Traces Properties

Semantics and applications to verification

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Program of this lecture

Goal of verification

Prove that $[\![P]\!] \subseteq \mathcal{S}$

(i.e., all behaviors of P satisfy specification $\mathcal{S}$)

where $[\![P]\!]$ is the **program semantics** and $\mathcal{S}$ the **desired specification**

Last week, we studied a form of $[\![P]\!]$...

Today's lecture: we look back at program's properties

- **families of properties:**
  what properties can be considered "similar" ? in what sense ?
- **proof techniques:**
  how can those kinds of properties be established ?
- **specification of properties:**
  are there languages to describe properties ?

A high level overview

- In this lecture we look at **trace properties**
- A property is **a set of traces**, defining the **admissible** executions

Safety properties:

- **something (e.g., bad) will never happen**
- proof by invariance

Liveness properties:

- **something (e.g., good) will eventually happen**
- proof by variance

Some interesting program properties do not fit in this classification

State properties

As usual, we consider $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$

First approach: properties as sets of states

- A property $\mathcal{P}$ is **a set of states** $\mathcal{P} \subseteq \mathbb{S}$
- $\mathcal{P}$ is satisfied if and only if all reachable states belong to $\mathcal{P}$, i.e., $[\![\mathcal{S}]\!]_{\mathcal{R}} \subseteq \mathcal{P}$ where $[\![\mathcal{S}]\!]_{\mathcal{R}} = \{s_n \in \mathbb{S} \mid \exists \langle s_0, \ldots, s_n \rangle \in [\![\mathcal{S}]\!]^*,\ s_0 \in \mathbb{S}_{\mathcal{I}}\}$

Examples:

- **Absence of runtime errors**:

$$\mathcal{P} = \mathbb{S} \setminus \{\Omega\} \quad \text{where } \Omega \text{ is the error state}$$

- **Non termination** (e.g., for an operating system):

$$\mathcal{P} = \{s \in \mathbb{S} \mid \exists s' \in \mathbb{S}, s \rightarrow s'\}$$

Trace properties

Second approach: properties as sets of traces

- A property $\mathcal{T}$ is **a set of traces** $\mathcal{T} \subseteq \mathbb{S}^{\propto}$
- $\mathcal{T}$ is satisfied if and only if all traces belong to $\mathcal{T}$, i.e., $[\![\mathcal{S}]\!]^{\propto} \subseteq \mathcal{T}$

Examples:

- Obviously, **state properties** are trace properties
- **Functional properties**:
  e.g., "program P takes one integer input x and returns its absolute value"
- **Termination**: $\mathcal{T} = \mathbb{S}^*$ (i.e., the system should have no infinite execution)

Monotonicity

Property 1

Let $\mathcal{P}_0, \mathcal{P}_1 \subseteq \mathbb{S}$ be two state properties, such that $\mathcal{P}_0 \subseteq \mathcal{P}_1$.
Then $\mathcal{P}_0$ **is stronger than** $\mathcal{P}_1$, i.e. if program $\mathcal{S}$ satisfies $\mathcal{P}_0$, then it also satisfies $\mathcal{P}_1$.

Property 2

Let $\mathcal{T}_0, \mathcal{T}_1 \subseteq \mathbb{S}$ be two trace properties, such that $\mathcal{T}_0 \subseteq \mathcal{T}_1$.
Then $\mathcal{T}_0$ **is stronger than** $\mathcal{T}_1$, i.e. if program $\mathcal{S}$ satisfies $\mathcal{T}_0$, then it also satisfies $\mathcal{T}_1$.

Proofs:
straightforward application of the definition of state (resp., trace) properties

Outline

Safety properties

Informal definition: safety properties

A safety property is a property which specifies that some (bad) behavior **will never occur**

- **Absence of runtime errors** is a safety property (“bad thing”: error)
- **State properties** is a safety property (“bad thing”: reaching $\mathbb{S} \setminus \mathcal{P}$)
- **Non termination** is a safety property (“bad thing”: reaching a blocking state)
- **“Not reaching state b after visiting state a”** is a safety property (and **not** a state property)
- **Termination** is **not** a safety property

We now intend to provide a **formal definition** of safety.

Towards a formal definition

How to refute a safety property ?

- We assume $\mathcal{S}$ does **not** satisfy safety property $\mathcal{P}$
- Thus, there exists a **counter-example trace** $\sigma = \langle s_0, \ldots, s_n, \ldots \rangle \in [\![\mathcal{S}]\!] \setminus \mathcal{P}$; it may be finite or infinite...
- The intuitive definition says this trace **eventually exhibits some bad behavior**
- Thus, there exists a rank $i \in \mathbb{N}$, such that the bad behavior has been observed before reaching s_i
- Therefore, trace $\sigma' = \langle s_0, \ldots, s_i \rangle$ violates $\mathcal{P}$, i.e. $\sigma' \notin \mathcal{P}$
- We remark σ' **is finite**

A safety property that does not hold can always be refuted with a finite counter-example

Limit

Definition: upper closure operator (uco)

Function $\phi : \mathcal{S} \to \mathcal{S}$ is an **upper closure operator** iff:

- **monotone**
- **extensive:** $\forall x \in \mathcal{S},\ x \sqsubseteq \phi(x)$
- **idempotent:** $\forall x \in \mathcal{S},\ \phi(\phi(x)) = \phi(x)$

Definition: limit

The **limit operator** is defined by:

$$\begin{array}{rccl} \mathbf{Lim} : & \mathcal{P}(\mathbb{S}^{\propto}) & \longrightarrow & \mathcal{P}(\mathbb{S}^{\propto}) \\ & X & \longmapsto & X \cup \{\sigma \in \mathbb{S}^{\propto} \mid \forall i \in \mathbb{N},\ \sigma_{\lceil i} \in X\} \end{array}$$

Operator **Lim** **is an upper-closure operator**

Proof: exercise!

