Provable Security in the Computational Model

II – Encryption

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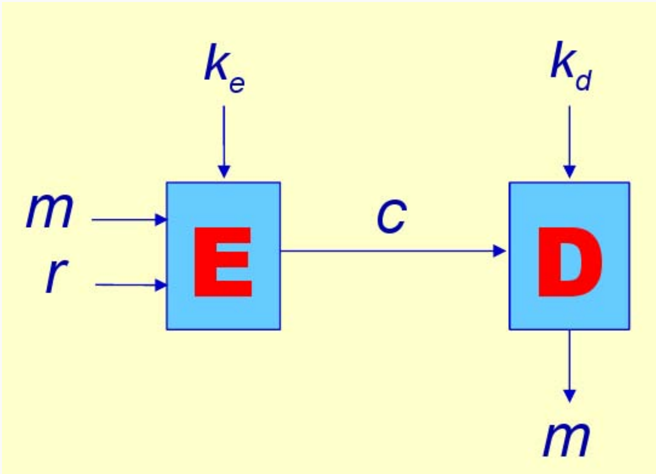
Outline

Basic Security Notions

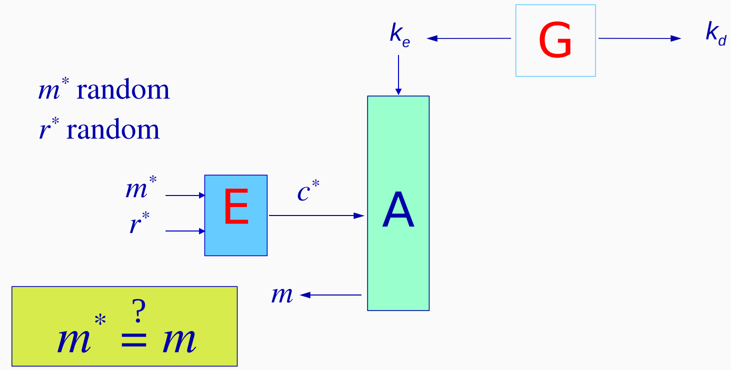
Outline

Public-Key Encryption



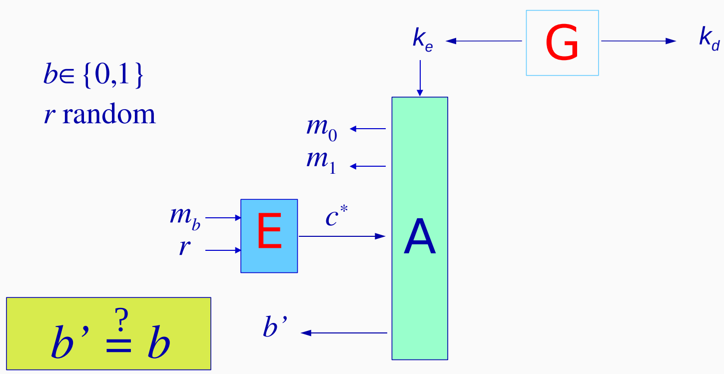
Goal: Privacy/Secrecy of the plaintext

OW – CPA Security Game



$$\mathbf{Succ}_{\mathcal{S}}^{\mathsf{ow}}(\mathcal{A}) = \Pr[(sk, pk) \leftarrow \mathcal{K}(); m \stackrel{R}{\leftarrow} \mathcal{M}; c = \mathcal{E}_{pk}(m) : \mathcal{A}(pk, c) \rightarrow m]$$

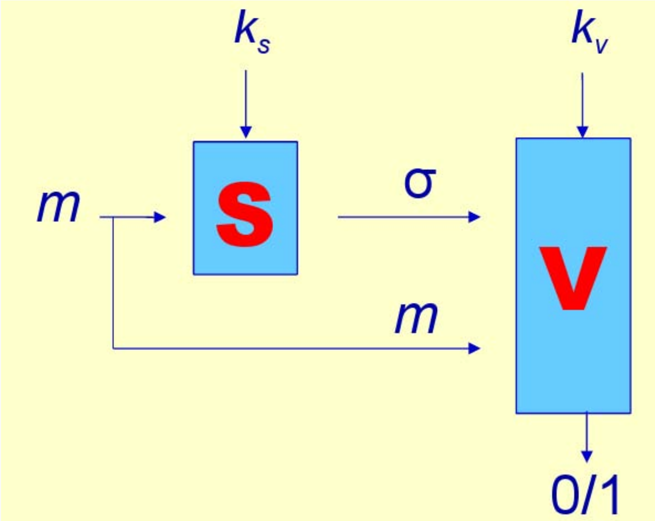
IND – CPA Security Game



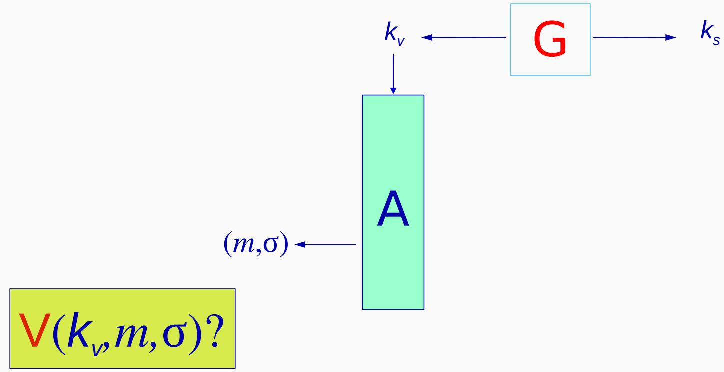
$$(sk, pk) \leftarrow \mathcal{K}(); (m_0, m_1, \text{state}) \leftarrow \mathcal{A}(pk);$$
$$b \stackrel{R}{\leftarrow} \{0, 1\}; c = \mathcal{E}_{pk}(m_b); b' \leftarrow \mathcal{A}(\text{state}, c)$$

$$\mathbf{Adv}_{\mathcal{S}}^{\mathsf{ind-cpa}}(\mathcal{A}) = \left|\Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0]\right| = \left|2 \times \Pr[b' = b] - 1\right|$$

Outline



Goal: Authentication of the sender



$$\mathbf{Succ}^{\mathsf{euf}}_{\mathcal{SG}}(\mathcal{A}) = \Pr[(sk, pk) \leftarrow \mathcal{K}(); (m, \sigma) \leftarrow \mathcal{A}(pk) : \mathcal{V}_{pk}(m, \sigma) = 1]$$

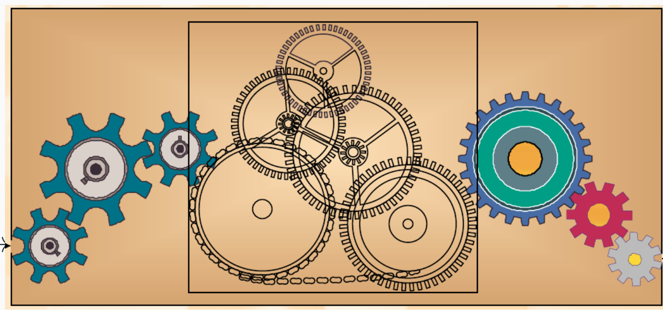
Game-based Proofs

Provable Security

One can prove that:

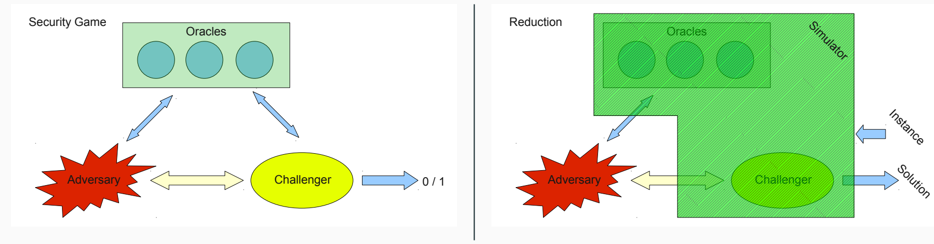
- if an adversary is able to break the cryptographic scheme
- then one can break the underlying problem (integer factoring, discrete logarithm, 3-SAT, etc)

hard instance →



→ solution

Direct Reduction



Unfortunately

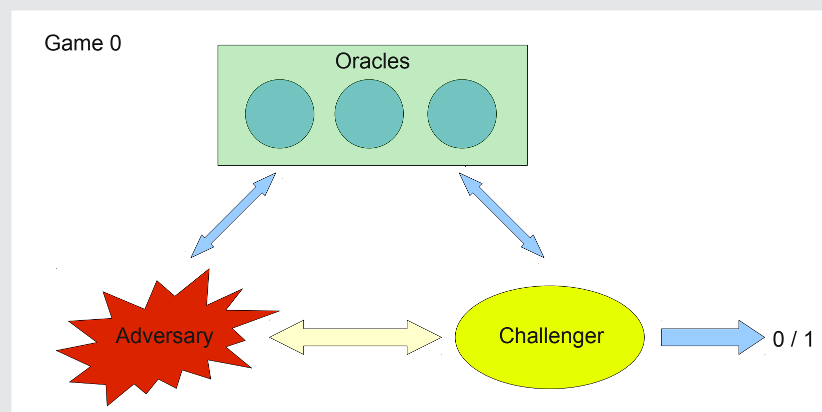
- Security may rely on several assumptions
- Proving that the view of the adversary, generated by the simulator, in the reduction is the same as in the real attack game is not easy to do in such a one big step

Outline

Sequence of Games

Real Attack Game

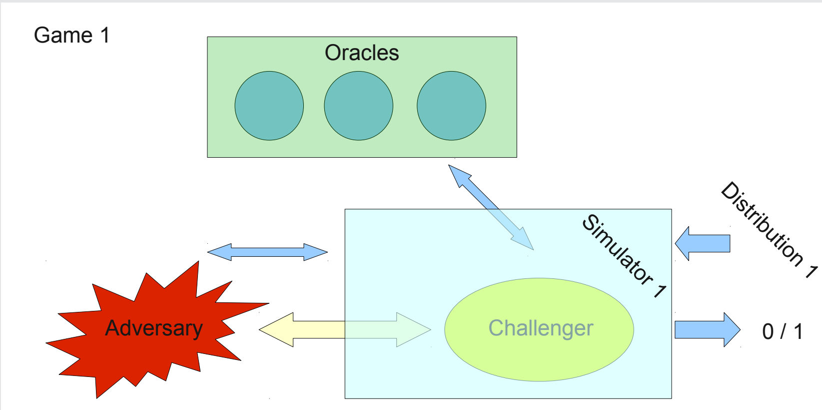
The adversary plays a game, against a challenger (security notion)



Sequence of Games

Simulation

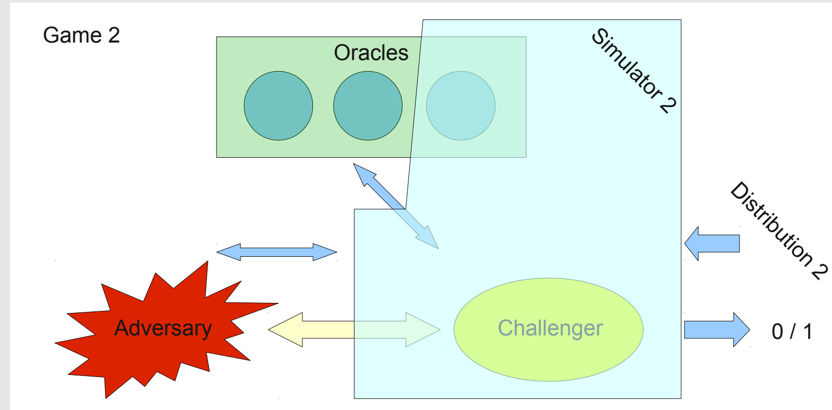
The adversary plays a game, against a sequence of simulators



Sequence of Games

Simulation

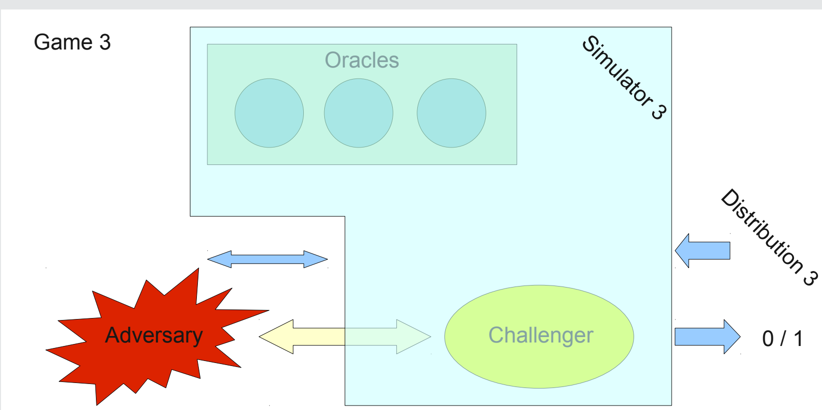
The adversary plays a game, against a sequence of simulators



Sequence of Games

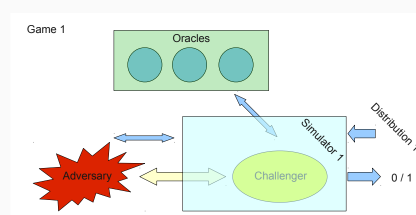
Simulation

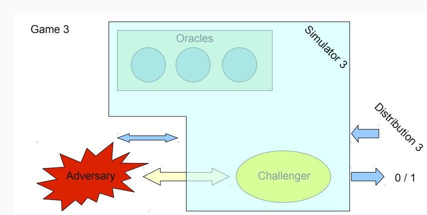
The adversary plays a game, against a sequence of simulators



Output

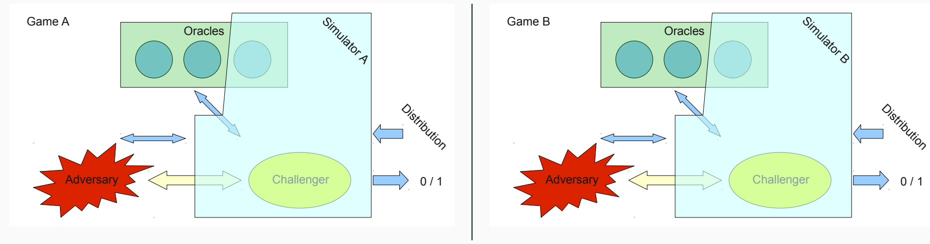
- The output of the simulator in Game 1 is related to the output of the challenger in Game 0 (adversary's winning probability)
- The output of the simulator in Game 3 is easy to evaluate (e.g. always zero, always 1, probability of one-half)
- The gaps (Game 1 $\leftrightarrow$ Game 2, Game 2 $\leftrightarrow$ Game 3, etc) are clearly identified with specific events





