Operational Semantics

Semantics and applications to verification

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Program of this first lecture

Operational semantics

Mathematical description of the executions of a program

1. a **model** of programs: **transition systems**
   - definition, a **small step semantics**
   - a few common examples
2. **trace semantics:** a kind of **big step** semantics
   - **finite** and **infinite** executions
   - **fixpoint**-based definitions
   - notion of **compositional semantics**

Outline

Definition

We will characterize a program by:

- **states**: photography of the program status at an instant of the execution
- **execution steps**: how do we move from one state to the next one

Definition: transition systems (TS)

A **transition system** is a tuple $(\mathbb{S}, \rightarrow)$ where:

- $\mathbb{S}$ is the **set of states** of the system
- $\rightarrow \subseteq \mathcal{P}(\mathbb{S} \times \mathbb{S})$ is the **transition relation** of the system

Note:

- the set of states **may be infinite**

Transition systems: properties of the transition relation

A **deterministic** system is such that a state fully determines the next state

$$\forall s_0, s_1, s_1' \in \mathbb{S},\ (s_0 \to s_1 \wedge s_0 \to s_1') \Longrightarrow s_1 = s_1'$$

Otherwise, a transition system is **non deterministic**, i.e.:

$$\exists s_0, s_1, s_1' \in \mathbb{S},\ s_0 \to s_1 \wedge s_0 \to s_1' \wedge s_1 \neq s_1'$$

Notes:

- transition relation $\to$ defines atomic execution steps;
  it is often called **small-step semantics** or **structured operational semantics**
- steps are **discrete** (not continuous)
  to describe both discrete and continuous behaviors, we would need to look at *hybrid systems* (beyond the scope of this lecture)

Transition systems: special states

Initial / final states:
we often consider transition systems with a set of initial and final states:

- a set of **initial states** $\mathbb{S}_{\mathcal{I}} \subseteq \mathbb{S}$ denotes states where the execution should start
- a set of **final states** $\mathbb{S}_{\mathcal{F}} \subseteq \mathbb{S}$ denotes states where the execution should reach the end of the program

When needed, we add these to the definition of the transition systems $((\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}}, \mathbb{S}_{\mathcal{F}})$.

Blocking state (not the same as final state):

- a state $s_0 \in \mathbb{S}$ is **blocking** when it is the origin of no transition: $\forall s_1 \in \mathbb{S},\ \neg(s_0 \rightarrow s_1)$
- example: we often introduce an **error state** (usually noted Ω to denote the erroneous, blocking configuration)

Outline

Finite automata as transition systems

We can clearly formalize the **word recognition** by a finite automaton using a transition system:

- we consider automaton $\mathcal{A} = (Q, q_{\mathrm{i}}, q_{\mathrm{f}}, \rightarrow)$
- a "state" is defined by:
  - the **remaining of the word to recognize**
  - the **automaton state** that has been reached so far

  thus, $\mathbb{S} = Q \times L^{\star}$
- the **transition relation** $\rightarrow$ of the transition system is defined by:

$$(q_0, aw) \rightarrow (q_1, w) \iff q_0 \xrightarrow{a} q_1$$

- the **initial** and **final states** are defined by:

$$\mathbb{S}_{\mathcal{I}} = \{(q_{\mathrm{i}}, w) \mid w \in L^{\star}\} \qquad \mathbb{S}_{\mathcal{F}} = \{(q_{\mathrm{f}}, \epsilon)\}$$

Pure λ-calculus

A **bare bones model of functional programing**:

λ-terms

The set of λ-terms is defined by:

$$\begin{array}{llll} t, u, \ldots & ::= & x & \text{variable} \\ & | & \lambda x \cdot t & \text{abstraction} \\ & | & t\, u & \text{application} \end{array}$$

β-reduction

- $(\lambda x \cdot t)\, u \rightarrow_\beta t[x \leftarrow u]$
- if $u \rightarrow_\beta v$ then $\lambda x \cdot u \rightarrow_\beta \lambda x \cdot v$
- if $u \rightarrow_\beta v$ then $u\, t \rightarrow_\beta v\, t$
- if $u \rightarrow_\beta v$ then $t\, u \rightarrow_\beta t\, v$

The λ-calculus defines a transition system:

- $\mathbb{S}$ is the set of λ-terms and $\rightarrow_\beta$ the transition relation
- $\rightarrow_\beta$ is **non-deterministic**; example ?
  though, ML fixes an execution order
- given a lambda term t_0, we may consider $(\mathbb{S}, \rightarrow_\beta, \mathbb{S}_\mathcal{I})$ where $\mathbb{S}_\mathcal{I} = \{t_0\}$
- **blocking states** are terms with no redex $(\lambda x \cdot u)\, v$

A MIPS like assembly language: syntax

We now consider a (very simplified) **assembly language**

- machine integers: sequences of 32-bits (set: $\mathbb{B}^{32}$)
- instructions are encoded over 32-bits (set: $\mathbb{I}_{\mathrm{MIPS}}$) and stored into the same space as data (i.e., $\mathbb{I}_{\mathrm{MIPS}} \subseteq \mathbb{B}^{32}$)

Memory configurations

- **program counter** **pc**
  current instruction
- **general purpose registers**
  $\mathbf{r}_0 \ldots \mathbf{r}_{31}$
- **main memory** (RAM)
  $\mathbf{mem} : \mathbf{Addrs} \to \mathbb{B}^{32}$
  where $\mathbf{Addrs} \subseteq \mathbb{B}^{32}$

