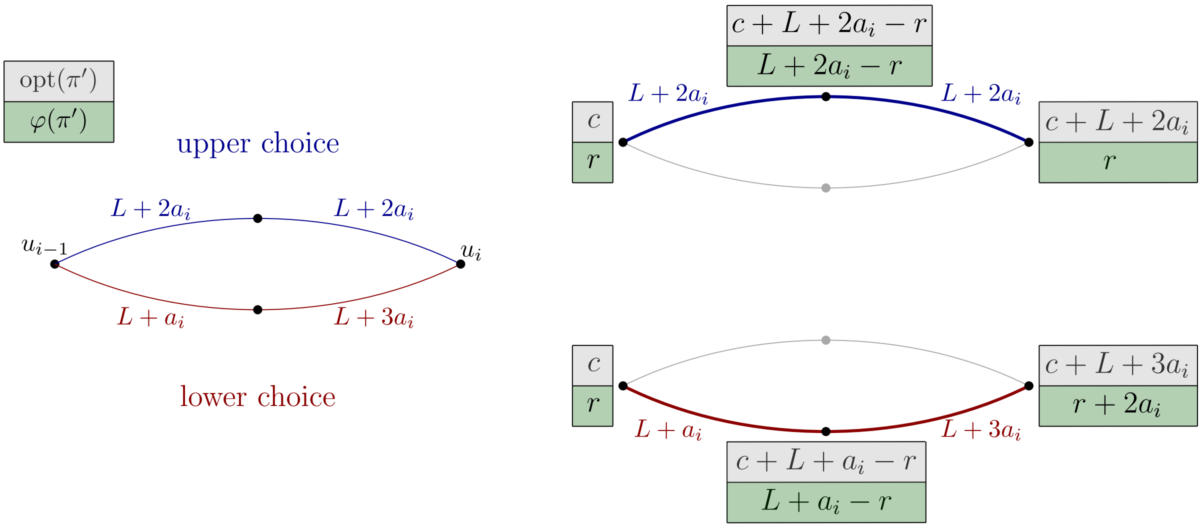

Figure 2: Left: Upper and lower choice at i. Right: the change in $\text{opt}(\pi')$ and $\varphi(\pi')$ depending on whether the path is extended by the upper or the lower choice.

The problem SUBSET SUM is one of the standard weakly NP-hard problems that can be solved in pseudopolynomial time via dynamic programming [7, Section 4.2]. In particular, when the numbers a_i are bounded by a polynomial in n, the problem can be solved in polynomial time.

Set L to be an integer strictly larger than $2\sum_{i\in[n]} a_i$. Then, for each $I \subseteq [n]$ we have $2\sum_{i\in I} a_i < L$.

We construct a graph $G = G(a_1,\dots,a_n,b)$ as follows (see Figure 1). G will include vertices s, t, $u_1,\dots,u_n$. Let us use the notation $u_0 = s$. For each $i \in [n]$, we put between u_{i-1} and u_i two paths, each of length two, one path with weights $L+2a_i$ and $L+2a_i$, and the other path with weights $L+a_i$ and $L+3a_i$, as we go from u_{i-1} to u_i. Finally, we put the edge $u_n t$ with weight $2b$. This finishes the construction of G.

Lemma 7. *There exists a path π from s to u_n in G with $\text{opt}(\pi) = c$ and $\varphi(\pi) = r$ if and only if there exists $I \subseteq [n]$ such that*

$$r = 2\sum_{i\in I} a_i \quad \text{and} \quad c = nL + 2\sum_{i\in[n]} a_i + \sum_{i\in I} a_i.$$

Proof. Consider the two paths, each of length two, connecting u_{i-1} to u_i. The *upper choice at i* is the path with weights $L+2a_i$; similarly, the *lower choice at i* is the path with weights $L+a_i$ and $L+3a_i$. See Figure 2.

Assume that we have a path π' that goes from $s = u_0$ to u_{i-1} with $\varphi(\pi') \le L$. Let π'_u be the concatenation of π' with the upper choice, and let π'_ℓ be the concatenation of π' with the lower choice. Because of Lemma 5, we obtain that $\text{opt}(\pi'_u) = \text{opt}(\pi') + L + 2a_i$ and $\varphi(\pi'_u) = \varphi(\pi')$, while $\text{opt}(\pi'_\ell) = \text{opt}(\pi') + L + 3a_i$ and $\varphi(\pi_\ell) = \varphi(\pi') + 2a_i$. See Figure 2. Here, the assumption $\varphi(\pi') \le L$ has been important to ensure that in using Lemma 5 the maximum defining $\varphi(\cdot)$ is not at 0. It easily follows by induction on i that, for each path π' from $s = u_0$ to u_i, we indeed have $\varphi(\pi') \le \sum_{j=1}^{i} 2a_i$, and thus the hypothesis is fulfilled for each $i \in [n]$.

The intuition here is that the lower choice has a larger cost, but keeps more power at the extreme of the prefix path for later use. See Figure 3 for a concrete example showing the values $\text{opt}(\pi')$ and

