Divide and recycle: types and compilation for a hybrid synchronous language [a]

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[a] Joint work with Albert Benveniste, Timothy Bourke and Benoit Caillaud

Motivation and Context

- **Explicit** *vs* **Implicit** hybrid system modelers: Simulink, Scicos *vs* Modelica.
- In this talk, we consider only explicit ones.
- A lot of work on the formal verification of hybrid systems but relatively few on programming language aspects.

Objective:

- Extend a Lustre-like language where dataflow equations are mixed with ODE.
- Make it conservative, i.e., nothing must change for the discrete subset (same typing, same code generation).

Contribution:

- **Divide** with a novel type system.
- **Recycle** existing tools, synchronous compilers and numerical solvers to execute them.

Parallel composition: homogeneous case

Two equations with discrete time:

```
f = 0.0 -> pre f + s and s = 0.2 * (x - pre f)
```

and the initial value problem:

```
der(y') = -9.81 init 0.0 and der(y) = y' init 10.0
```

The first program can be written in any synchronous language, e.g. LUSTRE.

$$\forall n \in \mathbb{N}^*, f_n = f_{n-1} + s_n \text{ and } f_0 = 0 \qquad \forall n \in \mathbb{N}, s_n = 0.2 * (x_n - f_{n-1})$$

The second program can be written in any hybrid modeler, e.g. SIMULINK.

$$\forall t \in \mathbb{R}_+, y'(t) = 0.0 + \int_0^t -9.81\, dt = -9.81\, t$$

$$\forall t \in \mathbb{R}_+, y(t) = 10.0 + \int_0^t y'(t)\, dt = 10.0 - 9.81 \int_0^t t\, dt$$

Parallel composition is clear since equations **share the same time scale**.

Parallel composition: heterogeneous case

Two equations: a signal defined at discrete instants, the other continuously.

```
der(time) = 1.0 init 0.0 and x = 0.0 fby x + time
```

or:

```
x = 0.0 fby x +. 1.0 and der(y) = x init 0.0
```

It would be tempting to define the first equation as: $\forall n \in \mathbb{N}, x_n = x_{n-1} + \texttt{time}(n)$

And the second as:

$$\forall n \in \mathbb{N}^*, x_n = x_{n-1} + 1.0 \text{ and } x_0 = 1.0$$

$$\forall t \in \mathbb{R}_+, y(t) = 0.0 + \int_0^t x(t)\, dt$$

i.e., $x(t)$ as a piecewise constant function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\forall t \in \mathbb{R}_+, x(t) = x_{\lfloor t \rfloor}$.

In both cases, this would be a mistake. `x` is defined on a discrete, logical time; `time` on an continuous, absolute time.

Equations with reset

Two independent groups of equations.

```
  der(p) = 1.0 init 0.0 reset 0.0 every up(p - 1.0)
and
  x = 0.0 fby x + p
and
  der(time) = 1.0 init 0.0
and
  z = up(sin (freq * time))
```

Properly translated in Simulink, changing `freq` changes the output of `x`!

If `f` is running on a continuous time basis, what would be the meaning of:

```
  y = f(x) every up(z) init 0
```

All these programs are **wrongly typed** and should be statically rejected. Simulink does it.

Discrete *vs* Continuous time signals

A signal is discrete if it is activated on a discrete clock.

A clock is termed *discrete* if it has been declared so or if it is the result of a zero-crossing or a sub-sampling of a discrete clock. Otherwise, it is termed *continuous*.

Notation

- `up`(e) tests the zero-crossing of expression e (from negative to positive).
- If $x =$ `up`(e), all handlers using x are governed by the same zero-crossing.
- Handlers have priorities.

```
z = 1 every up(x) | 2 every up(y) init 0
```

- `last`(x) for the left-limit of signal x.

```
z = last z + 1 every up(x) | last z - 1 every up(y) init 0
```

Examples

Combinatorial and sequential function (discrete time).

```
let add (x,y) = x + y

let node counter(top, tick) = o where
      o = if top then i else 0 fby o + 1
  and i = if tick then 1 else 0

let edge x = true -> pre x <> x
```