Prefix closure

We write $\sigma_{\lceil i}$ for the prefix of length i of trace σ:

$$\begin{array}{rcl} \langle s_0, \ldots, s_n \rangle_{\lceil 0} & = & \epsilon \\ \langle s_0, \ldots, s_n \rangle_{\lceil i+1} & = & \left\{ \begin{array}{ll} \langle s_0, \ldots, s_i \rangle & \text{if } i < n \\ \langle s_0, \ldots, s_n \rangle & \text{otherwise} \end{array} \right. \\ \langle s_0, \ldots \rangle_{\lceil i+1} & = & \langle s_0, \ldots, s_i \rangle \end{array}$$

If σ is finite, of length n, $|\sigma|i = \min(n, i)$; if σ is infinite, $|\sigma|i = i$.

Definition: prefix closure

The prefix closure operator is defined by:

$$\begin{array}{rrcl} \mathbf{PCl}: & \mathcal{P}(\mathbb{S}^{\propto}) & \longrightarrow & \mathcal{P}(\mathbb{S}^*) \\ & X & \longmapsto & \{\sigma_{\lceil i} \mid \sigma \in X, i \in \mathbb{N}\} \end{array}$$

Properties:

- **PCl** is monotone
- **PCl** is idempotent, i.e., $\mathbf{PCl} \circ \mathbf{PCl}(X) = \mathbf{PCl}(X)$

Safety properties: formal definition

An upper closure operator

Operator **Safe** is defined by $\mathbf{Safe} = \mathbf{Lim} \circ \mathbf{PCl}$.
It is an upper closure operator over $\mathcal{P}(\mathbb{S}^{\propto})$

Proof:

Safe is monotone since **Lim** and **PCl** are monotone

Safe is extensive:
indeed if $X \subseteq \mathbb{S}^{\propto}$ and $\sigma \in X$, we can show that $\sigma \in \mathbf{Safe}(X)$:

- if σ is a finite trace, it is one of its prefixes, so $\sigma \in \mathbf{PCl}(X) \subseteq \mathbf{Lim}(\mathbf{PCl}(X))$
- if σ is an infinite trace, all its prefixes belong to $\mathbf{PCl}(X)$, so $\sigma \in \mathbf{Lim}(\mathbf{PCl}(X))$

Safety properties: formal definition

Proof (continued):

Safe is idempotent:

- as **Safe** is extensive and monotone $\mathbf{Safe} \subseteq \mathbf{Safe} \circ \mathbf{Safe}$, so we simply need to show that $\mathbf{Safe} \circ \mathbf{Safe} \subseteq \mathbf{Safe}$
- let $X \subseteq \mathbb{S}^{\propto}, \sigma \in \mathbf{Safe}(\mathbf{Safe}(X))$; then:

$$\begin{array}{lll} & \sigma \in \mathbf{Safe}(\mathbf{Safe}(X)) & \\ \Rightarrow & \forall i,\ \sigma_{\lceil i} \in \mathbf{PCl} \circ \mathbf{Safe}(X) & \text{by def. of } \mathbf{Lim} \\ \Rightarrow & \forall i,\ \exists \sigma', j,\ \sigma_{\lceil i} = \sigma'_{\lceil j} \wedge \sigma' \in \mathbf{Safe}(X) & \text{by def. of } \mathbf{PCl} \\ \Rightarrow & \forall i,\ \exists \sigma', j,\ \sigma_{\lceil i} = \sigma'_{\lceil j} \wedge \forall k,\ \sigma'_{\lceil k} \in \mathbf{PCl}(X) & \text{by def. of } \mathbf{Lim} \\ \Rightarrow & \forall i,\ \exists \sigma', j,\ \sigma_{\lceil i} = \sigma'_{\lceil j} \wedge \sigma'_{\lceil i} \in \mathbf{PCl}(X) & \text{with } i = j \end{array}$$

  - if σ is finite, we let $i = |\sigma|$, thus j has to be equal to n as well and $\sigma = \sigma'_{\lceil i} \in \mathbf{PCl}(X)$, thus $\sigma \in \mathbf{Lim}(\mathbf{PCl}(X))$
  - if σ is infinite, $|\sigma_{\lceil i}| = i$ and we may let $i = k$ so

$$\forall i,\ \sigma_{\lceil i} = \sigma'_{\lceil i} \in \mathbf{PCl}(X)$$

  thus $\sigma \in \mathbf{Lim}(\mathbf{PCl}(X))$

Safety properties: formal definition

Safety: definition

A trace property $\mathcal{T}$ is a **safety** property if and only if $\textbf{Safe}(\mathcal{T}) = \mathcal{T}$

Theorem

If $\mathcal{T}$ is a trace property, then **$\textbf{Safe}(\mathcal{T})$ is a safety property**

Proof:

Straightforward, by idempotence of **Safe**

Example

We assume that:

- $\mathbb{S} = \{a, b\}$
- $\mathcal{T}$ states that a **should not be visited after state b is visited**; elements of $\mathcal{T}$ are of the general form

$$\langle a, a, a, \ldots, a, b, b, b, b, \ldots\rangle \textbf{ or } \langle a, a, a, \ldots, a, a, \ldots\rangle$$

Then:

- **PCl**($\mathcal{T}$) elements are all finite traces which are of the above form (i.e., made of n occurrences of a followed by m occurrences of b, where n, m are positive integers)
- **Lim**(**PCl**($\mathcal{T}$)) adds to this set the trace made made of infinitely many occurrences of a and the infinite traces made of n occurrences of a followed by infinitely many occurrences of b
- thus, **Safe**($\mathcal{T}$) = **Lim**(**PCl**($\mathcal{T}$)) = $\mathcal{T}$

Therefore $\mathcal{T}$ is indeed formally **a safety property**.

State properties are safety properties

Theorem

Any **state property** is also a **safety property**.

Proof:

Let us consider **state property** $\mathcal{P}$.
It is equivalent to **trace property** $\mathcal{T} = \mathcal{P}^{\propto}$:

$$\begin{array}{rcl}
\textbf{Safe}(\mathcal{T}) & = & \textbf{Lim}(\textbf{PCl}(\mathcal{P}^{\propto})) \\
 & = & \textbf{Lim}(\mathcal{P}^{*}) \\
 & = & \mathcal{P}^{*} \cup \mathcal{P}^{\omega} \\
 & = & \mathcal{P}^{\propto} \\
 & = & \mathcal{T}
\end{array}$$

Therefore $\mathcal{T}$ is indeed a safety property.