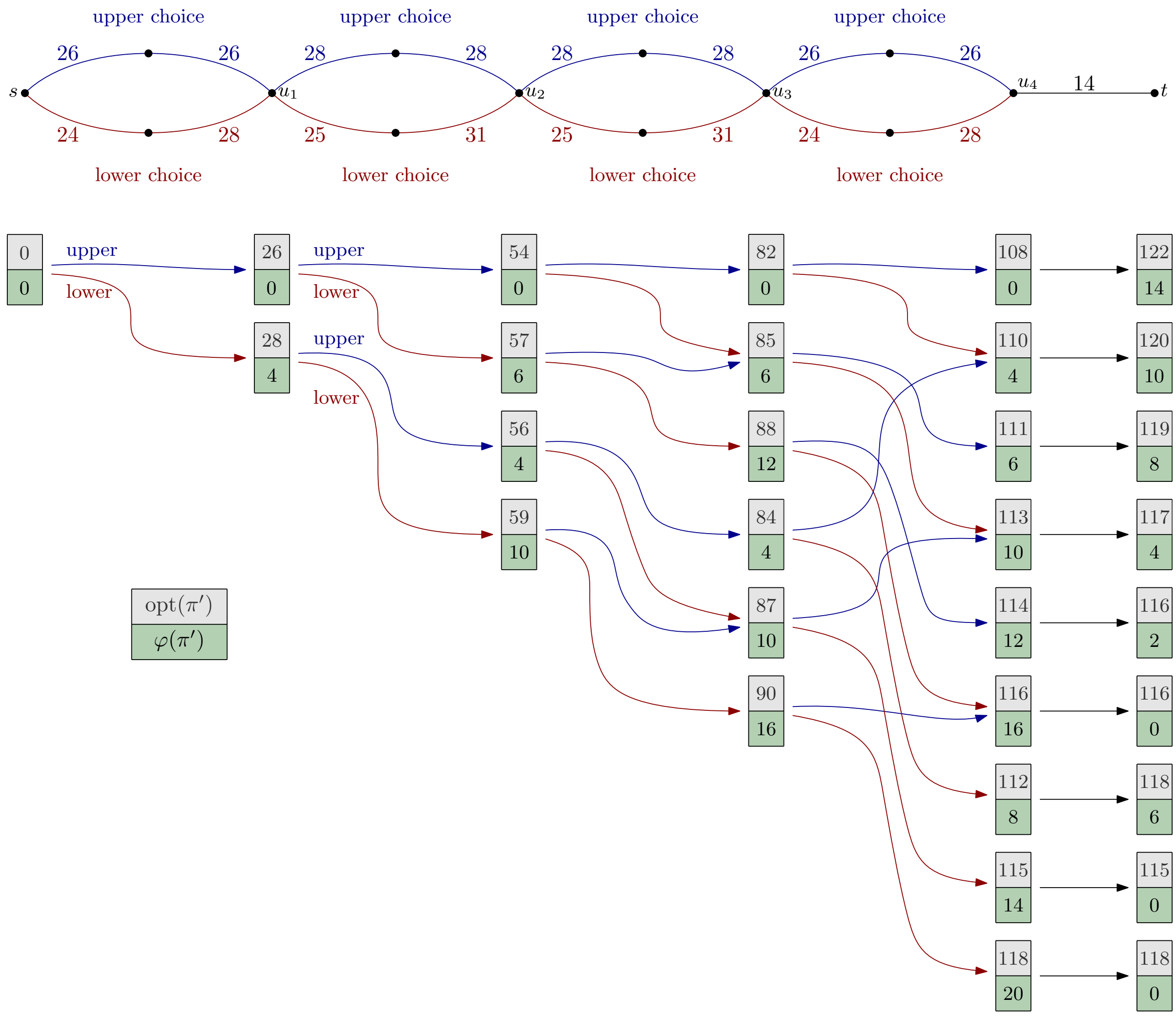

Figure 3: Top: The graph G for $n = 4$ with $a_1, \dots, a_4 = 2, 3, 3, 2$ and $b = 7$, when we take $L = 22$. We have to decide whether there is an assignment of power with cost $nL + 2\sum_i a_i + b = 115$ that activates some s-t path. Bottom: pairs $(\mathrm{opt}(\pi'), \varphi(\pi'))$ for all the s-u_i paths π'.

$\varphi(\pi')$ for paths π' from $s = u_0$ to u_i. It also helps understanding the idea behind the reduction.

Consider now a path π from $s = u_0$ to u_n. Let I be the set of indices $i \in [n]$ where the path takes the lower choice at i. From the previous discussion and a simple induction we have

$$\mathrm{opt}(\pi) \;=\; \sum_{i \in [n] \setminus I} (L + 2a_i) + \sum_{i \in I} (L + 3a_i) \;=\; nL + \sum_{i \in [n]} 2a_i + \sum_{i \in I} a_i$$

and

$$\varphi(\pi) \;=\; 2 \sum_{i \in I} a_i \;\leq\; L.$$

Since all the paths from s to u_n must follow the upper or lower choice at each $i \in [n]$, the result follows. ☐

Lemma 8. *For any real numbers A and B we have*

$$A + \max\{2B - 2A, 0\} \leq B \quad \Longrightarrow \quad A = B.$$

Proof. If $A \leq B$, then $B - A \geq 0$ and the assumption implies $A + (2B - 2A) \leq B$, which implies $B \leq A$, and thus $A = B$. If $A > B$, then $B - A < 0$ and the assumption implies $A + 0 \leq B$, which implies $A \leq B$. Thus this cannot happen. ☐

Theorem 9. *The problem* MINIMUM INSTALLATION PATH *is NP-hard.*

Proof. We show that the instance for SUBSET SUM has a positive answer if and only if in the graph $G = G(a_1, \dots, a_n, b)$ there is a power assignment with cost at most $C := nL + 2\sum_{i\in[n]} a_i + b$ that activates some path from s to t.

Assume that the exists a solution for the instance to the SUBSET SUM problem. This means that we have some $I \subseteq [n]$ such that $\sum_{i\in I} a_i = b$. Because of Lemma 7, there exists a path π from $s = u_0$ to u_n with optimal installation cost $\mathrm{opt}(\pi) = nL + 2\sum_{i\in[n]} a_i + \sum_{i\in I} a_i = C$ and $\varphi(\pi) = 2b$. Because of Lemma 4, this means that the power assignment p^*_π has cost $\mathrm{cost}(p^*_\pi) = C$, activates all edges of π, and assigns power $p^*_\pi(u_n) = \varphi(\pi) = 2b$ to vertex u_n. Such power assignment p^*_π also activates the edge $u_n t$ because it has weight $2b = p^*_\pi(u_n)$. (In particular, the vertex t gets power 0.)