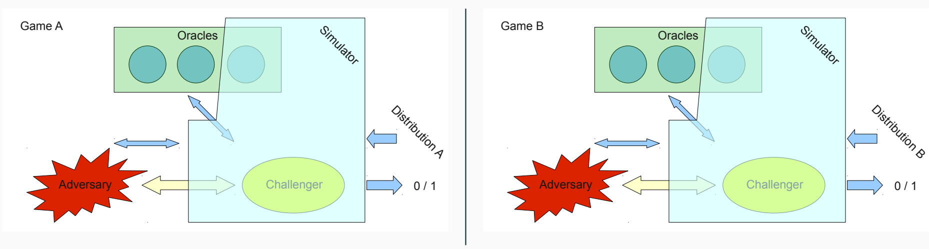
Outline

Two Simulators



- perfectly identical behaviors [**Hop-S-Perfect**]
- different behaviors, only if event **Ev** happens
  - **Ev** is negligible [**Hop-S-Negl**]
  - **Ev** is non-negligible (but not overwhelming) and independent of the output in $\mathbf{Game}_A$ [**Hop-S-Non-Negl**]
    $\rightarrow$ Simulator B terminates in case of event **Ev**

Two Distributions



- perfectly identical input distributions [**Hop-D-Perfect**]
- different distributions
  - statistically close [**Hop-D-Stat**]
  - computationally close [**Hop-D-Comp**]

Two Simulations

- Identical behaviors: $\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B] = 0$
- The behaviors differ only if **Ev** happens:
  - **Ev** is negligible, one can ignore it
    Shoup's Lemma: $|\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B]| \leq \Pr[\mathbf{Ev}]$

$$
\begin{aligned}
&|\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B]| \\
&= \left| \begin{array}{l} \Pr[\mathbf{Game}_A|\mathbf{Ev}]\Pr[\mathbf{Ev}] + \Pr[\mathbf{Game}_A|\neg\mathbf{Ev}]\Pr[\neg\mathbf{Ev}] \\ -\Pr[\mathbf{Game}_B|\mathbf{Ev}]\Pr[\mathbf{Ev}] - \Pr[\mathbf{Game}_B|\neg\mathbf{Ev}]\Pr[\neg\mathbf{Ev}] \end{array} \right| \\
&= \left| \begin{array}{l} (\Pr[\mathbf{Game}_A|\mathbf{Ev}] - \Pr[\mathbf{Game}_B|\mathbf{Ev}]) \times \Pr[\mathbf{Ev}] \\ +(\Pr[\mathbf{Game}_A|\neg\mathbf{Ev}] - \Pr[\mathbf{Game}_B|\neg\mathbf{Ev}]) \times \Pr[\neg\mathbf{Ev}] \end{array} \right| \\
&\leq |1 \times \Pr[\mathbf{Ev}] + 0 \times \Pr[\neg\mathbf{Ev}]| \leq \Pr[\mathbf{Ev}]
\end{aligned}
$$

  - **Ev** is non-negligible and independent of the output in $\mathbf{Game}_A$, Simulator B terminates in case of event **Ev**

Two Simulations

- Identical behaviors: $\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B] = 0$
- The behaviors differ only if **Ev** happens:
  - **Ev** is negligible, one can ignore it
  - **Ev** is non-negligible and independent of the output in $\mathbf{Game}_A$, Simulator B terminates and outputs 0, in case of event **Ev**:

$$\begin{aligned}\Pr[\mathbf{Game}_B] &= \Pr[\mathbf{Game}_B|\mathbf{Ev}]\Pr[\mathbf{Ev}] + \Pr[\mathbf{Game}_B|\neg\mathbf{Ev}]\Pr[\neg\mathbf{Ev}]\\ &= 0 \times \Pr[\mathbf{Ev}] + \Pr[\mathbf{Game}_A|\neg\mathbf{Ev}] \times \Pr[\neg\mathbf{Ev}]\\ &= \Pr[\mathbf{Game}_A] \times \Pr[\neg\mathbf{Ev}]\end{aligned}$$

  Simulator B terminates and flips a coin, in case of event **Ev**:

$$\begin{aligned}\Pr[\mathbf{Game}_B] &= \Pr[\mathbf{Game}_B|\mathbf{Ev}]\Pr[\mathbf{Ev}] + \Pr[\mathbf{Game}_B|\neg\mathbf{Ev}]\Pr[\neg\mathbf{Ev}]\\ &= \tfrac{1}{2} \times \Pr[\mathbf{Ev}] + \Pr[\mathbf{Game}_A|\neg\mathbf{Ev}] \times \Pr[\neg\mathbf{Ev}]\\ &= \tfrac{1}{2} + (\Pr[\mathbf{Game}_A] - \tfrac{1}{2}) \times \Pr[\neg\mathbf{Ev}]\end{aligned}$$

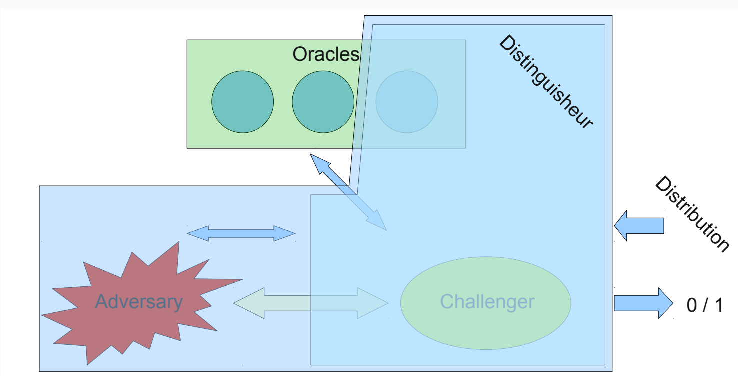
Two Simulations

- Identical behaviors: $\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B] = 0$
- The behaviors differ only if **Ev** happens:
  - **Ev** is negligible, one can ignore it
  - **Ev** is non-negligible and independent of the output in $\mathbf{Game}_A$, Simulator B terminates in case of event **Ev**

Event Ev

- Either **Ev** is negligible, or the output is independent of **Ev**
- For being able to terminate simulation B in case of event **Ev**, this event must be *efficiently* detectable
- For evaluating $\Pr[\mathbf{Ev}]$, one re-iterates the above process, with an initial game that outputs 1 when event **Ev** happens

Two Distributions



$$\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B] \leq \mathbf{Adv}(\mathcal{D}^{\text{oracles}})$$

Two Distributions

$$\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B] \leq \mathbf{Adv}(\mathcal{D}^{\text{oracles}})$$

- For identical/statistically close distributions, for any oracle:

$$\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B] = \mathbf{Dist}(\mathbf{Distrib}_A, \mathbf{Distrib}_B) = \text{negl}()$$

- For computationally close distributions, in general, we need to exclude additional oracle access:

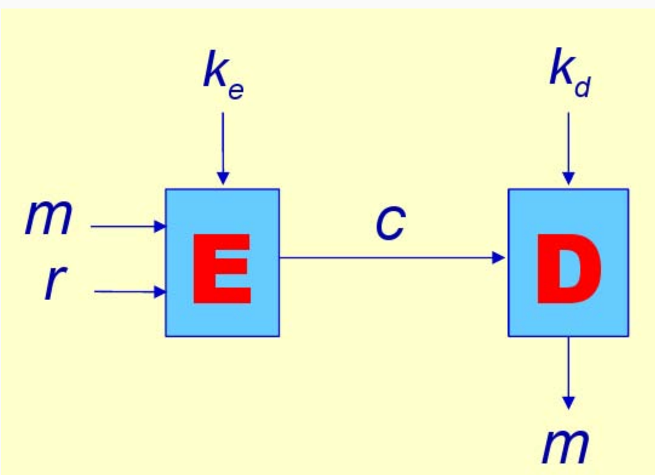
$$\Pr[\mathbf{Game}_A] - \Pr[\mathbf{Game}_B] \leq \mathbf{Adv}^{\mathsf{Distrib}}(t)$$

  where t is the computational time of the distinguisheur

Advanced Security for Encryption

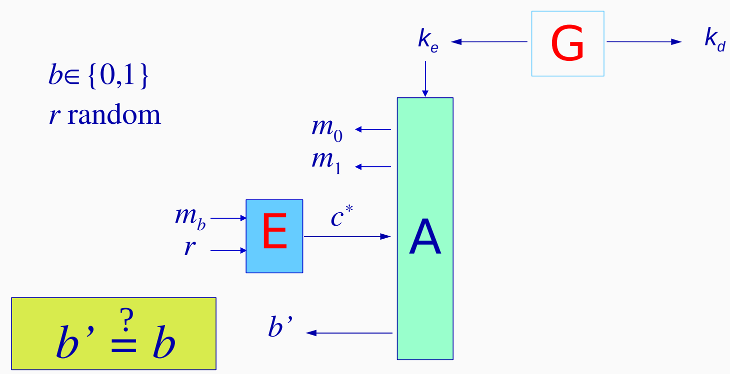
Outline

Public-Key Encryption



Goal: Privacy/Secrecy of the plaintext

IND – CPA Security Game



The adversary cannot get any information about a plaintext of a specific ciphertext (validity, partial value, etc)

Malleability

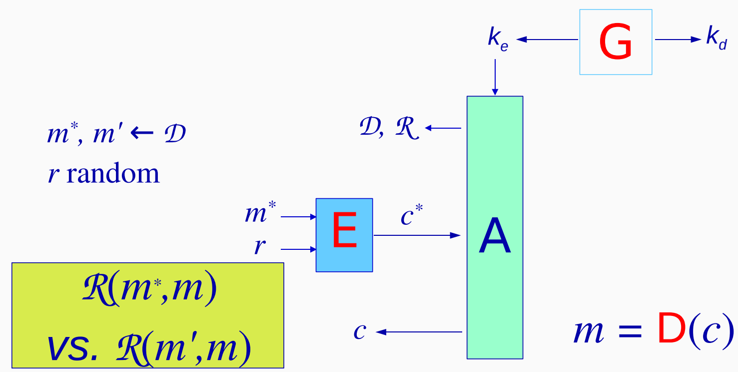
Semantic security (ciphertext indistinguishability) guarantees that no information is leaked from c about the plaintext m

But it may be possible to derive a ciphertext c' such that the plaintext m' is related to m in a meaningful way:

- ElGamal ciphertext: $c_1 = g^r$ and $c_2 = m \times y^r$
- Malleability: $c'_1 = c_1 = g^r$ and $c'_2 = 2 \times c_2 = (2m) \times y^r$

From an encryption of m, one can build an encryption of $2m$, or a random ciphertext of m, etc.

Non-Malleability: NM – CPA Security Game



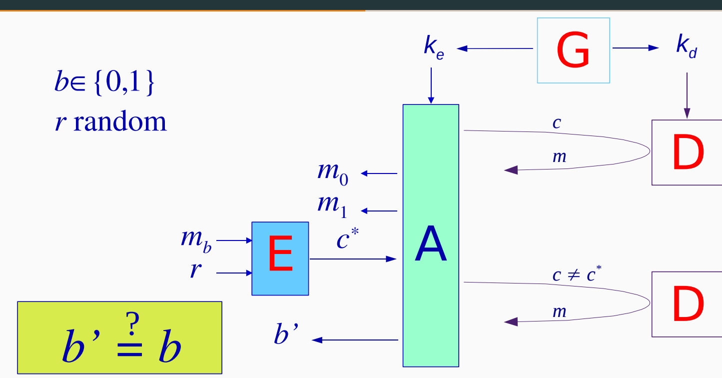
$$\mathbf{Adv}_{\mathcal{S}}^{\mathsf{nm-cpa}}(\mathcal{A}) = \left|\Pr[\mathcal{R}(m^*, m)] - \Pr[\mathcal{R}(m', m)]\right|$$

Additional Information

More information modelled by oracle access

- reaction attacks: oracle which answers, on c, whether the ciphertext c is valid or not
- plaintext-checking attacks: oracle which answers, on a pair (m, c), whether the plaintext m is really encrypted in c or not (whether $m = \mathcal{D}_{sk}(c)$)
- chosen-ciphertext attacks ($\mathbf{CCA}$): decryption oracle (with the restriction not to use it on the challenge ciphertext) $\Longrightarrow$ the adversary can obtain the plaintext of any ciphertext of its choice (excepted the challenge)
  - non-adaptive ($\mathbf{CCA} - 1$) [Naor-Yung – STOC '90]
    only before receiving the challenge
  - adaptive ($\mathbf{CCA} - 2$) [Rackoff-Simon – Crypto '91]
    unlimited oracle access

IND – CCA Security Game



The adversary can ask any decryption of its choice: Chosen-Ciphertext Attacks (oracle access)

$$(sk, pk) \leftarrow \mathcal{K}(); (m_0, m_1, \text{state}) \leftarrow \mathcal{A}^{\mathcal{D}}(pk);$$
$$b \stackrel{R}{\leftarrow} \{0,1\}; c = \mathcal{E}_{pk}(m_b); b' \leftarrow \mathcal{A}^{\mathcal{D}}(\text{state}, c)$$

$$\mathbf{Adv}_{\mathcal{S}}^{\mathsf{ind-cca}}(\mathcal{A}) = \left|\Pr[b' = 1 | b = 1] - \Pr[b' = 1 | b = 0]\right| = \left|2 \times \Pr[b' = b] - 1\right|$$

Relations

[Bellare-Desai-Pointcheval-Rogaway – Crypto '98]

$$\begin{array}{ccccc}
\text{NM-CPA} & \Leftarrow & \text{NM-CCA1} & \Leftarrow & \boxed{\text{NM-CCA2}} \\
\Downarrow & & \Downarrow & & \Updownarrow \\
\boxed{\text{IND-CPA}} & \Leftarrow & \text{IND-CCA1} & \Leftarrow & \boxed{\text{IND-CCA2}} \\
\Downarrow & & & & \\
\boxed{\text{OW-CPA}} & & & &
\end{array}$$

minimal security

weak security

strong security: CCA

Outline

Cramer-Shoup Encryption Scheme

[Cramer-Shoup – Crypto '98]

Key Generation

- $\mathbb{G} = (\langle g \rangle, \times)$ group of order q
- $sk = (x_1, x_2, y_1, y_2, z)$, where $x_1, x_2, y_1, y_2, z \overset{R}{\leftarrow} \mathbb{Z}_q$
- $pk = (g_1, g_2, \mathcal{H}, c, d, h)$, where
  - g_1, g_2 are independent elements in $\mathbb{G}$
  - $\mathcal{H}$ a hash function (second-preimage resistant)
  - $c = g_1^{x_1} g_2^{x_2}$, $d = g_1^{y_1} g_2^{y_2}$, and $h = g_1^z$

Encryption

$u_1 = g_1^r,\ u_2 = g_2^r,\ e = m \times h^r,\ v = c^r d^{r\alpha}$ where $\alpha = \mathcal{H}(u_1, u_2, e)$