- `add` get type signature: $\texttt{int} \times \texttt{int} \xrightarrow{\texttt{A}} \texttt{int}$
- `counter` get type signature: $\texttt{bool} \times \texttt{bool} \xrightarrow{\texttt{D}} \texttt{int}$
- `edge` get type signature: $\forall\alpha.\alpha \xrightarrow{\texttt{D}} \texttt{bool}$

Connecting a discrete to continuous time

```
let hybrid counter_ten(top, tick) = o where
  (* a periodic timer *)
     der(time) = 1.0 /. 10.0 init 0.0 reset 0.0 every zero
  and zero = up(time -. 1.0)
  (* discrete function *)
  and o = counter(top, tick) every zero init 0
```

The type signature is: $\texttt{bool} \times \texttt{bool} \xrightarrow{\texttt{C}} \texttt{int}$.

Remark: provide ad-hoc programming constructs for periodic timers.

The Bouncing ball

```
let hybrid bouncing(x0,y0,x'0,y'0) = (x,y) where
   der(x) = x' init x0
 and
   der(x') = 0.0 init x'0
 and
   der(y) = y' init y0
 and
   der(y') = -. g init y'0 reset -. 0.9 *. last y' every up(-. y)
```

Its type signature is: $\texttt{float} \times \texttt{float} \times \texttt{float} \xrightarrow{\texttt{c}} \texttt{float} \times \texttt{float}$

The language kernel

- Synchronous (discrete) Lustre-like functions.
- Ordinary Differential Equations (ODE) with reset handlers

$$
\begin{array}{lcl}
d & ::= & \texttt{let}\ k\ f(p) = e \mid d; d \\
e & ::= & x \mid v \mid op(e) \mid e\ \texttt{fby}\ e \mid \texttt{last}(x) \\
 & & \mid \texttt{up}(e) \mid f(e) \mid (e, e) \mid \texttt{let}\ E\ \texttt{in}\ e \\
p & ::= & (p, p) \mid x \\
h & ::= & e\ \texttt{every}\ e \parallel \ldots \parallel e\ \texttt{every}\ e \\
E & ::= & x = e \mid \texttt{der}(x) = e\ \texttt{init}\ e\ \texttt{reset}\ h \\
 & & \mid x = h\ \texttt{default}\ e\ \texttt{init}\ e \\
 & & \mid x = h\ \texttt{init}\ e \mid E\ \texttt{and}\ E
\end{array}
$$

Typing

The type language

$$
\begin{array}{lcl}
\sigma & ::= & \forall \beta_1, ..., \beta_n . t \xrightarrow{k} t \\
t & ::= & t \times t \mid \beta \mid bt \\
k & ::= & \texttt{D} \mid \texttt{C} \mid \texttt{A} \\
bt & ::= & \texttt{float} \mid \texttt{int} \mid \texttt{bool} \mid \texttt{zero}
\end{array}
$$

We restrict to a first order language. Extension to higher-order later (but simple).

Initial conditions

$$
\begin{array}{lcl}
(+) & : & \texttt{int} \times \texttt{int} \xrightarrow{\texttt{A}} \texttt{int} \\
(=) & : & \forall \beta . \beta \times \beta \xrightarrow{\texttt{A}} \texttt{bool} \\
\texttt{if} & : & \forall \beta . \texttt{bool} \times \beta \times \beta \xrightarrow{\texttt{A}} \beta \\
\texttt{pre}(.) & : & \forall \beta . \beta \xrightarrow{\texttt{D}} \beta \\
\texttt{.fby.} & : & \forall \beta . \beta \times \beta \xrightarrow{\texttt{D}} \beta \\
\texttt{up}(.) & : & \texttt{float} \xrightarrow{\texttt{C}} \texttt{zero}
\end{array}
$$

The Type system

Global and local environment

$$G ::= [f_1 : \sigma_1; ...; f_n : \sigma_n] \qquad H ::= [\,] \mid H, x : t \mid H, \texttt{last}(x) : t$$

Typing predicates

- $G, H \vdash_k e : t$: Expression e has type t and kind k. $G, H \vdash_k e : t$
- $H, H \vdash_k E : H'$: Equation E produces environment H' and has kind k.