Assume now that there is a power assignment $p' \geq 0$ with cost at most C that activates a path π' from s to t. Let π be the restriction of π' from s to u_n. Because of Lemma 5 and using that the power assignment p' activates π', we have

$$\mathrm{opt}(\pi) + \max\{2b - \varphi(\pi), 0\} \;=\; \mathrm{opt}(\pi') \;\leq\; \mathrm{cost}(p') \;\leq\; C. \tag{2}$$

Because of Lemma 7, there exists some $I \subseteq [n]$ such that

$$\mathrm{opt}(\pi) \;=\; nL + 2\sum_{i\in[n]} a_i + \sum_{i\in I} a_i \quad \text{and} \quad \varphi(\pi) \;=\; 2\sum_{i\in I} a_i.$$

Substituting in (2), for such $I \subseteq [n]$ we have

$$nL + 2\sum_{i\in[n]} a_i + \sum_{i\in I} a_i + \max\left\{2b - 2\sum_{i\in I} a_i, 0\right\} \;\leq\; C \;=\; nL + 2\sum_{i\in[n]} a_i + b.$$

This means that

$$\sum_{i\in I} a_i + \max\left\{2b - 2\sum_{i\in I} a_i, 0\right\} \;\leq\; b.$$

Because of Lemma 8 we conclude that $\sum_{i\in I} a_i = b$, and the given instance to SUBSET SUM problem has a solution. □

3 Minimum Barrier Shrinkage

In this section, we show that the MINIMUM BARRIER SHRINKAGE problem is NP-hard. The structure of the proof is very similar to the proof given in Section 2.3 for the NP-hardness of the problem MINIMUM INSTALLATION PATH.

We first give the construction assuming that we can compute algebraic numbers to infinite precision. Then we explain how an approximate construction with enough precision suffices and can be computed in polynomial time.

The ***penetration depth*** of a pair $(D(c, r), D(c', r'))$ of open disks $D(c, r)$ and $D(c', r')$ is $r + r' - |c - c'|$, where $|c - c'|$ is the distance between the centers c and c'. When no disk contains the center of the other disk, and they intersect, then the intersection $D(c, r) \cap D(c', r')$ is a lens of width equal to the penetration depth. See Figure 4. If we shrink the disks to $D(c, r - \delta)$ and $D(c', r' - \delta')$, the disks intersect if and only if $\delta + \delta'$ is strictly smaller than the penetration depth. (Recall that disks are taken as open sets.) Thus, the penetration depth equals the minimum total shrinking of the disks so that a curve can pass between the two disks.

We reduce again from SUBSET SUM. Consider an instance I of SUBSET SUM given by a sequence $a_1, \dots, a_n$ of positive integers and a positive integer b. Set L to be an integer strictly larger than $2\sum_{i\in[n]} a_i$. Then, for each $I \subseteq [n]$ we have $2\sum_{i\in I} a_i < L$. Set $C = nL + 2\sum_{i\in[n]} a_i + b$ and $\lambda = 10C$.

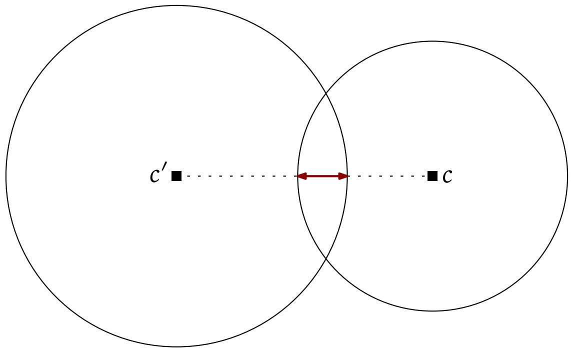

Figure 4: The penetration depth of the pair of drawn disks is the length of the arrow.

We will construct an instance to MINIMUM BARRIER SHRINKAGE problem such that it has a solution if and only if the instance I for SUBSET SUM has a solution.

Figure 5 shows the overall idea of the construction. Most of the action is happening around the filled (blue and green) disks. The remaining white disks create corridors to communicate from one side to the other of the filled disks. To provide a feasible solution of cost at most C, we have to indicate how to shrink the disks for a total radius of at most C and provide an x-y curve in the plane that does not touch the (interior of the) shrunken disks.

In our construction, no point of the plane will be covered by more than two disks. In such a case, the x-y curve can be described combinatorially by a sequence of pairs of disks such that, for each pair (D, D'), the curve passes between D and D', after shrinking.

If the penetration depth of two disks is at least $\lambda = 10C$, then, in any shrinking of the disks with total cost at most C, those two disks keep intersecting, which means that we cannot route the x-y curve between those two disks. More precisely, the segment connecting the centers of such disks cannot be crossed by the x-y curve. In the drawings we indicate this with a thick segment connecting the centers of the disks.

The main part to encode the instance, around the filled disks, consists of the following disks. See Figures 6 and 7.

- For $i = 0, \dots, n$, a disk D_i of radius 4λ centered at $((4\lambda) \cdot 2i, 0)$;
- for each $i \in [n]$, a disk D'_i of radius λ centered at $((4\lambda) \cdot (2i - 1), 0)$;
- for each $i \in [n]$, a disk A_i (for *above*) of radius 3λ placed such that the center is above the x-axis, and the penetration depth of (A_i, D_{i-1}) and (A_i, D_i) is $L + 2a_i$; this means that the distance between center(A_i) and center(D_{i-1}) is $7\lambda - (L + 2a_i)$, and the distance between center(A_i) and center(D_i) is $7\lambda - (L + 2a_i)$;
- for each $i \in [n]$, a disk B_i (for *below*) of radius 3λ placed such that the center is below the x-axis, the penetration depth of (B_i, D_{i-1}) is $L + a_i$ and the penetration depth of (B_i, D_i) is $L + 3a_i$; this means that the distance between center(B_i) and center(D_{i-1}) is $7\lambda - (L + a_i)$ and the distance between center(B_i) and center(D_i) is $7\lambda - (L + 3a_i)$;
- a disk A_{n+1} of radius 3λ placed such that the center is above the x-axis, the x-coordinate of center(A_{n+1}) is $(4\lambda) \cdot (2n + 1)$, and the penetration depth of (A_{n+1}, D_n) is $2b$; A_{n+1} is one of the green disks in the figures;
- a disk B_{n+1} of radius 3λ placed such that the center is below the x-axis, the x-coordinate of center(B_{n+1}) is $(4\lambda) \cdot (2n + 1)$, and the penetration depth of (B_{n+1}, D_n) is $2b$; B_{n+1} is another of the green disks in the figures;