Cramer-Shoup Encryption Scheme vs. ElGamal

$$u_1 = g_1^r,\ u_2 = g_2^r,\ e = m \times h^r,\ v = c^r d^{r\alpha} \text{ where } \alpha = \mathcal{H}(u_1, u_2, e)$$

(u_1, e) is an ElGamal ciphertext, with public key $h = g_1^z$

Decryption

- since $h = g_1^z$, $h^r = u_1^z$, thus $m = e/u_1^z$
- since $c = g_1^{x_1} g_2^{x_2}$ and $d = g_1^{y_1} g_2^{y_2}$

$$c^r = g_1^{rx_1} g_2^{rx_2} = u_1^{x_1} u_2^{x_2} \qquad d^r = u_1^{y_1} u_2^{y_2}$$

One thus first checks whether

$$v = u_1^{x_1 + \alpha y_1} u_2^{x_2 + \alpha y_2} \text{ where } \alpha = \mathcal{H}(u_1, u_2, e)$$

Security of the Cramer-Shoup Encryption Scheme

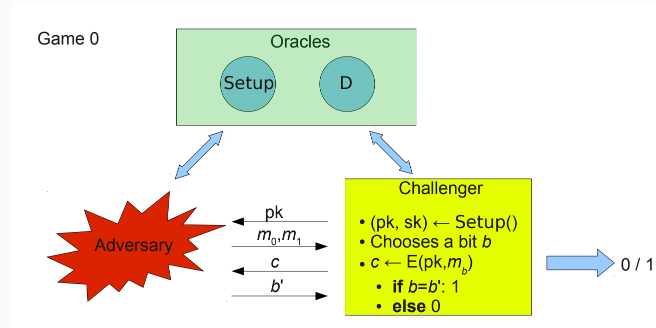
Theorem

The Cramer-Shoup encryption scheme achieves $\mathbf{IND-CCA}$ *security, under the* ***DDH*** *assumption, and the second-preimage resistance of* $\mathcal{H}$*:*

$$\mathbf{Adv}^{\mathsf{ind-cca}}_{\mathcal{CS}}(t) \le 2 \times \mathbf{Adv}^{\mathbf{ddh}}_{\mathbb{G}}(t) + \mathbf{Succ}^{\mathcal{H}}(t) + 3q_D/q$$

Let us prove this theorem, with a sequence of games, in which $\mathcal{A}$ is an $\mathbf{IND-CCA}$ adversary against the Cramer-Shoup encryption scheme.

Real Attack Game



Key Generation Oracle

$x_1, x_2, y_1, y_2, z \stackrel{R}{\leftarrow} \mathbb{Z}_q$, $g_1, g_2 \stackrel{R}{\leftarrow} \mathbb{G}$: $sk = (x_1, x_2, y_1, y_2, z)$
$c = g_1^{x_1} g_2^{x_2}$, $d = g_1^{y_1} g_2^{y_2}$, and $h = g_1^z$: $pk = (g_1, g_2, \mathcal{H}, c, d, h)$

Decryption Oracle

If $v = u_1^{x_1+\alpha y_1} u_2^{x_2+\alpha y_2}$ where $\alpha = \mathcal{H}(u_1, u_2, e)$: $m = e/u_1^z$

Proof: Invalid ciphertexts

- **Game**$_0$: use of the oracles $\mathcal{K}$, $\mathcal{D}$
- **Game**$_1$: we abort (with a random output b') in case of bad (invalid) accepted ciphertext, where invalid ciphertext means $\log_{g_1} u_1 \neq \log_{g_2} u_2$

Event F

$\mathcal{A}$ submits a bad accepted ciphertext
(note: this is not computationally detectable)

The advantage in **Game**$_1$ is: $\Pr_1[b' = b|\mathbf{F}] = 1/2$

$$\Pr_{\mathbf{Game}_0}[\mathbf{F}] = \Pr_{\mathbf{Game}_1}[\mathbf{F}] \qquad \Pr_{\mathbf{Game}_1}[b' = b|\neg\mathbf{F}] = \Pr_{\mathbf{Game}_0}[b' = b|\neg\mathbf{F}]$$

$\Longrightarrow$ **Hop-S-Negl**: $\mathbf{Adv}_{\mathbf{Game}_1} \ge \mathbf{Adv}_{\mathbf{Game}_0} - \Pr[\mathbf{F}]$

Details: Shoup's Lemma

$$\begin{aligned}
\mathbf{Adv}_{\mathbf{Game}_1} &= 2 \times \Pr_{\mathbf{Game}_1}[b' = b] - 1 \\
&= 2 \times \Pr_{\mathbf{Game}_1}[b' = b|\neg\mathbf{F}] \Pr_{\mathbf{Game}_1}[\neg\mathbf{F}] \\
&\quad + 2 \times \Pr_{\mathbf{Game}_1}[b' = b|\mathbf{F}] \Pr_{\mathbf{Game}_1}[\mathbf{F}] - 1 \\
&= 2 \times \Pr_{\mathbf{Game}_0}[b' = b|\neg\mathbf{F}] \Pr_{\mathbf{Game}_0}[\neg\mathbf{F}] + \Pr_{\mathbf{Game}_0}[\mathbf{F}] - 1 \\
&= 2 \times \Pr_{\mathbf{Game}_0}[b' = b] - 2 \times \Pr_{\mathbf{Game}_0}[b' = b|\mathbf{F}] \Pr_{\mathbf{Game}_0}[\mathbf{F}] \\
&\quad + \Pr_{\mathbf{Game}_0}[\mathbf{F}] - 1 \\
&= \mathbf{Adv}_{\mathbf{Game}_0} - \Pr_{\mathbf{Game}_0}[\mathbf{F}](2 \times \Pr_{\mathbf{Game}_0}[b' = b|\mathbf{F}] - 1) \\
&\ge \mathbf{Adv}_{\mathbf{Game}_0} - \Pr_{\mathbf{Game}_0}[\mathbf{F}]
\end{aligned}$$

Details: Bad Accept

In order to evaluate $\Pr[\mathbf{F}]$, we study the probability that

- $r_1 = \log_{g_1} u_1 \neq \log_{g_2} u_2 = r_2$,
- whereas $v = u_1^{x_1+\alpha y_1} u_2^{x_2+\alpha y_2}$

The adversary just knows the public key:

$$c = g_1^{x_1} g_2^{x_2} \qquad d = g_1^{y_1} g_2^{y_2}$$

Let us move to the exponents, in basis g_1, with $g_2 = g_1^s$:

$$\begin{aligned} \log c &= x_1 + sx_2 \\ \log d &= y_1 + sy_2 \\ \log v &= r_1(x_1 + \alpha y_1) + sr_2(x_2 + \alpha y_2) \end{aligned}$$

The system is under-defined: for any v, there are (x_1, x_2, y_1, y_2) that satisfy the system $\Longrightarrow$ v is unpredictable

$\Longrightarrow \Pr[\mathbf{F}] \leq q_D/q \qquad \Longrightarrow \mathbf{Adv}_{\textsf{Game}_1} \geq \mathbf{Adv}_{\textsf{Game}_0} - q_D/q$

Proof: Simulations

- **Game**$_2$: we use the simulations

Key Generation Simulation

$x_1, x_2, y_1, y_2, z_1, z_2 \stackrel{R}{\leftarrow} \mathbb{Z}_q$, $g_1, g_2 \stackrel{R}{\leftarrow} \mathbb{G}$: $sk = (x_1, x_2, y_1, y_2, z_1, z_2)$

$g_2 = g_1^s$

$c = g_1^{x_1} g_2^{x_2}$, $d = g_1^{y_1} g_2^{y_2}$, and $h = g_1^{z_1} g_2^{z_2}$: $pk = (g_1, g_2, \mathcal{H}, c, d, h)$