Subtyping

An combinatorial function can be passed where a discrete or continuous one is expected:

$$\forall k, \mathtt{A} \leq k$$

A sketch of Typing rules

(DER)

$$\frac{G, H \vdash_{\mathtt{c}} e_1 : \mathtt{float} \qquad G, H \vdash_{\mathtt{c}} e_2 : \mathtt{float} \qquad G, H \vdash h : \mathtt{float}}{G, H \vdash_{\mathtt{c}} \mathtt{der}(x) = e_1 \ \mathtt{init}\ e_2 \ \mathtt{reset}\ h : [\mathtt{last}(x) : \mathtt{float}]}$$

(AND)

$$\frac{G, H \vdash_k E_1 : H_1 \qquad G, H \vdash_k E_2 : H_2}{G, H \vdash_k E_1 \ \mathtt{and}\ E_2 : H_1 + H_2}$$

(EQ)

$$\frac{G, H \vdash_k e : t}{G, H \vdash_k x = e : [x : t]}$$

(APP)

$$\frac{t \xrightarrow{k} t' \in Inst(G(f)) \qquad G, H \vdash_k e : t}{G, H \vdash_k f(e) : t'}$$

(VAR)

$$G, H + [x : t] \vdash_k x : t$$

(VAR-LAST)

$$G, H + [\texttt{last}(x) : t] \vdash_k x : t$$

(EQ-DISCRETE)

$$\frac{G, H \vdash h : t \qquad G, H \vdash_{\texttt{C}} e : t}{G, H \vdash_{\texttt{C}} x \;=\; h \;\texttt{init}\; e : [\texttt{last}(x) : t]}$$

(HANDLER)

$$\frac{\forall i \in \{1, .., n\} \qquad G, H \vdash_{\texttt{D}} e_i : t \qquad G, H \vdash_{\texttt{C}} z_i : \texttt{zero}}{G, H \vdash e_1 \;\texttt{every}\; z_1 \parallel ... \parallel e_n \;\texttt{every}\; z_n : t}$$

Property 1 (Subtyping) *The following property holds:*

$$G, H \vdash_{\texttt{A}} e : t \Rightarrow (G, H \vdash_{\texttt{C}} e : t) \wedge (G, H \vdash_{\texttt{D}} e : t)$$

A sketch of the semantics

The sets ${}^\star\mathbb{R}$ and ${}^\star\mathbb{N}$ as the non-standard extensions of $\mathbb{R}$ and $\mathbb{N}$.

- ${}^\star\mathbb{N}$ contains elements that are infinitely large (${}^\star n > n$ for any $n \in \mathbb{N}$).
- ${}^\star\mathbb{R}$ contains elements that are *infinitesimal*, $0 < \partial < t$ for any $t \in \mathbb{R}_+$.

The base clock: ∂ infinitesimal, the set

$$BaseClock(\partial) = \{n\partial \mid n \in {}^\star\mathbb{N}\}$$

is isomorphic to ${}^\star\mathbb{N}$ as a total order. For every $t \in \mathbb{R}_+$ and any $\epsilon > 0$, there exists $t' \in BaseClock(\partial)$ such that $|t' - t| < \epsilon$ expressing that $BaseClock(\partial)$ is dense in $\mathbb{R}_+$.

$BaseClock(\partial)$ is a natural candidate for a time index set and ∂ is the corresponding time basis.

For $t = t_n = n\partial \in BaseClock(\partial)$, ${}^\bullet t = t_{n-1}$ and $t^\bullet = t_{n+1}$.

A sketch of the semantics

Reason "as if" the time was discrete and global. The idea of using non standard analysis for the semantics of systems has been recognized by Bliudze et Krob.

Clock and signals A *clock* T is a subset of $BaseClock(\partial)$. A *signal* s is a total function $s : T \mapsto V$.