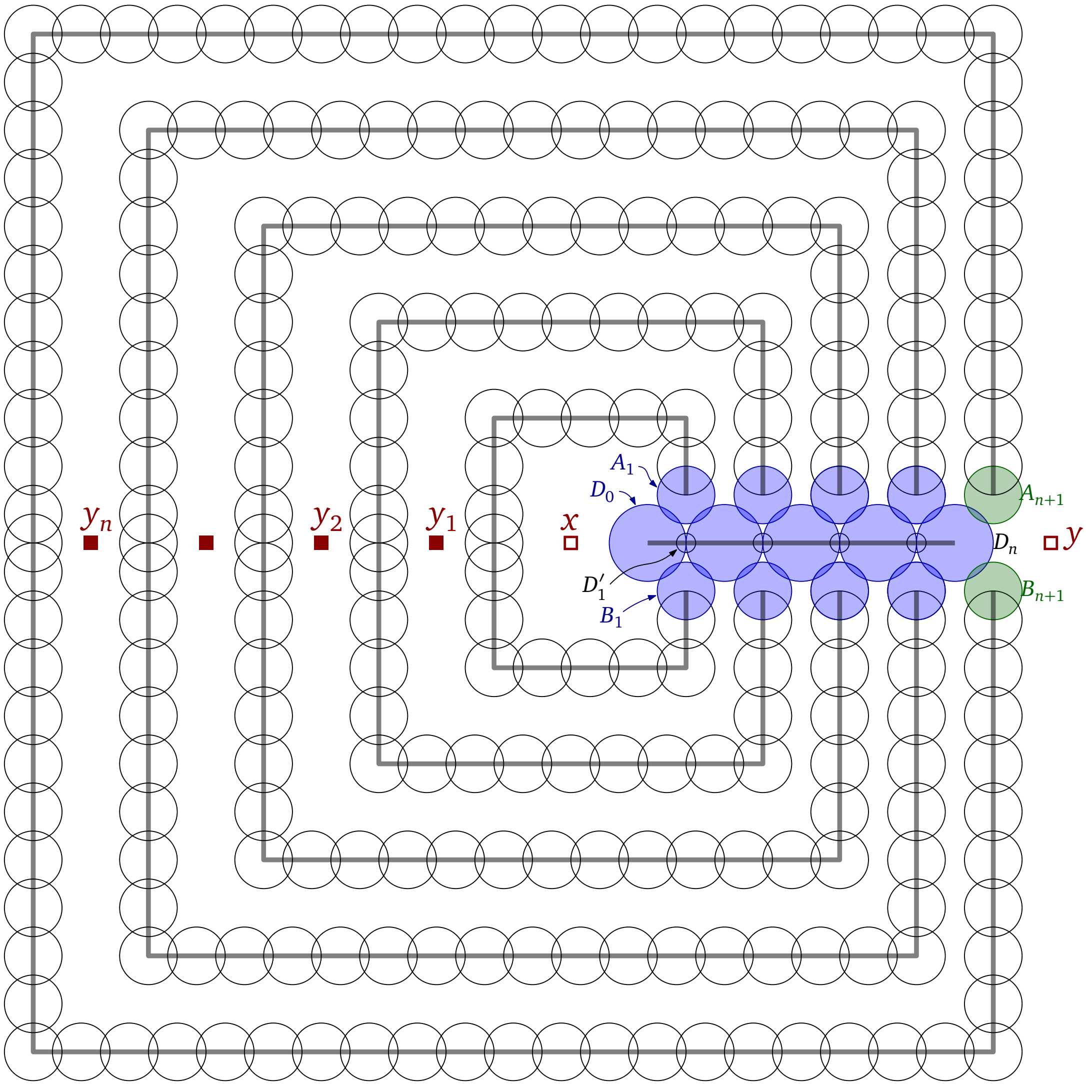

Figure 5: Basic idea of the construction for $n = 4$. All the shrinking of disks and the decisions on how to route the x-y curve are happening around the (blue and green) filled disks. The thick lines will not be crossed by any x-y curve that is disjoint from the shrunken disks in a solution with the desired cost.

- for each $i \in [n+1]$, a disk A'_i of radius 3λ centered at $((4\lambda)\cdot(2i-1), 8\lambda)$ and a disk B'_i of radius 3λ centered at $((4\lambda)\cdot(2i-1), -8\lambda)$.

For $i \in [n]$, the ***block*** $\mathbb{B}_i$ consists of the disks D_{i-1}, D_i, D'_i, A_i, A'_i, B_i and B'_i. We also define the block $\mathbb{B}_{n+1}$ as the group of disks D_n, A_{n+1}, A'_{n+1}, B_{n+1} and B'_{n+1}. Note that the blocks $\mathbb{B}_i$ and $\mathbb{B}_{i+1}$, for $i \in [n]$, share the disk D_i.

For each $i \in [n+1]$, we make a *path* of disks of radius 3λ, starting from A'_i and finishing with B'_i, where any two consecutive disks have penetration depth at least 3λ. The disks in these paths are pairwise disjoint for different indices i, and disjoint from the rest of the construction. The disks in each such path can be centered along a 5-link axis-parallel path, and it uses $O(i)$ disks. See Figure 5. We denote the path of disks for the index $i \in [n+1]$ by Π_i. For later use, we place a point y_i in the "tunnel" between the paths Π_i and Π_{i+1}. See Figure 5.