Instructions

i	$::=$	$(\in \mathbb{I}_{\mathrm{MIPS}})$	
	$\mid$	$\mathbf{add}\ \mathbf{r}_d, \mathbf{r}_{s0}, \mathbf{r}_{s1}$	addition
	$\mid$	$\mathbf{addi}\ \mathbf{r}_d, \mathbf{r}_{s0}, v$	add. $v \in \mathbb{B}^{32}$
	$\mid$	$\mathbf{sub}\ \mathbf{r}_d, \mathbf{r}_{s0}, \mathbf{r}_{s1}$	subtraction
	$\mid$	$\mathbf{b}\ dst$	branch
	$\mid$	$\mathbf{blt}\ \mathbf{r}_{s0}, \mathbf{r}_{s1}, dst$	cond. branch
	$\mid$	$\mathbf{ld}\ \mathbf{r}_d, o, \mathbf{r}_x$	relative load
	$\mid$	$\mathbf{st}\ \mathbf{r}_d, o, \mathbf{r}_x$	relative store

$v, dst, o \in \mathbb{B}^{32}$

A MIPS like assembly language: states

Definition: state

A state is a tuple (pc, ρ, μ) which comprises:

- a **program counter** value $pc \in \mathbb{B}^{32}$
- a function mapping each **general purpose register** to its value $\rho : \{0, \ldots, 31\} \rightarrow \mathbb{B}^{32}$
- a function mapping each **memory cell** to its value $\mu : \textbf{Addrs} \rightarrow \mathbb{B}^{32}$

What would a **dangerous state** be ?

- writing **over an instruction**
- reading or writing **outside the program's memory**

$\Rightarrow$ we cannot fully formalize these yet...
as we need to formalize the behavior of each instruction first

A MIPS like assembly language: transition relation

We assume a state $s = (pc, \rho, \mu)$ and that $\mu(pc) = i$; then:

- if $i =$ **add** $\mathbf{r}_d, \mathbf{r}_{s0}, \mathbf{r}_{s1}$, then:

$$s \rightarrow (pc + 4, \rho[d \leftarrow \rho(s0) + \rho(s1)], \mu)$$

- if $i =$ **addi** $\mathbf{r}_d, \mathbf{r}_{s0}, v$, then:

$$s \rightarrow (pc + 4, \rho[d \leftarrow \rho(s0) + v], \mu)$$

- if $i =$ **sub** $\mathbf{r}_d, \mathbf{r}_{s0}, \mathbf{r}_{s1}$, then:

$$s \rightarrow (pc + 4, \rho[d \leftarrow \rho(s0) - \rho(s1)], \mu)$$

- if $i =$ **b** dst, then:

$$s \rightarrow (dst, \rho, \mu)$$

A MIPS like assembly language: transition relation

We assume a state $s = (pc, \rho, \mu)$ and that $\mu(pc) = i$; then:

- if $i = \textbf{blt}\ \mathbf{r}_{s0}, \mathbf{r}_{s1}, dst$, then:

$$s \rightarrow \begin{cases} (dst, \rho, \mu) & \text{if } \rho(s0) < \rho(s1) \\ (pc + 4, \rho, \mu) & \text{otherwise} \end{cases}$$

- if $i = \textbf{ld}\ \mathbf{r}_d, o, \mathbf{r}_x$, then:

$$s \rightarrow \begin{cases} (pc + 4, \rho[d \leftarrow \mu(\rho(x) + o)], \mu) & \text{if } \mu(\rho(x) + o) \text{ is defined} \\ \Omega & \text{otherwise} \end{cases}$$

- if $i = \textbf{st}\ \mathbf{r}_d, o, \mathbf{r}_x$, then:

$$s \rightarrow \begin{cases} (pc + 4, \rho, \mu[\rho(x) + o \leftarrow \rho(d)]) & \text{if } \mu(\rho(x) + o) \text{ is defined} \\ \Omega & \text{otherwise} \end{cases}$$

A simple imperative language: syntax

We now look at a more classical **imperative language** (intuitively, a bare-bone subset of C):

- **variables** $\mathbb{X}$: finite, predefined set of variables
- **labels** $\mathbb{L}$: before and after each statement
- **values** $\mathbb{V}$: $\mathbb{V}_{\text{int}} \cup \mathbb{V}_{\text{float}} \cup \ldots$

Syntax

$\mathtt{e}$	$::=$	$v \in \mathbb{V}_{\text{int}} \cup \mathbb{V}_{\text{float}} \cup \ldots \mid \mathtt{e} + \mathtt{e} \mid \mathtt{e} * \mathtt{e} \mid \ldots$	expressions
$\mathtt{c}$	$::=$	$\mathtt{TRUE} \mid \mathtt{FALSE} \mid \mathtt{e} < \mathtt{e} \mid \mathtt{e} = \mathtt{e}$	conditions
$\mathtt{i}$	$::=$	$x := \mathtt{e};$	assignment
	$\mid$	$\textbf{if}(\mathtt{c})\ \mathtt{b}\ \textbf{else}\ \mathtt{b}$	condition
	$\mid$	$\textbf{while}(\mathtt{c})\ \mathtt{b}$	loop
$\mathtt{b}$	$::=$	$\{\mathtt{i}; \ldots; \mathtt{i};\}$	block, program($\mathbb{P}$)

A simple imperative language: states

A **non-error state** should fully describe the configuration at one instant of the program execution:

- the **memory state** defines the current contents of the memory

$$m \in \mathbb{M} = \mathbb{X} \longrightarrow \mathbb{V}$$

- the **control state** defines *where* the program currently is
  - analoguous to the program counter
  - can be defined by adding **labels** $\mathbb{L} = \{\ell_0, \ell_1, \ldots\}$ between each pair of consecutive statements; then:

$$\mathbb{S} = \mathbb{L} \times \mathbb{M} \uplus \{\Omega\}$$

  - or by the program remaining to be executed; then:

$$\mathbb{S} = \mathbb{P} \times \mathbb{M} \uplus \{\Omega\}$$

A simple imperative language: semantics of expressions

- The **semantics $[\![\mathrm{e}]\!]$ of expression** e should evaluate each expression into a value, given a memory state
- **Evaluation errors** may occur: division by zero...
  error value is also noted Ω

Thus: $[\![\mathrm{e}]\!] : \mathbb{M} \longrightarrow \mathbb{V} \uplus \{\Omega\}$

Definition, by **induction over the syntax:**

$$\begin{array}{rcl}
[\![v]\!](m) & = & v \\
[\![x]\!](m) & = & m(x) \\
[\![\mathrm{e}_0 + \mathrm{e}_1]\!](m) & = & [\![\mathrm{e}_0]\!](m) \underline{+} [\![\mathrm{e}_1]\!](m) \\
[\![\mathrm{e}_0 / \mathrm{e}_1]\!](m) & = & \left\{ \begin{array}{ll} \Omega & \text{if } [\![\mathrm{e}_1]\!](m) = 0 \\ [\![\mathrm{e}_0]\!](m) \underline{/} [\![\mathrm{e}_1]\!](m) & \text{otherwise} \end{array} \right.
\end{array}$$

where $\underline{\oplus}$ is the machine implementation of operator $\oplus$, and is Ω-strict, i.e., $\forall v \in \mathbb{V},\ v \underline{\oplus} \Omega = \Omega \underline{\oplus} v = \Omega$.

A simple imperative language: semantics of conditions

- The **semantics $[\![\mathsf{c}]\!]$ of condition** c should return a *boolean value*
- It follows a similar definition to that of the semantics of expressions: $[\![\mathsf{c}]\!] : \mathbb{M} \longrightarrow \mathbb{V}_{\mathrm{bool}} \uplus \{\Omega\}$

Definition, by **induction over the syntax**:

$$\begin{array}{rcl}
[\![\mathtt{TRUE}]\!](m) & = & \mathtt{TRUE} \\
[\![\mathtt{FALSE}]\!](m) & = & \mathtt{FALSE} \\
[\![\mathsf{e}_0 < \mathsf{e}_1]\!](m) & = & \left\{ \begin{array}{ll} \mathtt{TRUE} & \text{if } [\![\mathsf{e}_0]\!](m) < [\![\mathsf{e}_1]\!](m) \\ \mathtt{FALSE} & \text{if } [\![\mathsf{e}_0]\!](m) \geq [\![\mathsf{e}_1]\!](m) \\ \Omega & \text{if } [\![\mathsf{e}_0]\!](m) = \Omega \text{ or } [\![\mathsf{e}_1]\!](m) = \Omega \end{array} \right. \\
[\![\mathsf{e}_0 = \mathsf{e}_1]\!](m) & = & \left\{ \begin{array}{ll} \mathtt{TRUE} & \text{if } [\![\mathsf{e}_0]\!](m) = [\![\mathsf{e}_1]\!](m) \\ \mathtt{FALSE} & \text{if } [\![\mathsf{e}_0]\!](m) \neq [\![\mathsf{e}_1]\!](m) \\ \Omega & \text{if } [\![\mathsf{e}_0]\!](m) = \Omega \text{ or } [\![\mathsf{e}_1]\!](m) = \Omega \end{array} \right.
\end{array}$$

A simple imperative language: transitions

We now consider the transition induced by each statement.

Case of **assignment** $\ell_0 : x = \mathrm{e};\ \ell_1$

- if $[\![\mathrm{e}]\!](m) \neq \Omega$, then $(\ell_0, m) \rightarrow (\ell_1, m[x \leftarrow [\![\mathrm{e}]\!](m)])$
- if $[\![\mathrm{e}]\!](m) = \Omega$, then $(\ell_0, m) \rightarrow \Omega$

Case of **condition** $\ell_0 : \mathbf{if}(\mathrm{c})\{\ell_1 : \mathtt{b}_t\ \ell_2\}\ \mathbf{else}\{\ell_3 : \mathtt{b}_f\ \ell_4\}\ \ell_5$

- if $[\![\mathrm{c}]\!](m) = \mathtt{TRUE}$, then $(\ell_0, m) \rightarrow (\ell_1, m)$
- if $[\![\mathrm{c}]\!](m) = \mathtt{FALSE}$, then $(\ell_0, m) \rightarrow (\ell_3, m)$
- if $[\![\mathrm{c}]\!](m) = \Omega$, then $(\ell_0, m) \rightarrow \Omega$
- $(\ell_2, m) \rightarrow (\ell_5, m)$
- $(\ell_4, m) \rightarrow (\ell_5, m)$

A simple imperative language: transitions

Case of **loop** $\ell_0 : \textbf{while}(\mathtt{c})\{\ell_1 : \mathtt{b}_t\ \ell_2\}\ \ell_3$

- if $[\![\mathtt{c}]\!](m) = \mathtt{TRUE}$, then $\begin{cases} (\ell_0, m) \rightarrow (\ell_1, m) \\ (\ell_2, m) \rightarrow (\ell_1, m) \end{cases}$
- if $[\![\mathtt{c}]\!](m) = \mathtt{FALSE}$, then $\begin{cases} (\ell_0, m) \rightarrow (\ell_3, m) \\ (\ell_2, m) \rightarrow (\ell_3, m) \end{cases}$
- if $[\![\mathtt{c}]\!](m) = \Omega$, then $\begin{cases} (\ell_0, m) \rightarrow \Omega \\ (\ell_2, m) \rightarrow \Omega \end{cases}$

Case of $\{\ell_0 : \mathtt{i}_0; \ell_1 : \ldots; \ell_{n-1}\mathtt{i}_{n-1}; \ell_n\}$

- the transition relation is defined by the individual instructions

Extending the language with non-determinism

The language we have considered so far is a bit **limited**:

- it is **deterministic**: at most one transition possible from any state
- it does not support the **input of values**

Changes if we model non deterministic inputs...