$z = z_1 + sz_2$

Distribution of the public key: Identical

Decryption Simulation

If $v = u_1^{x_1+\alpha y_1} u_2^{x_2+\alpha y_2}$ where $\alpha = \mathcal{H}(u_1, u_2, e)$: $m = e/u_1^{z_1} u_2^{z_2}$

Under the assumption of $\neg\mathbf{F}$, perfect simulation

$\Longrightarrow$ **Hop-S-Perfect**: $\mathbf{Adv}_{\textsf{Game}_2} = \mathbf{Adv}_{\textsf{Game}_1}$

Proof: Computable Adversary

- **Game**$_3$: we do no longer exclude bad accepted ciphertexts

  $\Longrightarrow$ **Hop-S-Negl**:

  $\mathbf{Adv}_{\textsf{Game}_3} \geq \mathbf{Adv}_{\textsf{Game}_2} - \Pr[\mathbf{F}] \geq \mathbf{Adv}_{\textsf{Game}_2} - q_D/q$

  This is technical: to make the simulator/adversary computable

Proof: DDH Assumption

- **Game**$_4$: we modify the generation of the challenge ciphertext:

Original Challenge

Random choice: $b \stackrel{R}{\leftarrow} \{0, 1\}$, $r \stackrel{R}{\leftarrow} \mathbb{Z}_q$ $\qquad [\alpha = \mathcal{H}(u_1, u_2, e)]$

$$u_1 = g_1^r,\ u_2 = g_2^r,\ e = m_b \times h^r,\ v = c^r d^{r\alpha}$$

New Challenge 1

Given $(U = g_1^r, V = g_2^r)$ and random choice $b \stackrel{R}{\leftarrow} \{0, 1\}$

$$u_1 = U,\ u_2 = V,\ e = m_b \times U^{z_1} V^{z_2},\ v = U^{x_1+\alpha y_1} V^{x_2+\alpha y_2}$$

With $(U = g_1^r, V = g_2^r)$: $U^{z_1} V^{z_2} = h^r$ and $U^{x_1+\alpha y_1} V^{x_2+\alpha y_2} = c^r d^{r\alpha}$

$\Longrightarrow$ **Hop-S-Perfect**: $\mathbf{Adv}_{\textsf{Game}_4} = \mathbf{Adv}_{\textsf{Game}_3}$

Proof: DDH Assumption

- **Game$_5$**: we modify the generation of the challenge ciphertext:

Previous Challenge 1

Given ($U = g_1^r, V = g_2^r$) and random choice $b \overset{R}{\leftarrow} \{0,1\}$

$$u_1 = U,\ u_2 = V,\ e = m_b \times U^{z_1} V^{z_2},\ v = U^{x_1+\alpha y_1} V^{x_2+\alpha y_2}$$

New Challenge 2

Given ($U = g_1^{r_1}, V = g_2^{r_2}$) and random choice $b \overset{R}{\leftarrow} \{0,1\}$

$$u_1 = U,\ u_2 = V,\ e = m_b \times U^{z_1} V^{z_2},\ v = U^{x_1+\alpha y_1} V^{x_2+\alpha y_2}$$

The input changes from ($U = g_1^r, V = g_2^r$) to ($U = g_1^{r_1}, V = g_2^{r_2}$):
$\Longrightarrow$ **Hop-D-Comp**: $\mathbf{Adv}_{\mathbf{Game}_5} \geq \mathbf{Adv}_{\mathbf{Game}_4} - 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$

Proof: DDH Assumption

The input from outside changes from ($U = g_1^r, V = g_2^r$) (a CDH tuple) to ($U = g_1^{r_1}, V = g_2^{r_2}$) (a random tuple):

$$\Pr_{\mathbf{Game}_4}[b' = b] - \Pr_{\mathbf{Game}_5}[b' = b] \leq \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$$

$\Longrightarrow$ **Hop-D-Comp**: $\mathbf{Adv}_{\mathbf{Game}_5} \geq \mathbf{Adv}_{\mathbf{Game}_4} - 2 \times \mathbf{Adv}_{\mathbb{G}}^{\mathbf{ddh}}(t)$
(Since $\mathbf{Adv} = 2 \times \Pr[b' = b] - 1$)

Proof: Collision

- **Game$_6$**: we abort (with a random output b') in case of second pre-image with a decryption query

Event $\mathbf{F}_H$

$\mathcal{A}$ submits a ciphertext with the same α as the challenge ciphertext, but a different initial triple: $(u_1, u_2, e) \neq (u_1^*, u_2^*, e^*)$, but $\alpha = \alpha^*$, were "*" are for all the elements related to the challenge ciphertext.

Second pre-image of $\mathcal{H}$: $\Longrightarrow \Pr[\mathbf{F}_H] \leq \mathbf{Succ}^{\mathcal{H}}(t)$

The advantage in **Game$_6$** is: $\Pr_{\mathbf{Game}_6}[b' = b | \mathbf{F}_H] = 1/2$

$$\Pr_{\mathbf{Game}_5}[\mathbf{F}_H] = \Pr_{\mathbf{Game}_6}[\mathbf{F}_H] \quad \Pr_{\mathbf{Game}_6}[b' = b | \neg \mathbf{F}_H] = \Pr_{\mathbf{Game}_5}[b' = b | \neg \mathbf{F}_H]$$

$\Longrightarrow$ **Hop-S-Negl**: $\mathbf{Adv}_{\mathbf{Game}_6} \geq \mathbf{Adv}_{\mathbf{Game}_5} - \Pr[\mathbf{F}_H]$

$$\mathbf{Adv}_{\mathbf{Game}_6} \geq \mathbf{Adv}_{\mathbf{Game}_5} - \mathbf{Succ}^{\mathcal{H}}(t)$$

Proof: Invalid ciphertexts

- **Game$_7$**: we abort (with a random output b') in case of bad accepted ciphertext, we do as in **Game$_1$**

Event $\mathbf{F}'$

$\mathcal{A}$ submits a bad accepted ciphertext
(note: this is not computationally detectable)

The advantage in **Game$_7$** is: $\Pr_{\mathbf{Game}_7}[b' = b | \mathbf{F}'] = 1/2$

$$\Pr_{\mathbf{Game}_6}[\mathbf{F}'] = \Pr_{\mathbf{Game}_7}[\mathbf{F}'] \quad \Pr_{\mathbf{Game}_7}[b' = b | \neg \mathbf{F}'] = \Pr_{\mathbf{Game}_6}[b' = b | \neg \mathbf{F}']$$

$\Longrightarrow$ **Hop-S-Negl**: $\mathbf{Adv}_{\mathbf{Game}_7} \geq \mathbf{Adv}_{\mathbf{Game}_6} - \Pr[\mathbf{F}']$

Details: Bad Accept

In order to evaluate $\Pr[\mathbf{F}']$, we study the probability that

- $r_1 = \log_{g_1} u_1 \neq \log_{g_2} u_2 = r_2$,
- whereas $v = {u_1}^{x_1+\alpha y_1} {u_2}^{x_2+\alpha y_2}$

Let us use "*" for all the elements related to the challenge ciphertext.