Lemma 10. *For each $i \in [n]$, the disks D'_i, A_i and B_i are pairwise disjoint. Moreover, the penetration depth of the pairs (D_{i-1}, D'_i), (D_i, D'_i), (A_i, A'_i) and (B_i, B'_i) is at least λ. For $\mathbb{B}_{n+1}$, the disks A_{n+1} and B_{n+1} are disjoint and the penetration depth of the pairs (A_{n+1}, A'_{n+1}) and (B_{n+1}, B'_{n+1}) is at least λ.*

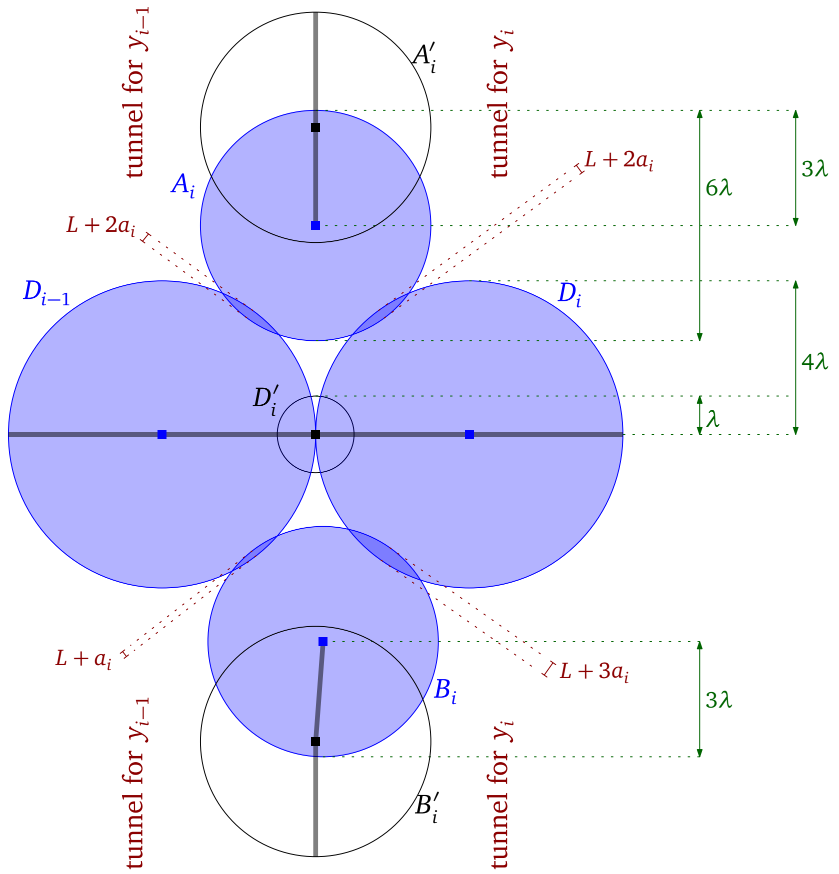

Figure 6: Block $\mathbb{B}_i$ for $1 < i < n$; the penetration depths are not to scale. Note that A_i has the same overlap with D_{i-1} and D_i, while B_i is moved closer to D_i. The center of B_i is to the right of the (vertical) line through the centers of A'_i, A_i, D'_i and B'_i.

Proof. We consider only the case $i \in [n]$. The arguments for $\mathbb{B}_{n+1}$ are similar. The penetration depth of the pairs (D_{i-1}, D'_i) and (D_i, D'_i) is λ by construction.

Consider the disk $\tilde{A}_i$ of radius 3λ centered at $((4\lambda)\cdot(2i-1), 5\lambda)$ and the disk $\tilde{B}_i$ of radius 3λ centered at $((4\lambda)\cdot(2i-1), -5\lambda)$. See Figure 8. We will compare B_i to $\tilde{B}_i$; note that they have the same size, just a different placement. The argument for A_i is the same.

The penetration depth of the pairs $(\tilde{B}_i, D_{i-1})$ and $(\tilde{B}_i, D_i)$ is

$$3\lambda + 4\lambda - \sqrt{(5\lambda)^2 + (4\lambda)^2} \;=\; (7-\sqrt{41})\lambda \;\approx\; 0.59687\lambda,$$

while the penetration depth of $(\tilde{B}_i, B'_i)$ is exactly 3λ. The disk D'_i is at distance λ from $\tilde{B}_i$.

Since the penetration depth of (B_i, D_{i-1}) and (B_i, D_i) is at most $L + 3a_i \leq C = \lambda/10$, these penetration depths are smaller than the penetration depths of $(\tilde{B}_i, D_{i-1})$ and $(\tilde{B}_i, D_i)$, namely, between 0 and 0.59687λ. See Figure 9. As can be seen on the figure (and proved by a slightly involved computation), this implies that B_i and D'_i are disjoint, and that the disk B'_i contains the center of B_i. The latter fact implies that the penetration depth of (B_i, B'_i) is at least 3λ. □

From Lemma 10 we conclude that, in any solution with cost under $\lambda = 10C$, the x-y curve cannot cross the segments connecting center(D_{i-1}) and center(D_i), the segments connecting center(A_i) and center(A'_i), nor the segments connecting center(B_i) and center(B'_i), for each $i \in [n+1]$. Furthermore, it cannot cross the path Π_i connecting A'_i to B'_i, for each $i \in [n+1]$. This implies that, at each block $\mathbb{B}_i$, we have to decide whether the x-y curve goes above (crossing A_i before shrinking) or below (crossing B_i before shrinking). See Figure 10 for one such choice.