Intuition of the formal definition

Operator **Safe saturates** a set of traces S with

- prefixes
- infinite traces all finite prefixes of which can be observed in S

Thus, if **Safe**$(S) = S$ and σ is a trace, to establish that **σ is not in S**, it is sufficient to discover a **finite prefix of σ** that cannot be observed in S.

Alternatively, if all finite prefixes of σ belong to S or can observed as a prefix of another trace in S, by definition of the limit operator, **σ belongs to S** (even if it is infinite).

Thus, our definition **indeed captures properties that can be disproved with a counter-example**.

Outline

Proof by invariance

- We consider transition system $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$, and safety property $\mathcal{T}$. Finite traces semantics is the least fixpoint of F_*.
- We seek a way of **verifying that $\mathcal{S}$ satisfies $\mathcal{T}$**, i.e., that $[\![\mathcal{S}]\!]^{\propto} \subseteq \mathcal{T}$

Principle of invariance proofs

Let $\mathbb{I}$ be a set of finite traces; it is said to be an **invariant** if and only if:

- $\forall s \in \mathbb{S}_{\mathcal{I}},\ \langle s \rangle \in \mathbb{I}$
- $F_*(\mathbb{I}) \subseteq \mathbb{I}$

It is stronger than $\mathcal{T}$ if and only if $\mathbb{I} \subseteq \mathcal{T}$.

The **"by invariance"** proof method is based on finding an invariant that is stronger than $\mathcal{T}$.

Soundness

Theorem: soundness

The invariance proof method is **sound**: if we can find an invariant for $\mathcal{S}$, that is stronger than $\mathcal{T}$, then $\mathcal{S}$ satisfies $\mathcal{T}$.

Proof:

We assume that $\mathbb{I}$ is an invariant of $\mathcal{S}$ and that it is stronger than $\mathcal{T}$, and we show that $\mathcal{S}$ satisfies $\mathcal{T}$:

- by induction over n, we can prove that $F_*^n(\{\langle s \rangle \mid s \in \mathbb{S}_{\mathcal{I}}\}) \subseteq F_*^n(\mathbb{I}) \subseteq \mathbb{I}$
- therefore $[\![\mathcal{S}]\!]^* \subseteq \mathbb{I}$
- thus, $\textbf{Safe}([\![\mathcal{S}]\!]^*) \subseteq \textbf{Safe}(\mathbb{I}) \subseteq \textbf{Safe}(\mathcal{T})$ since **Safe** is monotone
- we remark that $[\![\mathcal{S}]\!]^\propto = \textbf{Safe}([\![\mathcal{S}]\!]^*)$
- $\mathcal{T}$ is a safety property so $\textbf{Safe}(\mathcal{T}) = \mathcal{T}$
- we conclude $[\![\mathcal{S}]\!]^\propto \subseteq \mathcal{T}$, i.e., $\mathcal{S}$ **satisfies property** $\mathcal{T}$

Completeness

Theorem: completeness

The invariance proof method is **complete**: if $\mathcal{S}$ satisfies $\mathcal{T}$, then we can find an invariant $\mathbb{I}$ for $\mathcal{S}$, that is stronger than $\mathcal{T}$.

Proof:

We assume that $[\![\mathcal{S}]\!]^{\propto}$ satisfies $\mathcal{T}$, and show that we can exhibit an invariant.

Then, $\mathbb{I} = [\![\mathcal{S}]\!]^{\propto}$ is an invariant of $\mathcal{S}$ by definition of $[\![.]\!]^{\propto}$, and it is stronger than $\mathcal{T}$.

Caveat:

- $[\![\mathcal{S}]\!]^{\propto}$ is most likely **not** a very easy to express invariant
- it is just a convenient completeness argument
- so, completeness does not mean the proof is easy !

Example

We consider the proof that the program below **computes the sum of the elements of an array**, i.e., when the exit is reached, $\mathtt{s} = \sum_{k=0}^{n-1} \mathtt{t}[k]$:

```
        i, s integer variables
        t integer array of length $n$
$\ell_0$ :  $(\!|\mathbf{true}|\!)$
        s = 0;
$\ell_1$ :  $(\!|\mathtt{s} = 0|\!)$
        i = 0;
$\ell_2$ :  $(\!|\mathtt{i} = 0 \wedge \mathtt{s} = 0|\!)$
        while(i < $n$){
$\ell_3$ :  $(\!|0 \leq \mathtt{i} < n \wedge \mathtt{s} = \sum_{k=0}^{\mathtt{i}-1} \mathtt{t}[k]|\!)$
            s = s + t[i];
$\ell_4$ :  $(\!|0 \leq \mathtt{i} < n \wedge \mathtt{s} = \sum_{k=0}^{\mathtt{i}} \mathtt{t}[k]|\!)$
            i = i + 1;
$\ell_5$ :  $(\!|1 \leq \mathtt{i} \leq n \wedge \mathtt{s} = \sum_{k=0}^{\mathtt{i}-1} \mathtt{t}[k]|\!)$
        }
$\ell_6$ :  $(\!|\mathtt{i} = n \wedge \mathtt{s} = \sum_{k=0}^{n-1} \mathtt{t}[k]|\!)$
```

Principle of the proof:

- for each program point ℓ, we have a **local invariant** $\mathbb{I}_\ell$ (denoted by a logical formula instead of a set of states in the figure)
- the global **invariant** $\mathbb{I}$ is defined by:

$$\mathbb{I} = \{\langle(\ell_0, m_0), \ldots, (\ell_n, m_n)\rangle \mid \forall n,\ m_n \in \mathbb{I}_{\ell_n}\}$$

Outline

Liveness properties

Informal definition: liveness properties

A liveness property is a property which specifies that some (good) behavior **will eventually occur**.

- **Termination** is a liveness property
  "good behavior": reaching a blocking state (no more transition available)
- **"State *a* will eventually be reached by all execution"** is a liveness property
  "good behavior": reaching state *a*
- The **absence of runtime errors** is *not* a liveness property

As for safety properties, we intend to provide a **formal definition** of liveness.

Intuition towards a formal definition

How to refute a liveness property ?