... with an input instruction:

- $\texttt{i} ::= \ldots \mid x := \textbf{input}()$
- $\ell_0 : x := \textbf{input}();\, \ell_1$ generates transitions

$$\forall v \in \mathbb{V},\ (\ell_0, m) \rightarrow (\ell_1, m[x \leftarrow v])$$

- one instruction induces non determinism

... with a random function:

- $\texttt{e} ::= \ldots \mid \textbf{rand}()$
- **expressions** have a **non-deterministic** semantics:

$$\begin{array}{c} [\![\texttt{e}]\!] : \mathbb{M} \rightarrow \mathcal{P}(\mathbb{V} \uplus \{\Omega\}) \\ [\![\textbf{rand}()]\!](m) = \mathbb{V} \\ [\![v]\!](m) = \{v\} \\ [\![\texttt{c}]\!] : \mathbb{M} \rightarrow \mathcal{P}(\mathbb{V}_{\mathrm{bool}} \uplus \{\Omega\}) \end{array}$$

- all instructions induce non determinism

Semantics of real world programming languages

C language:

- several **norms**: ANSI C'99, ANSI C'11, K&R...
- not fully specified:
  - **undefined behavior**
  - **implementation dependent behavior**: architecture (ABI) or implementation (compiler...)
  - unspecified parts: leave room for implementation of compilers and optimizations
- **formalizations** in HOL (C'99), in Coq (CompCert C compiler)

OCaml language:

- more formal...
- ... but still with some unspecified parts, e.g., execution order

Outline

Execution traces

- So far, we considered only states and atomic transitions
- We now consider program **executions** as a whole

Definition: traces

- A **finite trace** is a finite sequence of states $s_0, \ldots, s_n$, noted $\langle s_0, \ldots, s_n \rangle$
- An **infinite trace** is an infinite sequence of states $\langle s_0, \ldots \rangle$

Besides, we write:

- $\mathbb{S}^\star$ for the **set of finite traces**
- $\mathbb{S}^\omega$ for the **set of infinite traces**
- $\mathbb{S}^\propto = \mathbb{S}^\star \cup \mathbb{S}^\omega$ for the **set of finite or infinite traces**

Operations on traces: concatenation

Definition: concatenation

The **concatenation operator** $\cdot$ is defined by:

$$
\begin{array}{rcl}
\langle s_0, \ldots, s_n \rangle \cdot \langle s'_0, \ldots, s'_{n'} \rangle & = & \langle s_0, \ldots, s_n, s'_0, \ldots, s'_{n'} \rangle \\
\langle s_0, \ldots, s_n \rangle \cdot \langle s'_0, \ldots \rangle & = & \langle s_0, \ldots, s_n, s'_0, \ldots \rangle \\
\langle s_0, \ldots, s_n, \ldots \rangle \cdot \sigma' & = & \langle s_0, \ldots, s_n, \ldots \rangle
\end{array}
$$

We also define:

- the **empty trace** ϵ, neutral element for $\cdot$
- the **length** operator $|.|$:

$$
\left\{
\begin{array}{lcl}
|\epsilon| & = & 0 \\
|\langle s_0, \ldots, s_n \rangle| & = & n + 1 \\
|\langle s_0, \ldots \rangle| & = & \omega
\end{array}
\right.
$$

Comparing traces: the prefix order relation

Definition: prefix order relation

Relation $\prec$ is defined by:

$$\langle s_0, \ldots, s_n \rangle \prec \langle s'_0, \ldots, s'_{n'} \rangle \iff \begin{cases} n \leq n' \\ \forall i \in [\![0, n]\!],\ s_i = s'_i \end{cases}$$

$$\langle s_0, \ldots \rangle \prec \langle s'_0, \ldots \rangle \iff \forall i \in \mathbb{N},\ s_i = s'_i$$

$$\langle s_0, \ldots, s_n \rangle \prec \langle s'_0, \ldots \rangle \iff \forall i \in [\![0, n]\!],\ s_i = s'_i$$

Proof: straightforward application of the definition of order relations

Outline

Semantics of finite traces

We consider a transition system $\mathcal{S} = (\mathbb{S}, \rightarrow)$

Definition

The **finite traces semantics** $[\![\mathcal{S}]\!]^\star$ is defined by:

$$[\![\mathcal{S}]\!]^\star = \{\langle s_0, \ldots, s_n \rangle \in \mathbb{S}^\star \mid \forall i,\ s_i \rightarrow s_{i+1}\}$$

Example:

- contrived transition system $\mathcal{S} = (\{a, b, c, d\}, \{(a, b), (b, a), (b, c)\})$
- finite traces semantics:

$$\begin{array}{rcll} [\![\mathcal{S}]\!]^\star & = & \{ & \epsilon, \\ & & & \langle a, b, \ldots, a, b, a \rangle, \quad \langle b, a, \ldots, a, b, a \rangle, \\ & & & \langle a, b, \ldots, a, b, a, b \rangle, \quad \langle b, a, \ldots, a, b, a, b \rangle, \\ & & & \langle a, b, \ldots, a, b, a, b, c \rangle, \quad \langle b, a, \ldots, a, b, a, b, c \rangle \\ & & & \langle c \rangle, \quad \langle d \rangle \quad \} \end{array}$$