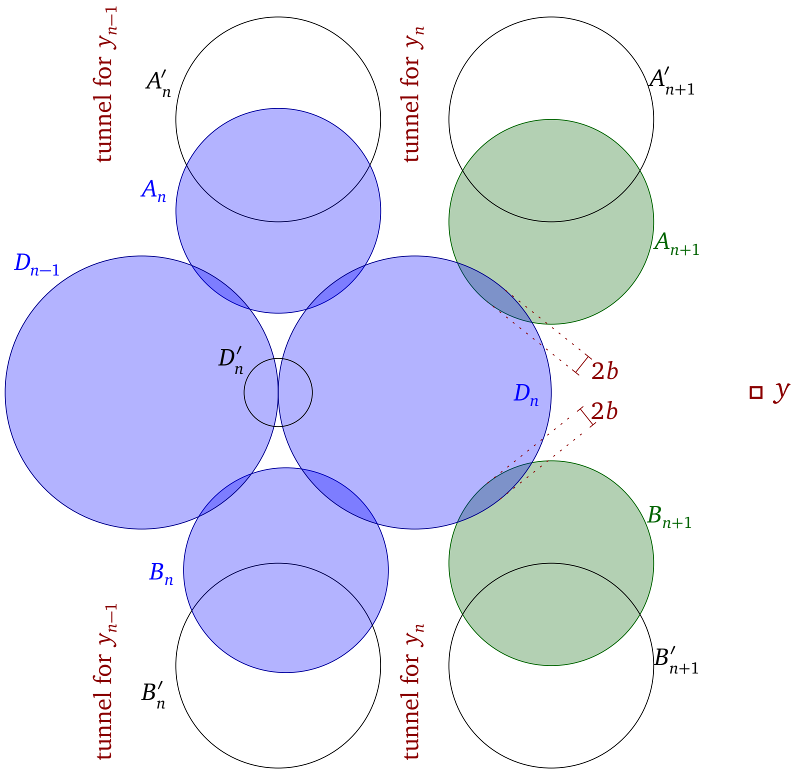

Figure 7: The blocks $\mathbb{B}_n$ and $\mathbb{B}_{n+1}$; the penetration depths are not to scale.

So, in a nutshell, the strategy is to reformulate the problem in terms of graphs, and to note that the instance is equivalent to the Minimum Installation Path in that graph. Let $X_i = A_i$ or $X_i = B_i$, depending on the choice of how to route the x-y curve. If $X_i = A_i$, then the x-y curve, after shrinking the disks, passes between D_{i-1} and A_i, and also between D_i and A_i. If $X_i = B_i$, then the x-y curve, after shrinking the disks, passes between D_{i-1} and B_i, and also between D_i and B_i. Note that we can assume that the x-y curve passes between two disks at most once. Moreover, for each disk D, the x-y curve passes between D and another disk at most twice. Once we decide the combinatorial routing of the x-y curve, that is, once we select $X_1, \dots, X_n, X_{n+1}$, then greedily shrinking the disks gives an optimal solution, similarly to Lemma 4: it pays off to push the shrinking towards disks that are crossed later by the x-y curve. That is, to pass between D_1 and X_1, it pays off to do not shrink D_1, as it is never crossed again, and shrink X_1 just enough to pass in between. Similarly, it pays off to shrink D_2 to pass between D_2 and X_1, because X_1 will not be crossed again later on. In general, to pass between D_{i-1} and X_i it pays off to reduce X_i just enough to pass between them, taking into account how much D_{i-1} was already shrunken, and to pass between X_i and D_i it pays off to reduce D_i just enough to pass between them, taking into account how much X_i was already reduced.

Let $\mathbb{D} = \mathbb{D}(I)$ be the set of all disks in the constructed instance.

Lemma 11. *The instance $I = (a_1, \dots, a_n, b)$ to* Subset Sum *has a solution if and only if the instance $(\mathbb{D}, x, y, C)$ to* Minimum Barrier Shrinkage *has a positive answer, where $C = nL + 2\sum_{i \in [n]} a_i + b$. Furthermore, for any $\alpha \in [0,1)$,* Minimum Barrier Shrinkage*$(\mathbb{D}, x, y, C)$ has a positive answer if and only if* Minimum Barrier Shrinkage*$(\mathbb{D}, x, y, C + \alpha)$ has a positive answer.*

Proof. We construct a graph G' as follows. We make a node for each connected component of $\mathbb{R}^2 \setminus \bigcup \mathbb{D}$ that may be crossed by the x-y curve after shrinking disks for a cost strictly smaller than $\lambda = 10C$. This means that we have the following nodes in the graph:

- a node for the cell containing x, which we call x also;

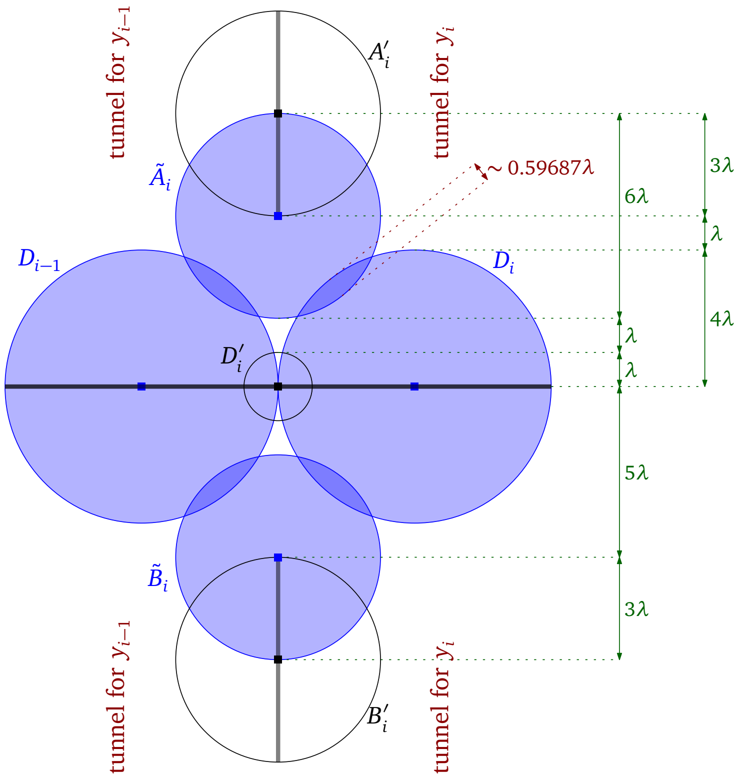

Figure 8: Disks $\tilde{A}_i$ and $\tilde{B}_i$ considered in the proof of Lemma 10.

- a node for the cell containing y, which we call y also;
- a node called α_i for the region bounded between the disks D_{i-1}, D_i, A_i $(i \in [n])$;
- a node called β_i for the region bounded between the disks D_{i-1}, D_i, B_i $(i \in [n])$;
- a node for the cell that contains y_i $(i \in [n])$, that is, the tunnel bounded by Π_i and Π_{i+1}; we call the node y_i also.