- We consider liveness property $\mathcal{T}$ (think $\mathcal{T}$ **is termination**)
- We assume $\mathcal{S}$ does **not** satisfy liveness property $\mathcal{T}$
- Thus, there exists a **counter-example trace** $\sigma \in [\![\mathcal{S}]\!] \setminus \mathcal{T}$;
- Let us assume σ is actually finite...
  the definition of liveness says some (good) behavior should eventually occur:
  - how do we know that σ cannot be extended into a trace $\sigma \cdot \sigma'$ that will satisfy this behavior ?
  - maybe that after a few more computation steps, σ **will reach a blocking state...**

Intuition towards a formal definition

To refute a liveness property, we need to look at infinite traces.

Example: if we run a program, and do not see it return...

- should we do `Ctrl+C` and conclude it does not terminate ?
- should we just wait a few more seconds minutes, hours, years ?

Towards a formal definition:

we expect any finite trace be the prefix of a trace in $\mathcal{T}$

... since finite executions cannot be used to disprove $\mathcal{T}$

Formal definition (incomplete)

$$\mathbf{PCl}(\mathcal{T}) = \mathbb{S}^*$$

Definition

Formal definition

Operator **Live** is defined by $\mathbf{Live}(\mathcal{T}) = \mathcal{T} \cup (\mathbb{S}^{\propto} \setminus \mathbf{Safe}(\mathcal{T}))$. Given property $\mathcal{T}$, the following three statements are equivalent:

(*i*) $\mathbf{Live}(\mathcal{T}) = \mathcal{T}$

(*ii*) $\mathbf{PCl}(\mathcal{T}) = \mathbb{S}^*$

(*iii*) $\mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T}) = \mathbb{S}^{\propto}$

When they are satisfied, $\mathcal{T}$ is said to be a **liveness property**

Example: **termination**

- The property is $\mathcal{T} = \mathbb{S}^*$
  (i.e., there should be no infinite execution)
- Clearly, it satisfies (*ii*): $\mathbf{PCl}(\mathcal{T}) = \mathbb{S}^*$
  thus termination indeed satisfies this definition

Proof of equivalence

Proof of equivalence:

(*i*) implies (*ii*):

We assume that $\mathbf{Live}(\mathcal{T}) = \mathcal{T}$, i.e., $\mathcal{T} \cup (\mathbb{S}^{\propto} \setminus \mathbf{Safe}(\mathcal{T})) = \mathcal{T}$
therefore, $\mathbb{S}^{\propto} \setminus \mathbf{Safe}(\mathcal{T}) \subseteq \mathcal{T}$;
let $\sigma \in \mathbb{S}^*$, and let us show that $\sigma \in \mathbf{PCl}(\mathcal{T})$; clearly, $\sigma \in \mathbb{S}^{\propto}$, thus:

- either $\sigma \in \mathbf{Safe}(\mathcal{T}) = \mathbf{Lim}(\mathbf{PCl}(\mathcal{T}))$, so all its prefixes are in $\mathbf{PCl}(\mathcal{T})$ and $\sigma \in \mathbf{PCl}(\mathcal{T})$
- or $\sigma \in \mathcal{T}$, which implies that $\sigma \in \mathbf{PCl}(\mathcal{T})$

(*ii*) implies (*iii*):

If $\mathbf{PCl}(\mathcal{T}) = \mathbb{S}^*$, then $\mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T}) = \mathbb{S}^{\propto}$

(*iii*) implies (*i*):

If $\mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T}) = \mathbb{S}^{\propto}$, then
$\mathbf{Live}(\mathcal{T}) = \mathcal{T} \cup (\mathbb{S}^{\propto} \setminus (\mathcal{T} \cup \mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T}))) = \mathcal{T} \cup (\mathbb{S}^{\propto} \setminus \mathbb{S}^{\propto}) = \mathcal{T}$

Example

We assume that:

- $\mathbb{S} = \{a, b, c\}$
- $\mathcal{T}$ states that **b should eventually be visited, after a has been visited**; elements of $\mathcal{T}$ can be described by

$$\mathcal{T} = \mathbb{S}^* \cdot a \cdot \mathbb{S}^* \cdot b \cdot \mathbb{S}^\propto$$

Then $\mathcal{T}$ **is a liveness property:**

- let $\sigma \in \mathbb{S}^*$; then $\sigma \cdot a \cdot b \in \mathcal{T}$, so $\sigma \in \mathbf{PCl}(\mathcal{T})$
- thus, $\mathbf{PCl}(\mathcal{T}) = \mathbb{S}^*$

A property of **Live**

Theorem

If $\mathcal{T}$ is a trace property, then **Live**($\mathcal{T}$) **is a liveness property** (i.e., operator **Live** is **idempotent**).

Proof: we show that $\mathbf{PCl} \circ \mathbf{Live}(\mathcal{T}) = \mathbb{S}^*$, by considering $\sigma \in \mathbb{S}^*$ and proving that $\sigma \in \mathbf{PCl} \circ \mathbf{Live}(\mathcal{T})$; we first note that:

$$\begin{array}{rcl} \mathbf{PCl} \circ \mathbf{Live}(\mathcal{T}) & = & \mathbf{PCl}(\mathcal{T}) \cup \mathbf{PCl}(\mathbb{S}^\propto \setminus \mathbf{Safe}(\mathcal{T})) \\ & = & \mathbf{PCl}(\mathcal{T}) \cup \mathbf{PCl}(\mathbb{S}^\propto \setminus \mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T})) \end{array}$$

- if $\sigma \in \mathbf{PCl}(\mathcal{T})$, this is obvious.
- if $\sigma \notin \mathbf{PCl}(\mathcal{T})$, then:
  - $\sigma \notin \mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T})$ by definition of the limit
  - thus, $\sigma \in \mathbb{S}^\propto \setminus \mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T})$
  - $\sigma \in \mathbf{PCl}(\mathbb{S}^\propto \setminus \mathbf{Lim} \circ \mathbf{PCl}(\mathcal{T}))$ as **PCl** is extensive, which proves the above result

Outline

Termination proof with ranking function

- We consider only **termination**
- We consider transition system $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$, and liveness property $\mathcal{T}$
- We seek a way of **verifying that $\mathcal{S}$ satisfies termination**, i.e., that $[\![\mathcal{S}]\!]^{\propto} \subseteq \mathbb{S}^*$

Definition: ranking function

A **ranking function** is a function $\phi : \mathbb{S} \rightarrow E$ where:

- $(E, \sqsubseteq)$ is a **well-founded ordering**
- $\forall s_0, s_1 \in \mathbb{S},\ s_0 \rightarrow s_1 \Longrightarrow \phi(s_1) \sqsubset \phi(s_0)$

Theorem

If $\mathcal{S}$ has a ranking function ϕ, it satisfies termination.