Interesting subsets of the finite trace semantics

We consider a transition system $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_{\mathcal{I}}, \mathbb{S}_{\mathcal{F}})$

- the **initial traces**, i.e., starting from an initial state:

$$\{\langle s_0, \ldots, s_n \rangle \in [\![\mathcal{S}]\!]^\star \mid s_0 \in \mathbb{S}_{\mathcal{I}}\}$$

- the **traces reaching a blocking state**:

$$\{\sigma \in [\![\mathcal{S}]\!]^\star \mid \forall \sigma' \in [\![\mathcal{S}]\!]^\star, \sigma \prec \sigma' \Longrightarrow \sigma = \sigma'\}$$

- the **traces ending in a final state**:

$$\{\langle s_0, \ldots, s_n \rangle \in [\![\mathcal{S}]\!]^\star \mid s_n \in \mathbb{S}_{\mathcal{F}}\}$$

Example (same transition system, with $\mathbb{S}_{\mathcal{I}} = \{a\}$ and $\mathbb{S}_{\mathcal{F}} = \{c\}$):

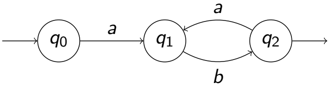
- traces from an initial state ending in a final state:

$$\{\langle a, b, \ldots, a, b, a, b, c \rangle\}$$

Example: finite automaton

We consider the **example of the previous course:**

$$L = \{a, b\} \qquad Q = \{q_0, q_1, q_2\}$$
$$q_{\mathrm{i}} = q_0 \qquad q_{\mathrm{f}} = q_2$$
$$q_0 \stackrel{a}{\longrightarrow} q_1 \qquad q_1 \stackrel{b}{\longrightarrow} q_2 \qquad q_2 \stackrel{a}{\longrightarrow} q_1$$



Then, we have the following traces:

$$\begin{array}{rcl}
\tau_0 & = & \langle (q_0, ab), (q_1, b), (q_2, \epsilon) \rangle \\
\tau_1 & = & \langle (q_0, abab), (q_1, bab), (q_2, ab), (q_1, b), (q_2, \epsilon) \rangle \\
\tau_2 & = & \langle (q_0, ababab), (q_1, babab), (q_2, abab), (q_1, bab) \rangle \\
\tau_3 & = & \langle (q_0, abaaa), (q_1, baaa), (q_2, aaa), (q_1, aa) \rangle
\end{array}$$

Then:

- τ_0, τ_1 are initial traces, reaching a final state
- τ_2 is an initial trace, and is not maximal
- τ_3 reaches a blocking state, but not a final state

Example: λ-term

We consider λ-term $\lambda y \cdot ((\lambda x \cdot y)((\lambda x \cdot x\,x)\,(\lambda x \cdot x\,x)))$, and show two traces generated from it (at each step the reduced lambda is shown in red):

$$\tau_0 = \langle \begin{array}{l} \lambda y \cdot ((\lambda x \cdot y)((\lambda x \cdot x\,x)\,(\lambda x \cdot x\,x))) \\ \lambda y \cdot y \quad \rangle \end{array}$$

$$\tau_1 = \langle \begin{array}{l} \lambda y \cdot ((\lambda x \cdot y)((\lambda x \cdot x\,x)\,(\lambda x \cdot x\,x))) \\ \lambda y \cdot ((\lambda x \cdot y)((\lambda x \cdot x\,x)\,(\lambda x \cdot x\,x))) \\ \lambda y \cdot ((\lambda x \cdot y)((\lambda x \cdot x\,x)\,(\lambda x \cdot x\,x))) \quad \rangle \end{array}$$

Then:

- τ_0 is a maximal trace; it reaches a blocking state (no more reduction can be done)
- τ_1 can be extended for arbitrarily many steps ;
  the second part of the course will study **infinite traces**

Example: imperative program

Similarly, we can write the traces of a simple imperative program:

$$
\begin{array}{ll}
l_0: & \mathrm{x} := 1; \\
l_1: & \mathrm{y} := 0; \\
l_2: & \textbf{while}(\mathrm{x} < 4)\{ \\
l_3: & \quad \mathrm{y} := \mathrm{y} + \mathrm{x}; \\
l_4: & \quad \mathrm{x} := \mathrm{x} + 1; \\
l_5: & \} \\
l_6: & \text{(final program point)}
\end{array}
$$

$$
\begin{array}{rcll}
\tau & = & \langle & (l_0, (\!|\mathrm{x} = 6, \mathrm{y} = 8|\!)), (l_1, (\!|\mathrm{x} = 1, \mathrm{y} = 8|\!)), \\
& & & (l_2, (\!|\mathrm{x} = 1, \mathrm{y} = 0|\!)), (l_3, (\!|\mathrm{x} = 1, \mathrm{y} = 0|\!)), \\
& & & (l_4, (\!|\mathrm{x} = 1, \mathrm{y} = 1|\!)), (l_5, (\!|\mathrm{x} = 2, \mathrm{y} = 1|\!)), \\
& & & (l_3, (\!|\mathrm{x} = 2, \mathrm{y} = 1|\!)), (l_4, (\!|\mathrm{x} = 2, \mathrm{y} = 3|\!)), \\
& & & (l_5, (\!|\mathrm{x} = 3, \mathrm{y} = 3|\!)), (l_3, (\!|\mathrm{x} = 3, \mathrm{y} = 3|\!)), \\
& & & (l_4, (\!|\mathrm{x} = 3, \mathrm{y} = 6|\!)), (l_5, (\!|\mathrm{x} = 4, \mathrm{y} = 6|\!)), \\
& & & (l_6, (\!|\mathrm{x} = 4, \mathrm{y} = 6|\!)) \quad \rangle
\end{array}
$$