We put an edge between two nodes whenever we can pass from one region to the other passing between two disks with penetration strictly below $\lambda = 10C$. See Figure 11 for the resulting graph, G'. This graph G' is essentially the graph $G(a_1, \ldots, a_n, b)$ used in Section 2.3. (The only difference is that, in G', we have two parallel edges from y_n to y, instead of a single edge.)

We assign a weight to each edge of G' equal to the penetration depth of the pair of disks that separate the cell. For example, the edges $y_{i-1}\alpha_i$ and $\alpha_i y_i$ have weight $L + 2a_i$ $(i \in [n])$, the edge $\beta_i y_i$ has weight $L + 3a_i$ $(i \in [n])$, and the two parallel edges $y_n y$ have weight $2b$.

There is a simple correspondence between power assignments $p(\cdot)$ that give a feasible solution for Minimum Installation Path(G', x, y, C) and the reduction in radii for feasible solutions for Minimum Barrier Shrinkage$(\mathbb{D}, x, y, C)$, as follows:

- the decrease in radius of D_i corresponds to the power $p(y_i)$ $(i \in [n])$;
- the decrease in radius of A_i corresponds to the power $p(\alpha_i)$ $(i \in [n])$;
- the decrease in radius of B_i corresponds to the power $p(\beta_i)$ $(i \in [n])$;
- the decrease in radius of D_0 corresponds to the power $p(x)$;
- we may assume that at most one of the disks A_{n+1} and B_{n+1} is shrunken; the decrease in radius of A_{n+1} or B_{n+1}, whichever is larger, corresponds to the power $p(y)$;

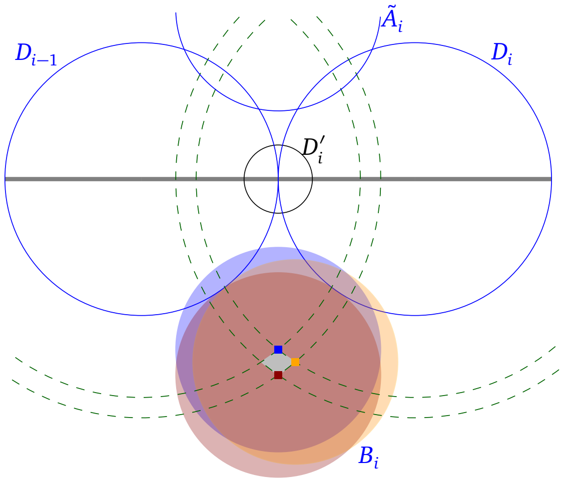

Figure 9: The gray region shows the position of the centers where the disk B_i may be placed, more precisely, the positions for center(B_i) where the penetration depth of (B_i, D_{i-1}) and (B_i, D_i) lies in the interval $[0, 0.59687\lambda]$. The blue mark denotes the center of $\tilde{B}_i$.

- we may assume that all other disks are not shrunken.

This correspondence transforms feasible solutions for MINIMUM INSTALLATION PATH(G', x, y, C) into feasible solutions for MINIMUM BARRIER SHRINKAGE$(\mathbb{D}, x, y, C)$, and conversely. So the instances MINIMUM INSTALLATION PATH(G', x, y, C) and MINIMUM BARRIER SHRINKAGE$(\mathbb{D}, x, y, C)$ are equivalent.

The second part of the lemma follows from the above correspondence and from Lemma 6. □

The disks $\mathbb{D}$, as described, cannot be constructed in polynomial time in a Turing machine because the centers of the disks do not have integer (or rational) coordinates. More precisely, the centers of A_i and B_i ($i \in [n+1]$) are solutions to a system of equations with degree-two polynomials. However, we can scale up the numbers involved in the construction, and then round the non-integer numbers, to obtain a polynomial-time construction, doable in a Turing machine:

Theorem 12. *The* MINIMUM BARRIER SHRINKAGE *problem is NP-hard.*

Proof. Consider an instance $(a_1, \dots, a_n, b)$ for SUBSET SUM and the associated instance $(\mathbb{D}, x, y, C)$ for MINIMUM BARRIER SHRINKAGE constructed above, with $C = nL + 2\sum_{i \in [n]} a_i + b$.

The centers of the disks in $\mathbb{D} \setminus \{A_1, \dots, A_{n+1}, B_1, \dots, B_{n+1}\}$ are integers bounded by $O(\lambda) = O(nL)$. For each $i \in [n+1]$, we compute the centers of the disks A_i and B_i up to a precision of at least $\varepsilon = \frac{1}{6(n+1)}$. Thus, the coordinates of the centers are multiples of ε. Let $\hat{A}_i$ and $\hat{B}_i$ be the resulting disks; they have the same radius, 3λ, but have been displaced by at most ε with respect to the original position in the construction. Let $\hat{\mathbb{D}}$ be the set of disks obtained from $\mathbb{D}$, where each A_i, B_i are replaced with $\hat{A}_i, \hat{B}_i$ ($i \in [n+1]$).

We consider instances of MINIMUM BARRIER SHRINKAGE. If the instance $(\mathbb{D}, x, y, C)$ is positive, then the instance $(\hat{\mathbb{D}}, x, y, C + 1/3)$ is also positive (because each of the $2(n+1)$ disks are moved by at most ε, so the total displacement is at most $1/3$), which implies that the instance $(\mathbb{D}, x, y, C + 2/3)$ is also positive (by the same argument), which in turn also implies that the instance $(\mathbb{D}, x, y, C)$ is positive (by Lemma 11). So, the instances $(\mathbb{D}, x, y, C)$ and $(\hat{\mathbb{D}}, x, y, C + 1/3)$ are equivalent.