Three cases may appear:

- Case 1: $(u_1, u_2, e) = (u_1^*, u_2^*, e^*)$, then necessarily

$$v \neq v^* = U^{x_1+\alpha^* y_1} V^{x_2+\alpha^* y_2} = {u_1^*}^{x_1+\alpha^* y_1} {u_2^*}^{x_2+\alpha^* y_2}$$

  Then, the ciphertext is rejected $\Longrightarrow \Pr[\mathbf{F}_1'] = 0$
- Case 2: $(u_1, u_2, e) \neq (u_1^*, u_2^*, e^*)$, but $\alpha = \alpha^*$:
  From the previous game, Aborts $\Longrightarrow \Pr[\mathbf{F}_2'] = 0$
- Case 3: $(u_1, u_2, e) \neq (u_1^*, u_2^*, e^*)$, and $\alpha \neq \alpha^*$

Details: Bad Accept (Case 3)

The adversary knows the public key, and the (invalid) challenge ciphertext:

$$c = g_1^{x_1} g_2^{x_2} \qquad d = g_1^{y_1} g_2^{y_2}$$

$$v^* = U^{x_1+\alpha^* y_1} V^{x_2+\alpha^* y_2} = g_1^{r_1^*(x_1+\alpha^* y_1)} g_2^{r_2^*(x_2+\alpha^* y_2)}$$

Let us move to the exponents, in basis g_1, with $g_2 = g_1^s$:

$$\begin{aligned}
\log c &= x_1 + s x_2 \\
\log d &= y_1 + s y_2 \\
\log v^* &= r_1^*(x_1 + \alpha^* y_1) + s r_2^*(x_2 + \alpha^* y_2) \\
\log v &= r_1(x_1 + \alpha y_1) + s r_2(x_2 + \alpha y_2)
\end{aligned}$$

Details: Bad Accept (Case 3)

The determinant of the system is

$$\begin{aligned}
\Delta &= \begin{vmatrix} 1 & s & 0 & 0 \\ 0 & 0 & 1 & s \\ r_1^* & s r_2^* & r_1^* \alpha^* & s r_2^* \alpha^* \\ r_1 & s r_2 & r_1 \alpha & s r_2 \alpha \end{vmatrix} \\
&= s^2 \times ((r_2 - r_1) \times (r_2^* - r_1^*) \times \alpha^* - (r_2^* - r_1^*) \times (r_2 - r_1) \times \alpha) \\
&= s^2 \times (r_2 - r_1) \times (r_2^* - r_1^*) \times (\alpha^* - \alpha) \\
&\neq 0
\end{aligned}$$

The system is under-defined:
for any v, there are (x_1, x_2, y_1, y_2) that satisfy the system

$\Longrightarrow v$ is unpredictable $\Longrightarrow \Pr[\mathbf{F}_3'] \leq q_D/q$

$\Longrightarrow \mathbf{Adv}_{\mathbf{Game}_7} \geq \mathbf{Adv}_{\mathbf{Game}_6} - q_D/q$

Proof: Analysis of the Final Game

In the final $\mathbf{Game}_7$:

- only valid ciphertexts are decrypted
- the challenge ciphertext contains

$$e = m_b \times U^{z_1} V^{z_2}$$

- the public key contains

$$h = g_1^{z_1} g_2^{z_2}$$

Again, the system is under-defined:
for any m_b, there are (z_1, z_2) that satisfy the system
$\Longrightarrow m_b$ is unpredictable $\Longrightarrow b$ is unpredictable
$\Longrightarrow \mathbf{Adv}_{\mathbf{Game}_7} = 0$

Conclusion

$$
\begin{array}{rcl}
\mathbf{Adv}_{\mathsf{Game}_7} & = & 0 \\
\mathbf{Adv}_{\mathsf{Game}_7} & \geq & \mathbf{Adv}_{\mathsf{Game}_6} - q_D/q \\
\mathbf{Adv}_{\mathsf{Game}_6} & \geq & \mathbf{Adv}_{\mathsf{Game}_5} - \mathbf{Succ}^{\mathcal{H}}(t) \\
\mathbf{Adv}_{\mathsf{Game}_5} & \geq & \mathbf{Adv}_{\mathsf{Game}_4} - 2 \times \mathbf{Adv}^{\mathsf{ddh}}_{\mathbb{G}}(t) \\
\mathbf{Adv}_{\mathsf{Game}_4} & = & \mathbf{Adv}_{\mathsf{Game}_3} \\
\mathbf{Adv}_{\mathsf{Game}_3} & \geq & \mathbf{Adv}_{\mathsf{Game}_2} - q_D/q \\
\mathbf{Adv}_{\mathsf{Game}_2} & = & \mathbf{Adv}_{\mathsf{Game}_1} \\
\mathbf{Adv}_{\mathsf{Game}_1} & \geq & \mathbf{Adv}_{\mathsf{Game}_0} - q_D/q \\
\mathbf{Adv}_{\mathsf{Game}_0} & = & \mathbf{Adv}^{\mathsf{ind-cca}}_{\mathcal{CS}}(\mathcal{A})
\end{array}
$$

$$\mathbf{Adv}^{\mathsf{ind-cca}}_{\mathcal{CS}}(\mathcal{A}) \leq 2 \times \mathbf{Adv}^{\mathsf{ddh}}_{\mathbb{G}}(t) + \mathbf{Succ}^{\mathcal{H}}(t) + 3q_D/q$$

Outline

First Generic Conversion

[Bellare-Rogaway – Eurocrypt '93]

For efficiency: random oracle model

Setup

- A trapdoor one-way permutation family $\{(f, g)\}$ onto the set X
- Two hash functions, for the security parameter k_1,
  $$\mathcal{G} : X \longrightarrow \{0,1\}^n \text{ and } \mathcal{H} : \{0,1\}^\star \longrightarrow \{0,1\}^{k_1},$$
  where n is the bit-length of the plaintexts.

Key Generation

One chooses a random element in the family

- f is the public key
- the inverse g is the private key

First Generic Conversion (Cont'ed)

Encryption

One chooses a random element $r \in X$

$$a = f(r), \quad b = m \oplus \mathcal{G}(r), \quad c = \mathcal{H}(m, r)$$

Decryption

Given (a, b, c), and the private key g,

- one first recovers $r = g(a)$
- one gets $m = b \oplus \mathcal{G}(r)$
- one then checks whether $c \stackrel{?}{=} \mathcal{H}(m, r)$

If the equality holds, one returns m,
otherwise one rejects the ciphertext

Security of the Bellare-Rogaway Conversion

Theorem

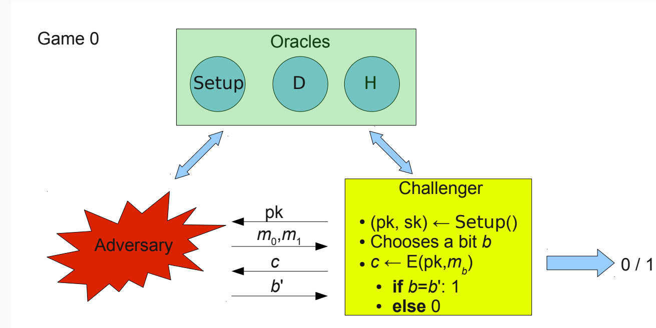
The Bellare-Rogaway conversion achieves $\mathbf{IND}-\mathbf{CCA}$ *security, under the one-wayness of the trapdoor permutation* f*:*

$$\mathbf{Adv}^{\mathsf{ind-cca}}_{\mathcal{BR}}(t) \leq 2 \times \mathbf{Succ}^{\mathsf{ow}}_{f}(T) + \frac{4q_D}{2^{k_1}},$$

where $T \leq t + (q_G + q_H) \cdot T_f$.

Let us prove this theorem, with a sequence of games, in which $\mathcal{A}$ is an $\mathbf{IND}-\mathbf{CCA}$ adversary against the Bellare-Rogaway conversion.