- very **precise** description of what the program does...
- ... but **quite cumbersome**

Outline

Towards a fixpoint definition

We consider again our contrived transition system

$$\mathcal{S} = (\{a, b, c, d\}, \{(a, b), (b, a), (b, c)\})$$

Traces **by length:**

i	traces of length i
0	ϵ
1	$\langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle$
2	$\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle$
3	$\langle a, b, a \rangle, \langle b, a, b \rangle, \langle a, b, c \rangle$
4	$\langle a, b, a, b \rangle, \langle b, a, b, a \rangle, \langle b, a, b, c \rangle$

Like the automaton in lecture 1, this suggests a least fixpoint definition: traces of length $i + 1$ can be derived from the traces of length i, by adding a transition

Trace semantics fixpoint form

We define a **semantic function**, that computes the traces of length $i+1$ from the traces of length i (where $i \geq 1$):

Finite traces semantics as a fixpoint

Let $\mathcal{I} = \{\epsilon\} \uplus \{\langle s \rangle \mid s \in \mathbb{S}\}$.
Let $F_\star$ be the function defined by:

$$\begin{array}{rccl} F_\star : & \mathcal{P}(\mathbb{S}^\star) & \longrightarrow & \mathcal{P}(\mathbb{S}^\star) \\ & X & \longmapsto & X \cup \{\langle s_0, \ldots, s_n, s_{n+1} \rangle \mid \langle s_0, \ldots, s_n \rangle \in X \wedge s_n \rightarrow s_{n+1}\} \end{array}$$

Then, $F_\star$ is **continuous** and thus has a least-fixpoint greater than $\mathcal{I}$; moreover:

$$\mathbf{lfp}_{\mathcal{I}} F_\star = [\![\mathcal{S}]\!]^\star = \bigcup_{n \in \mathbb{N}} F_\star^n(\mathcal{I})$$

Fixpoint definition: proof (1), fixpoint existence

First, we prove that $F_\star$ is **continuous**.

Let $\mathcal{X} \subseteq \mathcal{P}(\mathbb{S}^\star)$ and $A = \bigcup_{X \in \mathcal{X}} X$. Then:

$$\begin{array}{l}
F_\star(\bigcup_{X\in\mathcal{X}} X) \\
\quad = \quad A \cup \{\langle s_0, \ldots, s_n, s_{n+1}\rangle \mid (\langle s_0, \ldots, s_n\rangle \in \bigcup_{X\in\mathcal{X}} X) \wedge s_n \rightarrow s_{n+1}\} \\
\quad = \quad A \cup \{\langle s_0, \ldots, s_n, s_{n+1}\rangle \mid (\exists X \in \mathcal{X},\ \langle s_0, \ldots, s_n\rangle \in X) \wedge s_n \rightarrow s_{n+1}\} \\
\quad = \quad A \cup \{\langle s_0, \ldots, s_n, s_{n+1}\rangle \mid \exists X \in \mathcal{X},\ \langle s_0, \ldots, s_n\rangle \in X \wedge s_n \rightarrow s_{n+1}\} \\
\quad = \quad (\bigcup_{X\in\mathcal{X}} X) \cup (\bigcup_{X\in\mathcal{X}} \{\langle s_0, \ldots, s_n, s_{n+1}\rangle \mid \langle s_0, \ldots, s_n\rangle \in X \wedge s_n \rightarrow s_{n+1}\}) \\
\quad = \quad \bigcup_{X\in\mathcal{X}} (X \cup \{\langle s_0, \ldots, s_n, s_{n+1}\rangle \mid \langle s_0, \ldots, s_n\rangle \in X \wedge s_n \rightarrow s_{n+1}\}) \\
\quad = \quad \bigcup_{X\in\mathcal{X}} F_\star(X)
\end{array}$$

Function $F_\star$ is $\cup$-complete, hence continuous.

As $(\mathcal{P}(\mathbb{S}^\star), \subseteq)$ is a CPO, the continuity of $F_\star$ entails the **existence of a least-fixpoint** (Kleene theorem); moreover, it implies that:

$$\mathbf{lfp}_{\mathcal{I}} F_\star = \bigcup_{n\in\mathbb{N}} F_\star^n(\mathcal{I})$$

Fixpoint definition: proof (2), fixpoint equality

We now show **that $[\![\mathcal{S}]\!]^\star$ is equal to $\mathbf{lfp}_{\mathcal{I}} F_\star$**, by showing the property below, by induction over n:

$$\forall k \leq n,\ \langle s_0, \ldots, s_k \rangle \in F_\star^n(\mathcal{I}) \iff \langle s_0, \ldots, s_k \rangle \in [\![\mathcal{S}]\!]^\star$$

- at rank 0, only traces of length 1 need be considered:

$$\begin{array}{rcl} \langle s \rangle \in [\![\mathcal{S}]\!]^\star & \iff & s \in \mathbb{S} \\ & \iff & \langle s \rangle \in F_\star^0(\mathcal{I}) \end{array}$$