If T is a clock and b a signal $b : T \mapsto \mathbb{B}$, then T `on` b defines a subset of T comprising those instants where $b(t)$ is true:

$$T \texttt{ on } b = \{t \mid (t \in T) \wedge (b(t) = \texttt{true})\}$$

If $s : T \mapsto {}^{\star}\mathbb{R}$, we write T `on up`(s) for the instants when s crosses zero, that is:

$$T \texttt{ on up}(s) = \{t^{\bullet} \mid (t \in T) \wedge (s({}^{\bullet}t) \leq 0) \wedge (s(t) > 0)\}$$

The effect of `up`(e) is delayed by one cycle.

Discrete *vs* Continuous

Let x be a signal with clock domain T_x, it is typed *discrete* ($\mathtt{D}(T)$) either if it has been so declared, or if its clock is the result of a zero-crossing or a sub-clock of a discrete clock. Otherwise it is typed *continuous* ($\mathtt{C}(T)$). That is:

1. $\mathtt{C}(\mathit{BaseClock}(\partial))$
2. If $\mathtt{C}(T)$ and $s : T \mapsto {}^{\star}\mathbb{R}$ then $\mathtt{D}(T \texttt{ on up}(s))$
3. If $\mathtt{D}(T)$ and $s : T \mapsto \mathbb{B}$ then $\mathtt{D}(T \texttt{ on } s)$
4. If $\mathtt{C}(T)$ and $s : T \mapsto \mathbb{B}$ then $\mathtt{C}(T \texttt{ on } s)$

Correction of the type system:

When an is typed `D` (resp. `C`), it is indeed activated on a discrete (resp. continuous) clock.

Infinitesimal independence: For well-typed programs, the ideal semantics does not depend on the choice of the infinitesimal.

$$
\begin{array}{lcll}
integr^{\#}(T)(s)(s_0)(hs)(t) & = & s'(t) & \text{where} \\
s'(t) & = & s_0(t) & \text{if } t = \min(T) \\
s'(t) & = & s'(^{\bullet}t) + \partial s(^{\bullet}t) & \text{if } handler^{\#}(T)(hs)(t) = NoEvent \\
s'(t) & = & v & \text{if } handler^{\#}(T)(hs)(t) = Xcrossing(v) \\
 & & & \\
up^{\#}(T)(s)(t) & = & \texttt{false} & \text{if } t = \min(T) \\
up^{\#}(T)(s)(t^{\bullet}) & = & \texttt{true} & \text{if } (s(^{\bullet}t) \leq 0) \wedge (s(t) > 0) \text{ and } (t \in T) \\
up^{\#}(T)(s)(t^{\bullet}) & = & \texttt{false} & \text{otherwise}
\end{array}
$$

Compilation

The non-standard semantics is not operational. It serves as a reference to establish the correctness of the compilation. Two problems to address:

1. The compilation of the discrete part, that is, the synchronous subset of the language.
2. The compilation of the continuous part which is to be linked to a black-box numerical solver.

Principle

Translate the program into the discrete subset. Compile the result with an existing synchronous compiler such that it verifies the following invariant:

> The discrete state, i.e., the values of delays, will not change if all of the zero-crossing conditions are false.

Example (counter)

Add extra input and outputs.

- `up`(e) becomes a fresh boolean input z and generate an equation $up_z = e$.
- `der`$(x) = e$ `init` e_0 becomes $dx = e$ `init` e_0.
- A continuous state variable becomes an input.

```
let node counter_ten([z], [time], (top, tick)) = (o, [upz], [dtime])
 where
     dtime = 1.0 /. 10.0 init 0.0 reset 0.0 every z
 and o = counter(top, tick) every z init 0
 and upz = time -. 1.0
```

In practice, represent these extra inputs with arrays.