Example

We consider the termination of the array sum program:

```
        i, s integer variables
        t integer array of length n
ℓ0 :    s = 0;
ℓ1 :    i = 0;
ℓ2 :    while(i < n){
ℓ3 :        s = s + t[i];
ℓ4 :        i = i + 1;
ℓ5 :    }
ℓ6 :    ...
```

Ranking function:

$$\begin{array}{rlcl} \phi : & \mathbb{S} & \longrightarrow & \mathbb{N} \\ & (\ell_0, m) & \longmapsto & 3 \cdot n + 6 \\ & (\ell_1, m) & \longmapsto & 3 \cdot n + 5 \\ & (\ell_2, m) & \longmapsto & 3 \cdot n + 4 \\ & (\ell_3, m) & \longmapsto & 3 \cdot (n - m(\mathtt{i})) + 3 \\ & (\ell_4, m) & \longmapsto & 3 \cdot (n - m(\mathtt{i})) + 2 \\ & (\ell_5, m) & \longmapsto & 3 \cdot (n - m(\mathtt{i})) + 1 \\ & (\ell_6, m) & \longmapsto & 0 \end{array}$$

Proof by variance

- We consider transition system $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$, and liveness property $\mathcal{T}$; infinite traces semantics is the least fixpoint of F_ω.
- We seek a way of **verifying that $\mathcal{S}$ satisfies $\mathcal{T}$**, i.e., that $[\![\mathcal{S}]\!]^\propto \subseteq \mathcal{T}$

Principle of variance proofs

Let $(\mathbb{I}_n)_{n\in\mathbb{N}}$, $\mathbb{I}_\omega$ be elements of $\mathbb{S}^\propto$; these are said to form a variance proof of $\mathcal{T}$ if and only if:

- $\mathbb{S}^\propto \subseteq \mathbb{I}_0$
- for all $k \in \{1, 2, \ldots, \omega\}$, $\forall s \in \mathbb{S}$, $\langle s \rangle \in \mathbb{I}_k$
- for all $k \in \{1, 2, \ldots, \omega\}$, there exists $l < k$ such that $F_\omega(\mathbb{I}_l) \subseteq \mathbb{I}_k$
- $\mathbb{I}_\omega \subseteq \mathcal{T}$

Proofs of soundness and completeness: exercise

Outline

The decomposition theorem

Theorem

Let $\mathcal{T} \subseteq \mathbb{S}^{\propto}$; it can be decomposed into the **conjunction** of **safety property** **Safe**$(\mathcal{T})$ and **liveness property** **Live**$(\mathcal{T})$:

$$\mathcal{T} = \mathbf{Safe}(\mathcal{T}) \cap \mathbf{Live}(\mathcal{T})$$

- **Reading**: **Recognizing Safety and Liveness**.
  Bowen Alpern and **Fred B. Schneider**.
  In Distributed Computing, Springer, 1987.
- **Consequence of this result:**
  the proof of any trace property can be decomposed into
  - a proof of safety
  - a proof of liveness

Proof

- **Safety part:**
  Safe is idempotent, so **Safe**$(\mathcal{T})$ is a safety property.
- **Liveness part:**
  Live is idempotent, so **Live**$(\mathcal{T})$ is a liveness property.
- **Decomposition:**

$$\begin{array}{rcl}
\mathbf{Safe}(\mathcal{T}) \cap \mathbf{Live}(\mathcal{T}) & = & \mathbf{Safe}(\mathcal{T}) \cap (\mathbb{S}^{\propto} \setminus \mathbf{Safe}(\mathcal{T}) \cup \mathcal{T}) \\
& = & \mathbf{Safe}(\mathcal{T}) \cap (\mathbb{S}^{\propto} \setminus \mathbf{Safe}(\mathcal{T})) \\
& & \qquad \cup\, \mathbf{Safe}(\mathcal{T}) \cap \mathcal{T} \\
& = & \emptyset \cup \mathcal{T} \\
& = & \mathcal{T}
\end{array}$$

Example: verification of total correctness

```
        i, s integer variables
        t integer array of length $n$
$\ell_0$ :    s = 0;
$\ell_1$ :    i = 0;
$\ell_2$ :    while(i < $n$){
$\ell_3$ :        s = s + t[i];
$\ell_4$ :        i = i + 1;
$\ell_5$ :    }
$\ell_6$ :    ...
```

Property to prove:
total correctness

1. the program **terminates**
2. and it **computes the sum of the elements in the array**

Application of the decomposition principle

Conjunction of two proofs:

1. Proved with a **ranking function**
2. Proved with **local invariants**

Safety and Liveness Decomposition Example

We consider a very simple **greatest common divider** code function:

```
ℓ0 :  int f(int a, int b){
ℓ1 :      while(a > 0){
ℓ2 :          int d = b/a;
ℓ3 :          int r = b − a ∗ d;
ℓ4 :          b = a;
ℓ5 :          a = r;
ℓ6 :      }
ℓ7 :      return b;
ℓ8 :  }
```

Specification

When applied to positive integers, function `f` should always return their GCD.

Safety and Liveness Decomposition Example

We consider a very simple **greatest common divider** code function:

```
ℓ0 :  int f(int a, int b){
ℓ1 :      while(a > 0){
ℓ2 :          int d = b/a;
ℓ3 :          int r = b − a ∗ d;
ℓ4 :          b = a;
ℓ5 :          a = r;
ℓ6 :      }
ℓ7 :      return b;
ℓ8 :  }
```

Specification

When applied to positive integers, function `f` should always return their GCD.