- at rank $n+1$, and assuming the property holds at rank n (the equivalence is obvious for traces of length 1):

$$\begin{array}{l} \langle s_0, \ldots, s_k, s_{k+1} \rangle \in [\![\mathcal{S}]\!]^\star \\ \quad \iff \quad \langle s_0, \ldots, s_k \rangle \in [\![\mathcal{S}]\!]^\star \wedge s_k \to s_{k+1} \\ \quad \iff \quad \langle s_0, \ldots, s_k \rangle \in F_\star^n(\mathcal{I}) \wedge s_k \to s_{k+1} \quad (k \leq n \text{ since } k+1 \leq n+1) \\ \quad \iff \quad \langle s_0, \ldots, s_k, s_{k+1} \rangle \in F_\star^{n+1}(\mathcal{I}) \end{array}$$

Trace semantics fixpoint form: example

Example, with the same simple transition system $\mathcal{S} = (\mathbb{S}, \rightarrow)$:

- $\mathbb{S} = \{a, b, c, d\}$
- $\rightarrow$ is defined by $a \rightarrow b$, $b \rightarrow a$ and $b \rightarrow c$

Then, the first iterates are:

$$\begin{array}{rcl}
F_\star^0(\mathcal{I}) & = & \{\epsilon, \langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle\} \\
F_\star^1(\mathcal{I}) & = & F_\star^0(\mathcal{I}) \cup \{\langle b, a \rangle, \langle a, b \rangle, \langle b, c \rangle\} \\
F_\star^2(\mathcal{I}) & = & F_\star^1(\mathcal{I}) \cup \{\langle a, b, a \rangle, \langle b, a, b \rangle, \langle a, b, c \rangle\} \\
F_\star^3(\mathcal{I}) & = & F_\star^2(\mathcal{I}) \cup \{\langle b, a, b, a \rangle, \langle a, b, a, b \rangle, \langle b, a, b, c \rangle\} \\
F_\star^4(\mathcal{I}) & = & F_\star^3(\mathcal{I}) \cup \{\langle a, b, a, b, a \rangle, \langle b, a, b, a, b \rangle, \langle a, b, a, b, c \rangle\} \\
F_\star^5(\mathcal{I}) & = & \ldots
\end{array}$$

- the traces of $[\![\mathcal{S}]\!]^\star$ of length $n+1$ appear in $F_\star^n(\mathcal{I})$

Outline

Notion of compositional semantics

The traces semantics definition we have seen is **global:**

- the **whole system** defines a **transition relation**
- we **iterate** this relation until we get a fixpoint

Though, a **modular** definition would be nicer, to allow reasoning on program fragments, or derive properties of a program from properties of its pieces...

Can we derive a more modular expression of the semantics ?

Notion of compositional semantics

Observation: programs often have an inductive structure

- **λ-terms** are defined by induction over the syntax
- **imperative programs** are defined by induction over the syntax
- **there are exceptions:** our MIPS language does not naturally look that way

Definition: compositional semantics

A semantics $[\![.]\!]$ is said to be **compositional** when the semantics of a program can be defined as a function of the semantics of its parts, i.e.,
When program π writes down as $C[\pi_0, \ldots, \pi_k]$ where $\pi_0, \ldots, \pi_k$ are its components, there exists a function F_C such that
$[\![\pi]\!] = F_C([\![\pi_0]\!], \ldots, [\![\pi_k]\!])$, where F_C depends only on syntactic construction F_C.

Case of a simplified imperative language

Case of **a sequence of two instructions** $\mathtt{b} \equiv \ell_0 : \mathtt{i}_0; \ell_1 : \mathtt{i}_1; \ell_2$:

$$
\begin{array}{rcl}
[\![\mathtt{b}]\!]^\star & = & [\![\mathtt{i}_0]\!]^\star \cup [\![\mathtt{i}_1]\!]^\star \\
& \cup & \{\langle s_0, \ldots, s_m \rangle \mid \exists n \in [\![0, m]\!], \\
& & \quad \langle s_0, \ldots, s_n \rangle \in [\![\mathtt{i}_0]\!]^\star \wedge \langle s_n, \ldots, s_m \rangle \in [\![\mathtt{i}_1]\!]^\star\}
\end{array}
$$

This amounts to **concatenating** traces of $[\![\mathtt{i}_0]\!]^\star$ and $[\![\mathtt{i}_1]\!]^\star$ that share a state in common (necessarily at point ℓ_1).

Cases of **a condition, a loop: similar**

- by **concatenation** of traces around **junction points**
- by doing a **least-fixpoint computation** over loops

We can provide a compositional semantics for our simplified imperative language

Case of λ-calculus

Case of a **λ-term** $t = (\lambda x \cdot u)\, v$:

- executions may start with a reduction in u
- executions may start with a reduction in v
- executions may start with the reduction of the head redex
- an execution may mix reductions steps in u and v in an arbitrary order

No nice compositional trace semantics of λ-calculus...

Outline

Non termination

Can the finite traces semantics express non termination ?

Consider the case of our contrived system:

$$\mathbb{S} = \{a, b, c, d\} \qquad (\rightarrow) = \{(a, b), (b, a), (b, c)\}$$

- this system clearly **has non-terminating behaviors**:
  it can loop from a to b and back forever
- the finite traces semantics does show **the existence of this cycle** as there exists an **infinite chain of finite traces for the prefix order $\prec$**:
  $$\langle a, b\rangle, \langle a, b, a\rangle, \langle a, b, a, b\rangle, \langle a, b, a, b, a\rangle, \ldots \in [\![\mathcal{S}]\!]^\star$$
- though, the existence of this chain is **not very obvious**

Thus, we now define a semantics made of infinite traces

Semantics of infinite traces

We consider a transition system $\mathcal{S} = (\mathbb{S}, \rightarrow)$

Definition

The **infinite traces semantics** $[\![\mathcal{S}]\!]^\omega$ is defined by:

$$[\![\mathcal{S}]\!]^\omega = \{\langle s_0, \ldots \rangle \in \mathbb{S}^\omega \mid \forall i,\ s_i \rightarrow s_{i+1}\}$$