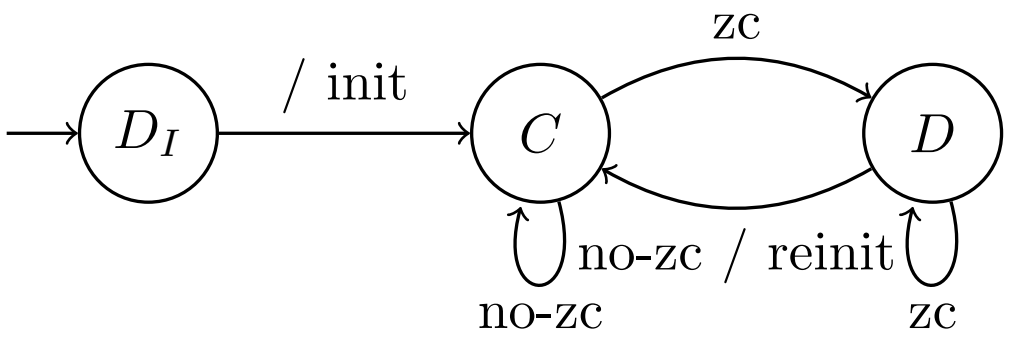
Now, ignoring details of syntax, the function `counter_ten` can be processed by any synchronous compiler, and the generated transition function verifies the invariant.

Interfacing with a numerical solver

We used the Sundials CVODE library. An Ocaml interface has been developed.

Structure of the execution: Run the transition function with two modes, a continuous one and a discrete one

- **Continuous phase:** processed by the numerical solver which stops when a zero-crossing event has been detected.
- **Discrete phase:** compute the consequence of (one or several) zero-crossing(s).



Delta-delayed synchrony *vs* Instantaneous synchrony

For cascaded zero-crossing, two interpretations of $\texttt{up}(e)$ lead to different results.

- **Delta-delay**: the effect of a zero-crossing is delayed by one instant.

$$T \texttt{ on up}(s) = \{t^{\bullet} \mid (t \in T) \wedge (s({}^{\bullet}t) \leq 0) \wedge (s(t) > 0)\}$$

- **Instantaneous**: the effect is immediate.

$$T \texttt{ on up}(s) = \{t \mid (t \in T) \wedge (s({}^{\bullet}t) \leq 0) \wedge (s(t) > 0)\}$$

We have considered the two solutions.

- The first one is simpler to compile. But the discrete state can last several instants.
- The second one is (a little) more complicated to compile. But all zero-crossing can be statically scheduled. Only one instant in the discrete state.

Simultaneous events A zero-crossing is a boolean signal; they are treated with a priority. Exactly what Simulink does.

Discussion: synchronous *vs* Asynchronous events

Zero-crossing can be ordered in a reset construct.

```
x = (last x + 1) every up(x1)
  | (last x - 1) every up(x2) init 0.0
```

`x` is incremented when `up(x1)` is true and decremented when `up(x2)` is true.

- When both are true, conditions are taken in sequence: the first branch is executed only so one zero-crossing is discarded.
- This makes the behavior reproducible (from one simulation to the other with the same program). Possible extension with boolean calculus on events.

```
x = (last x) every up(x1) & up(x2)
    | (last x + 1) every up(x1)
    | (last x - 1) every up(x2) init 0.0
```

- Is-it (physically) meaningfull? Numerical solvers indicate when several simultaneous zero-crossing occur
- Missing an event is questionnable (see Ramine Nikoukhah's discussion on the topic). Here, we prefer to forbid non determinism.

Conclusion

Proposal

- Mix signals on discrete and continuous time.
- A Lustre-like proposal to combine stream equations with ODE.
- Divide with a type-system, recycle a existing compiler to use a numerical solver as a black-box.
- A prototype implementation.

Extension

- Add timers (periodic clocks) as particular zero-crossing events.
- Hybrid (hierarchical) automata.
- Combination of solvers, i.e., use different solvers at the same time.

References

[1] Albert Benveniste, Timothy Bourke, Benoit Caillaud, and Marc Pouzet. Divide and recycle: types and compilation for a hybrid synchronous language. In *ACM SIGPLAN/SIGBED Conference on Languages, Compilers, Tools and Theory for Embedded Systems (LCTES'11)*, Chicago, USA, April 2011.

[2] Albert Benveniste, Benoit Caillaud, and Marc Pouzet. The Fundamentals of Hybrid Systems Modelers. In *49th IEEE International Conference on Decision and Control (CDC)*, Atlanta, Georgia, USA, December 15-17 2010.