Safety part

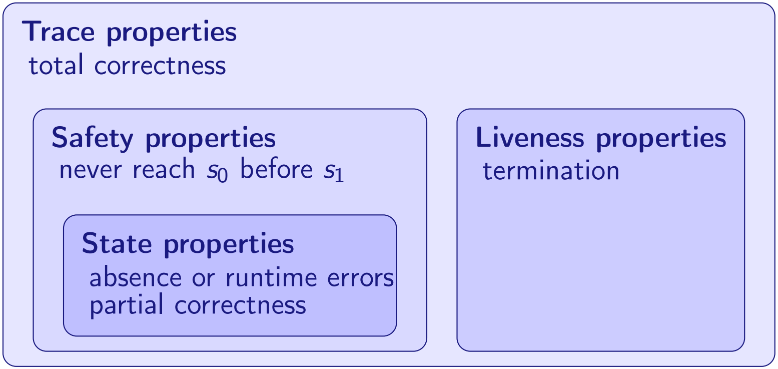
For all trace starting with positive inputs, a **conjunction of two properties:**

- **no runtime errors**
- **the value of `b` is the GCD**

Liveness part

Termination, on all traces starting with positive inputs

The Zoo of semantic properties: current status



- **Safety:** if wrong, can be refuted with a **finite trace**
  proof done by **invariance**
- **Liveness:** if wrong, has to be refuted with an **infinite trace**
  proof done by **variance**

Outline

Notion of specification language

- Ultimately, we would like to **verify or compute** properties
- So far, we simply describe properties with **sets of executions** or worse, with English / French / . . . statements
- Ideally, we would prefer to use a **mathematical language** for that
  - to **gain in concision**, **avoid ambiguity**
  - to **define sets of properties to consider**, fix **the form of inputs for verification tools...**

Definition: specification language

A **specification language** is a set of terms $\mathbb{L}$ with an **interpretation function** (or **semantics**)

$$[\![.]\!] : \quad \mathbb{L} \quad \longrightarrow \quad \mathcal{P}(\mathbb{S}^{\propto}) \qquad (\text{resp., } \mathcal{P}(\mathbb{S}))$$

- We are now going to consider specification languages **for states**, **for traces**...

A State specification language

A first **example** of a (simple) specification language:

A state specification language

- **Syntax:** we let terms of $\mathbb{L}_{\mathbb{S}}$ be defined by:

$$p \in \mathbb{L}_{\mathbb{S}} ::= @\ell \mid \mathrm{x} < \mathrm{x}' \mid \mathrm{x} < n \mid \neg p' \mid p' \wedge p'' \mid \Omega$$

- **Semantics:** $[\![p]\!] \subseteq \mathbb{S}_{\Omega}$ is defined by

$$\begin{array}{rcl}
[\![@\ell]\!] & = & \{\ell\} \times \mathbb{M} \\
[\![\mathrm{x} \leq \mathrm{x}']\!] & = & \{(\ell, m) \in \mathbb{S} \mid m(\mathrm{x}) \leq m(\mathrm{x}')\} \\
[\![\mathrm{x} \leq n]\!] & = & \{(\ell, m) \in \mathbb{S} \mid m(\mathrm{x}) \leq n\} \\
[\![\neg p]\!] & = & \mathbb{S}_{\Omega} \setminus [\![p]\!] \\
[\![p \wedge p']\!] & = & [\![p]\!] \cap [\![p']\!] \\
[\![\Omega]\!] & = & \{\Omega\}
\end{array}$$

Exercise: add $=$, $\vee$, $\Longrightarrow$...

State properties: examples

Unreachability of control state ℓ_0:

- **specification:** $\Omega \vee \neg @\ell_0$
- **property:** $[\![\Omega \vee \neg @\ell_0]\!] = \mathbb{S}_\Omega \setminus \{(\ell_0, m) \mid m \in \mathbb{M}\}$

Absence of runtime errors:

- **specification:** $\neg\Omega$
- **property:** $[\![\neg\Omega]\!] = \mathbb{S}_\Omega \setminus \{\Omega\} = \mathbb{S}$

Intermittent invariant:

- **principle:** attach a local invariant to each control state
- **example:**

$$\begin{array}{lllcl}
\ell_0 : & \mathbf{if}(\mathrm{x} \geq 0)\{ & & & \\
\ell_1 : & \quad \mathrm{y} = \mathrm{x}; & & & @\ell_1 \Longrightarrow \mathrm{x} \geq 0 \\
\ell_2 : & \}\mathbf{else}\{ & & \wedge & @\ell_2 \Longrightarrow \mathrm{x} \geq 0 \wedge \mathrm{y} \geq 0 \\
\ell_3 : & \quad \mathrm{y} = -\mathrm{x}; & & \wedge & @\ell_3 \Longrightarrow \mathrm{x} < 0 \\
\ell_4 : & \} & & \wedge & @\ell_4 \Longrightarrow \mathrm{x} < 0 \wedge \mathrm{y} > 0 \\
\ell_5 : & \ldots & & \wedge & @\ell_5 \Longrightarrow \mathrm{y} \geq 0
\end{array}$$

Propositional temporal logic: syntax

We now consider the **specification of trace properties**

- **Temporal logic:** specification of properties in terms of events that occur at distinct times in the execution (hence, the name "temporal")
- There are **many** instances of temporal logic
- We study a simple one: **Pnueli's Propositional Temporal Logic**

Definition: syntax of PTL (Propositional Temporal Logic)

Properties over traces are defined as terms of the form

$$
\begin{array}{rcll}
t (\in \mathbb{L}_{\mathbf{PTL}}) & ::= & p & \text{state property, i.e., } p \in \mathbb{L}_{\mathbb{S}} \\
 & | & t' \vee t'' & \text{disjunction} \\
 & | & \neg t' & \text{negation} \\
 & | & \bigcirc\, t' & \text{"next"} \\
 & | & t' \,\mathfrak{U}\, t'' & \text{"until", i.e., } t' \text{ until } t''
\end{array}
$$

Propositional temporal logic: semantics

Some operators on traces:

- $|\sigma|$ denotes the **length** of trace σ (either an integer or ∞)
- **"tail" operator** $\cdot_{i\rceil}$:

$$\begin{array}{rcll} \sigma_{i\rceil} & = & \epsilon & \text{if } |\sigma| < i \\ (\langle s_0, \ldots, s_i \rangle \cdot \sigma)_{i-1\rceil} & ::= & \sigma & \text{otherwise} \end{array}$$