Infinite traces starting from an initial state (considering $\mathcal{S} = (\mathbb{S}, \rightarrow, \mathbb{S}_\mathcal{I}, \mathbb{S}_\mathcal{F})$):

$$\{\langle s_0, \ldots \rangle \in [\![\mathcal{S}]\!]^\omega \mid s_0 \in \mathbb{S}_\mathcal{I}\}$$

Example:

- contrived transition system defined by

$$\mathbb{S} = \{a, b, c, d\} \qquad (\rightarrow) = \{(a, b), (b, a), (b, c)\}$$

- the infinite traces semantics contains **exactly two** traces

$$[\![\mathcal{S}]\!]^\omega = \{\langle a, b, \ldots, a, b, a, b, \ldots \rangle, \langle b, a, \ldots, b, a, b, a, \ldots \rangle\}$$

Fixpoint form

Can we also provide a fixpoint form for $[\![\mathcal{S}]\!]^\omega$?

Intuitively, $\langle s_0, s_1, \ldots \rangle \in [\![\mathcal{S}]\!]^\omega$ if and only if $\forall n,\ s_n \rightarrow s_{n+1}$, i.e.,

$$\forall n \in \mathbb{N},\ \forall k \leq n,\ s_k \rightarrow s_{k+1}$$

Let F_ω be defined by:

$$\begin{array}{rccl} F_\omega : & \mathcal{P}(\mathbb{S}^\omega) & \longrightarrow & \mathcal{P}(\mathbb{S}^\omega) \\ & X & \longmapsto & \{\langle s_0, s_1, \ldots, s_n, \ldots \rangle \mid \langle s_1, \ldots, s_n, \ldots \rangle \in X \wedge s_0 \rightarrow s_1\} \end{array}$$

Then, we can show by induction that:

$$\begin{array}{rcl} \sigma \in [\![\mathcal{S}]\!]^\omega & \Longleftrightarrow & \forall n \in \mathbb{N},\ \sigma \in F_\omega^n(\mathbb{S}^\omega) \\ & \Longleftrightarrow & \bigcap_{n \in \mathbb{N}} F_\omega^n(\mathbb{S}^\omega) \end{array}$$

Fixpoint form of the semantics of infinite traces

Infinite traces semantics as a fixpoint

Let F_ω be the function defined by:

$$F_\omega : \begin{array}{lcl} \mathcal{P}(\mathbb{S}^\omega) & \longrightarrow & \mathcal{P}(\mathbb{S}^\omega) \\ X & \longmapsto & \{\langle s_0, s_1, \ldots, s_n, \ldots\rangle \mid \langle s_1, \ldots, s_n, \ldots\rangle \in X \wedge s_0 \rightarrow s_1\} \end{array}$$

Then, F_ω is **$\cap$-continuous** and thus has a **greatest-fixpoint**; moreover:

$$\mathbf{gfp}_{\mathbb{S}^\omega} F_\omega = [\![\mathcal{S}]\!]^\omega = \bigcap_{n\in\mathbb{N}} F_\omega^n(\mathbb{S}^\omega)$$

Proof sketch:

- the $\cap$-contiunity proof is similar as for the $\cup$-continuity of $F_\star$
- by the dual version of Kleene's theorem, $\mathbf{gfp}_{\mathbb{S}^\omega} F_\omega$ exists and is equal to $\bigcap_{n\in\mathbb{N}} F_\omega^n(\mathbb{S}^\omega)$, i.e. to $[\![\mathcal{S}]\!]^\omega$ (similar induction proof)

Fixpoint form of the infinite traces semantics: iterates

Example, with the same simple transition system:

- $\mathbb{S} = \{a, b, c, d\}$
- $\rightarrow$ is defined by $a \rightarrow b$, $b \rightarrow a$ and $b \rightarrow c$

Then, the first iterates are:

$$\begin{array}{rcl}
F_\omega^0(\mathbb{S}^\omega) & = & \mathbb{S}^\omega \\
F_\omega^1(\mathbb{S}^\omega) & = & \langle a, b\rangle \cdot \mathbb{S}^\omega \cup \langle b, a\rangle \cdot \mathbb{S}^\omega \cup \langle b, c\rangle \cdot \mathbb{S}^\omega \\
F_\omega^2(\mathbb{S}^\omega) & = & \langle b, a, b\rangle \cdot \mathbb{S}^\omega \cup \langle a, b, a\rangle \cdot \mathbb{S}^\omega \cup \langle a, b, c\rangle \cdot \mathbb{S}^\omega \\
F_\omega^3(\mathbb{S}^\omega) & = & \langle a, b, a, b\rangle \cdot \mathbb{S}^\omega \cup \langle b, a, b, a\rangle \cdot \mathbb{S}^\omega \cup \langle b, a, b, c\rangle \cdot \mathbb{S}^\omega \\
F_\omega^4(\mathbb{S}^\omega) & = & \ldots
\end{array}$$

Intuition

- at iterate n, prefixes of length $n + 1$ match the traces in the infinite semantics
- only $\langle a, b, \ldots, a, b, a, b, \ldots\rangle$ and $\langle b, a, \ldots, b, a, b, a, \ldots\rangle$ belong to *all* iterates

Outline

Summary

We have discussed:

- **small-step / structural operational semantics:**
  individual program steps
- **big-step / natural semantics:**
  program executions as sequences of transitions
- their **fixpoint definitions** and properties

Next lectures:

- another family of semantics, **more compact** and **compositional**
- **semantic program** and **proof methods**