Scaling all values in the construction of $\hat{\mathbb{D}}$ (coordinates and radii) by $1/\varepsilon$, we get a construction where the disks have centers with integer coordinates, the radii are integers, and the whole construction can be constructed in polynomial time. □

Note that it is not clear whether the MINIMUM BARRIER SHRINKAGE problem belongs to NP. Indeed, if some triples of disks intersect, a priori it seems that a solution may have to reduce the

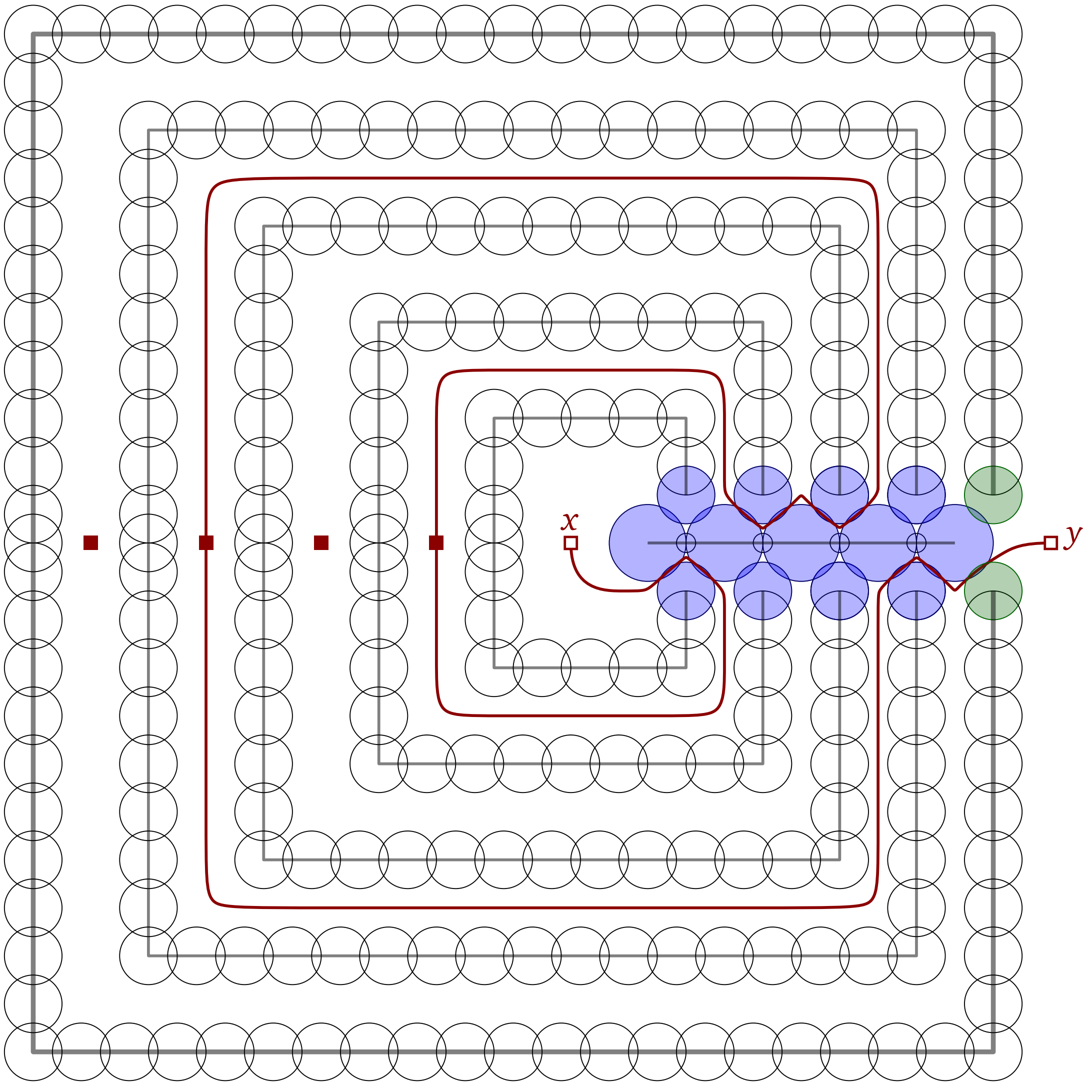

Figure 10: The red x-y curve shows the type of decisions that have to be made to make a feasible solution. In this example, we have to decide 5 times independently whether the x-y curve is routed above or below. Note that the curve can be routed to pass through each y_i, if desired.

radius of some disks by non-rational numbers, and decisions at different parts depend on each other, which could increase the algebraic degree of the numbers telling how much to decrease the radii.

Remark A similar statement can be done for axis-parallel squares. For this we have to place the overlapping squares in such a way that the overlap region, an axis-parallel rectangle, has width equal to the value we want to encode ($L+a_i$, $L+2a_i$, $L+3a_i$ or $2b$). In such a case we do not run into the numerical issues with the centers because all the coordinates can be taken directly to be integers.

Acknowledgments

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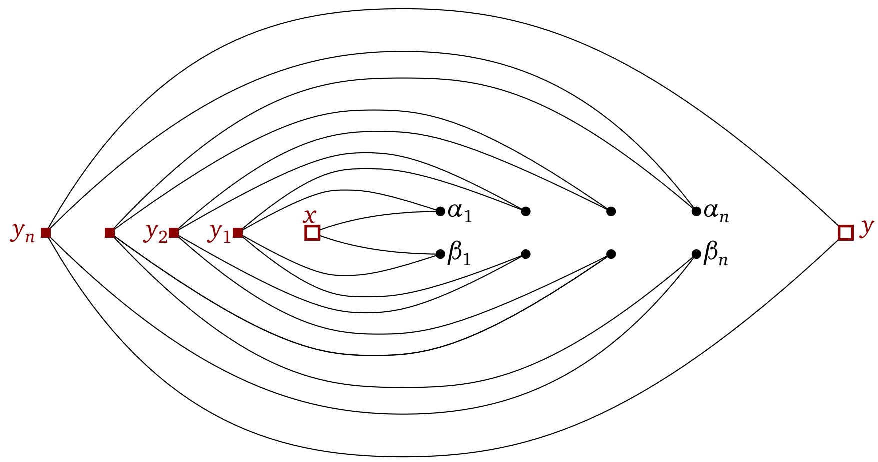

Figure 11: The combinatorially different x-y curves can be encoded in a graph, denoted G'. This is essentially the same graph $G(a_1, \ldots, a_n, b)$ used in Section 2.3, but with a different drawing; see Figure 1.

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