Semantics of Propositional Temporal Logic formulae

$$\begin{array}{rcl} [\![p]\!] & = & \{ s \cdot \sigma \mid s \in [\![p]\!] \wedge \sigma \in \mathbb{S}^{\propto} \} \\ [\![t_0 \vee t_1]\!] & = & [\![t_0]\!] \cup [\![t_1]\!] \\ [\![\neg t_0]\!] & = & \mathbb{S}^{\propto} \setminus [\![t_0]\!] \\ [\![\bigcirc\, t_0]\!] & = & \{ s \cdot \sigma \mid s \in \mathbb{S} \wedge \sigma \in [\![t_0]\!] \} \\ [\![t_0 \,\mathfrak{U}\, t_1]\!] & = & \{ \sigma \in \mathbb{S}^{\propto} \mid \exists n \in \mathbb{N},\ \forall i < n,\ \sigma_{i\rceil} \in [\![t_0]\!] \wedge \sigma_{n\rceil} \in [\![t_1]\!] \} \end{array}$$

Temporal logic operators as syntactic sugar

Many useful operators can be added:

- **Boolean constants:**

$$\begin{array}{l}\textbf{true} ::= (\mathtt{x} < 0) \vee \neg(\mathtt{x} < 0) \\ \textbf{false} ::= \neg\textbf{true}\end{array}$$

- **Sometime:**

$$\Diamond\, t ::= \textbf{true}\,\mathfrak{U}\, t$$

  intuition: there exists a rank n at which t holds

- **Always:**

$$\Box\, t ::= \neg(\Diamond(\neg t))$$

  intuition: there is no rank at which the negation of t holds

Exercise: what do $\Diamond\,\Box\, t$ and $\Box\,\Diamond\, t$ mean ?

Propositional temporal logic: examples

We consider the program below:

$$\begin{array}{ll}
\ell_0: & \textbf{int}\ \mathtt{x} = \textbf{input}(); \\
\ell_1: & \textbf{if}(\mathtt{x} < 8)\{ \\
\ell_2: & \qquad \mathtt{x} = 0; \\
\ell_3: & \}\ \textbf{else}\ \{ \\
\ell_4: & \qquad \mathtt{x} = 1; \\
\ell_5: & \} \\
\ell_6: & \ldots
\end{array}$$

Examples of properties:

- "when ℓ_4 is reached, $\mathtt{x}$ is positive"

$$\Box(@\ell_4 \Longrightarrow \mathtt{x} \geq 0)$$

- "if the value read at point ℓ_0 is negative, and when ℓ_6 is reached, $\mathtt{x}$ is equal to 0"

$$(@\ell_1 \wedge \mathtt{x} < 0) \Longrightarrow \Box(@\ell_6 \Longrightarrow \mathtt{x} = 0)$$

Outline

Security properties

We now consider other interesting properties of programs, and show that they do not all reduce to trace properties

Security

- Collects many kinds of properties
- So we consider just one:

  an unauthorized observer should not be able to guess anything about private information by looking at public information

- **Example:** another user should not be able to guess the content of an email sent to you
- We need to **formalize this property**

A few definitions

Assumptions:

- We let $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}})$ be a transition system
- States are of the form $(\ell, m) \in \mathbb{L} \times \mathbb{M}$
- Memory states are of the form $\mathbb{X} \rightarrow \mathbb{V}$
- We let $\ell, \ell' \in \mathbb{L}$ (program entry and exit)
  and $\mathrm{x}, \mathrm{x}' \in \mathbb{X}$ (private and public variables)

Security property we are looking at

Observing the value of x' at ℓ'
gives no information on the value of x at ℓ

A few examples

A secure program (no information flow, no way to guess $\mathtt{x}$):

$$\begin{array}{ll} \ell: & \mathtt{x}' = 84; \\ \ell': & \ldots \end{array}$$

An insecure program (explicit information flow, $\mathtt{x}'$ gives a lot of information about $\mathtt{x}$, so that we can simply recompute it):

$$\begin{array}{ll} \ell: & \mathtt{x}' = \mathtt{x} - 2; \\ \ell': & \ldots \end{array}$$

An insecure program (implicit information flow, through a test):

$$\begin{array}{ll} \ell: & \mathbf{if}(\mathtt{x} < 0)\{\mathtt{x}' = 0;\} \\ \ell': & \ldots \end{array}$$

How to characterize information flow in the semantic level ?

Non-interference

We consider the **transformer** Φ defined by:

$$\begin{array}{rccl} \Phi : & \mathbb{M} & \longrightarrow & \mathcal{P}(\mathbb{M}) \\ & m & \longmapsto & \{m' \in \mathbb{M} \mid \exists \sigma = \langle (\ell, m), \ldots, (\ell', m') \rangle \in [\![\mathcal{S}]\!]\} \end{array}$$

Definition: non-interference

There is **no interference** between (ℓ, x) and (ℓ', x') and we write $(l', x') \not\rightsquigarrow (l, x)$ if and only if the following property holds:

$$\begin{array}{l} \forall m \in \mathbb{M}, \forall v_0, v_1 \in \mathbb{V}, \\ \quad \{m'(\mathrm{x}') \mid m' \in \Phi(m[\mathrm{x} \leftarrow v_0])\} = \{m'(\mathrm{x}') \mid m' \in \Phi(m[\mathrm{x} \leftarrow v_1])\} \end{array}$$

Intuition:

- if two observations at point ℓ differ only in the value of x, there is no difference in observation of x' at ℓ'
- in other words, observing x' at ℓ' (even on many executions) gives no information about the value of x at point ℓ...

Non-interference is not a trace property

- We assume $\mathbb{V} = \{0, 1\}$ and $\mathbb{X} = \{\mathrm{x}, \mathrm{x}'\}$ (store m is defined by the pair $(m(\mathrm{x}), m(\mathrm{x}'))$, and denoted by it)
- We assume $\mathbb{L} = \{\ell, \ell'\}$ and consider two systems such that all transitions are of the form $(\ell, m) \to (\ell', m')$ (i.e., system $\mathcal{S}$ is isomorphic to its transformer $\Phi[\mathcal{S}]$)

$$\Phi[\mathcal{S}_0] : \begin{array}{rcl} (0,0) & \longmapsto & \mathbb{M} \\ (0,1) & \longmapsto & \mathbb{M} \\ (1,0) & \longmapsto & \mathbb{M} \\ (1,1) & \longmapsto & \mathbb{M} \end{array} \qquad \Phi[\mathcal{S}_1] : \begin{array}{rcl} (0,0) & \longmapsto & \mathbb{M} \\ (0,1) & \longmapsto & \mathbb{M} \\ (1,0) & \longmapsto & \{(1,1)\} \\ (1,1) & \longmapsto & \{(1,1)\} \end{array}$$

- $\mathcal{S}_1$ has fewer behaviors than $\mathcal{S}_0$: $[\![\mathcal{S}_1]\!]^* \subset [\![\mathcal{S}_0]\!]^*$
- $\mathcal{S}_0$ **has the non-interference property**, but $\mathcal{S}_1$ **does not**
- If non interference was a trace property, $\mathcal{S}_1$ should have it (monotony)

Thus, the non interference property is not a trace property

Dependence properties

Dependence property

- Many notions of dependences
- So we consider just one:

  what inputs may have an impact on the observation of a given output

- **Applications:**
  - **reverse engineering:** understand how an input gets computed
  - **slicing:** extract the fragment of a program that is relevant to a result
- This corresponds to the **negation** of non-interference

Interference

Definition: interference

There is **interference** between (ℓ, x) and (ℓ', x') and we write $(l', x') \rightsquigarrow (l, x)$ if and only if the following property holds:

$$\exists m \in \mathbb{M}, \exists v_0, v_1 \in \mathbb{V}, \\ \{m'(\mathrm{x}') \mid m' \in \Phi(m[\mathrm{x} \leftarrow v_0])\} \neq \{m'(\mathrm{x}') \mid m' \in \Phi(m[\mathrm{x} \leftarrow v_1])\}$$

- This expresses that there is at least one case, where the value of x at ℓ has an impact on that of x' at ℓ'
- It may not hold even if the computation of x' reads x:

$$\begin{array}{ll} \ell : & \mathrm{x}' = 0 \star \mathrm{x}; \\ \ell' : & \ldots \end{array}$$

Interference is not a trace property

- We assume $\mathbb{V} = \{0, 1\}$ and $\mathbb{X} = \{\mathrm{x}, \mathrm{x}'\}$ (store m is defined by the pair $(m(\mathrm{x}), m(\mathrm{x}'))$, and denoted by it)
- We assume $\mathbb{L} = \{\ell, \ell'\}$ and consider two systems such that all transitions are of the form $(\ell, m) \rightarrow (\ell', m')$ (i.e., system $\mathcal{S}$ is isomorphic to its transformer $\Phi[\mathcal{S}]$)

$$\Phi[\mathcal{S}_0] : \begin{array}{rcl} (0,0) & \longmapsto & \mathbb{M} \\ (0,1) & \longmapsto & \mathbb{M} \\ (1,0) & \longmapsto & \{(1,1)\} \\ (1,1) & \longmapsto & \{(1,1)\} \end{array} \qquad \Phi[\mathcal{S}_1] : \begin{array}{rcl} (0,0) & \longmapsto & \{(1,1)\} \\ (0,1) & \longmapsto & \{(1,1)\} \\ (1,0) & \longmapsto & \{(1,1)\} \\ (1,1) & \longmapsto & \{(1,1)\} \end{array}$$

- $\mathcal{S}_1$ has fewer behavior than $\mathcal{S}_0$: $[\![\mathcal{S}_1]\!]^* \subset [\![\mathcal{S}_0]\!]^*$
- $\mathcal{S}_0$ **has the interference property**, but $\mathcal{S}_1$ **does not**
- If interference was a trace property, $\mathcal{S}_1$ should have it (monotony)

Thus, the interference property is not a trace property

Hyperproperties

Conclusion:

- The absence of interference between (ℓ, x) and (ℓ', x') **is not a trace property**:
  we cannot describe as the set of programs the semantics of which is included into a given set of traces
- It can however **be described by a set of sets of traces**:
  we simply collect the set of program semantics that satisfy the property

This is what we call a **hyperproperty**:

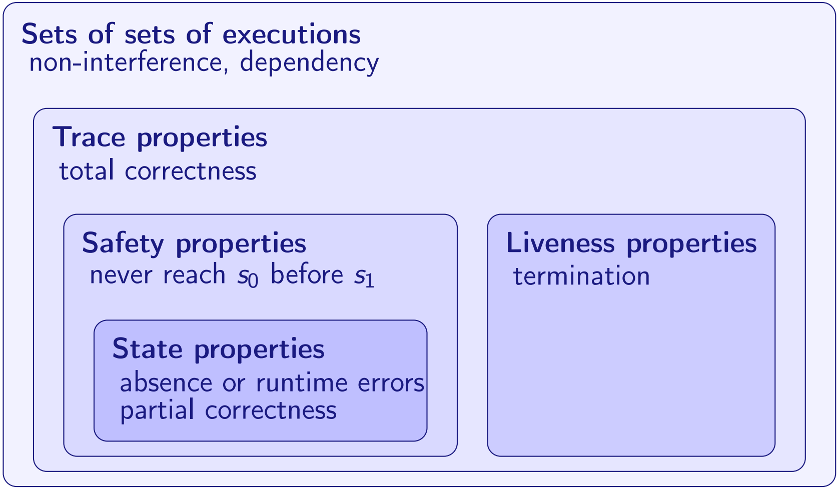
Hyperproperties

- **Trace hyperproperties** are described by sets of sets of executions
- **Trace properties** are described by sets of executions

2-safety: to disprove the absence of interference (i.e., to show there exists an interference), we simply need to exhibit **two finite traces**

Outline

The Zoo of semantic properties



Summary

To sum-up:

- **Trace properties** allow to express a large range of program properties
- **Safety = absence of bad behaviors**
- **Liveness = existence of good behaviors**
- Trace properties can be **decomposed** as conjunctions of safety and liveness properties, with **dedicated proof methods**
- Some interesting properties are **not trace properties** security properties are *sets of sets of executions*
- Notion of **specification languages** to describe program properties