Real Attack Game



Key Generation Oracle

Random permutation f, and its inverse g

Decryption Oracle

Compute $r = g(a)$, and then $m = b \oplus \mathcal{G}(r)$
if $c = \mathcal{H}(m, r)$, outputs m, otherwise reject

Simulation of the Random Oracles

- **Game**$_0$: use of the perfect oracles

Challenge Ciphertext

Random r, random bit b: $a = f(r)$, $b = m_b \oplus \mathcal{G}(r)$, $c = \mathcal{H}(m, r)$

$$\mathbf{Adv}_{\mathbf{Game}_0} = 2 \times \Pr_{\mathbf{Game}_0}[b' = b] - 1 = \varepsilon$$

- **Game**$_1$: use of the simulation of the random oracles

Random Oracles

For any new query, a new random output: management of lists

$$\mathbf{Adv}_{\mathbf{Game}_1} = \mathbf{Adv}_{\mathbf{Game}_0}$$

Simulation of the Challenge Ciphertext

- **Game**$_2$: use of an independent random value h^+

Challenge Ciphertext

Random r, random bit b: $a = f(r)$, $b = m_b \oplus \mathcal{G}(r)$, $c = h^+$

This game is indistinguishable from the previous one, unless (m_b, r) is queried to $\mathcal{H}$: event **AskMR** (it can only be asked by the adversary, since such a query by the decryption oracle would be for the challenge ciphertext).
Note that in case of **AskMR**, we stop the simulation with a random output:

$$\mathbf{Adv}_{\mathbf{Game}_2} \geq \mathbf{Adv}_{\mathbf{Game}_1} - 2 \times \Pr_{\mathbf{Game}_2}[\mathbf{AskMR}]$$

Simulation of the Decryption Oracle

- **Game**$_3$: reject if (m, r) not queried to $\mathcal{H}$

Decryption Oracle

Look in the $\mathcal{H}$-list for (m, r) such that $c = \mathcal{H}(m, r)$.
If not found: reject,
if for one pair, $a = f(r)$ and $b = m \oplus \mathcal{G}(r)$, output m

This makes a difference if this value c, without having been asked to $\mathcal{H}$, is correct: for each attempt, the probability is bounded by $1/2^{k_1}$:

$$\begin{aligned} \mathbf{Adv}_{\textbf{Game}_3} &\geq \mathbf{Adv}_{\textbf{Game}_2} - 2q_D/2^{k_1} \\ \Pr_{\textbf{Game}_3}[\textbf{AskMR}] &\geq \Pr_{\textbf{Game}_2}[\textbf{AskMR}] - q_D/2^{k_1} \end{aligned}$$

Simulation of the Challenge Ciphertext

- **Game**$_4$: use of an independent random value g^+ (and h^+)

Challenge Ciphertext

Random r, random bit b: $a = f(r)$, $b = m_b \oplus g^+$, $c = h^+$

This game is indistinguishable from the previous one, unless r is queried to $\mathcal{G}$ by the adversary or by the decryption oracle. We denote by **AskR** the event that r is asked to $\mathcal{G}$ or $\mathcal{H}$ by the adversary (which includes **AskMR**). But r cannot be asked to $\mathcal{G}$ by the decryption oracle without **AskR**: only possible if r is in the $\mathcal{H}$-list, and thus asked by the adversary:

$$\begin{aligned} \mathbf{Adv}_{\textbf{Game}_4} &\geq \mathbf{Adv}_{\textbf{Game}_3} - 2 \times \Pr_{\textbf{Game}_3}[\textbf{AskR} \wedge \neg\textbf{AskMR}] \\ \Pr_{\textbf{Game}_4}[\textbf{AskR}] &= \Pr_{\textbf{Game}_3}[\textbf{AskMR}] + \Pr_{\textbf{Game}_3}[\textbf{AskR} \wedge \neg\textbf{AskMR}] \end{aligned}$$

Simulation of the Challenge Ciphertext

- **Game**$_5$: use of an independent random value a^+ (and g^+, h^+)

Challenge Ciphertext

random bit b: $a = a^+$, $b = m_b \oplus g^+$, $c = h^+$

This determines r, the unique value such that $a^+ = f(r)$, which allows to detect event **AskR**.
This game is perfectly indistinguishable from the previous one:

$$\begin{aligned} \mathbf{Adv}_{\textbf{Game}_5} &= \mathbf{Adv}_{\textbf{Game}_4} \\ \Pr_{\textbf{Game}_5}[\textbf{AskR}] &= \Pr_{\textbf{Game}_4}[\textbf{AskR}] \end{aligned}$$

Inversion of the Permutation

Since we can assume that a^+ is a given challenge for inverting the permutation f, when one looks in the $\mathcal{G}$-list or the $\mathcal{H}$-list, one can find r, the pre-image of a^+:

$$\Pr_{\textbf{Game}_5}[\textbf{AskR}] \leq \mathbf{Succ}_f^{\mathsf{ow}}(t + (q_G + q_H) \cdot T_f)$$

But clearly, in the last game, because of g^+ that perfectly hides m_b:

$$\mathbf{Adv}_{\textbf{Game}_5} = 0$$

Conclusion

As a consequence, $0 = \mathbf{Adv}_{\mathsf{Game}_5}$

$$
\begin{aligned}
&= \mathbf{Adv}_{\mathsf{Game}_4} \geq \mathbf{Adv}_{\mathsf{Game}_3} - 2 \times \Pr_{\mathsf{Game}_3}[\mathbf{AskR} \wedge \neg\mathbf{AskMR}] \\
&\geq \mathbf{Adv}_{\mathsf{Game}_2} - 2 \times \Pr_{\mathsf{Game}_3}[\mathbf{AskR} \wedge \neg\mathbf{AskMR}] - 2q_D/2^{k_1} \\
&\geq \mathbf{Adv}_{\mathsf{Game}_1} - 2 \times \Pr_{\mathsf{Game}_2}[\mathbf{AskMR}] - 2 \times \Pr_{\mathsf{Game}_3}[\mathbf{AskR} \wedge \neg\mathbf{AskMR}] - 2q_D/2^{k_1} \\
&\geq \mathbf{Adv}_{\mathsf{Game}_0} - 2 \times \Pr_{\mathsf{Game}_3}[\mathbf{AskMR}] - 2 \times \Pr_{\mathsf{Game}_3}[\mathbf{AskR} \wedge \neg\mathbf{AskMR}] - 4q_D/2^{k_1} \\
&\geq \mathbf{Adv}_{\mathsf{Game}_0} - 2 \times \Pr_{\mathsf{Game}_4}[\mathbf{AskR}] - 4q_D/2^{k_1} \\
&\geq \mathbf{Adv}_{\mathsf{Game}_0} - 2 \times \Pr_{\mathsf{Game}_5}[\mathbf{AskR}] - 4q_D/2^{k_1}
\end{aligned}
$$

And then,

$$\mathbf{Adv}_{\mathsf{Game}_0} \leq 4q_D/2^{k_1} + 2 \times \mathbf{Succ}_f^{\mathsf{ow}}(T)$$

Conclusion

Outline

Conclusion

Game-based Methodology: the story of OAEP [Bellare-Rogaway EC '94]

- Reduction proven indistinguishable for an IND-CCA adversary (actually IND-CCA1, and not IND-CCA2) but widely believed for IND-CCA2, without any further analysis of the reduction
  The direct-reduction methodology
- [Shoup - Crypto '01]
  Shoup showed the gap for IND-CCA2, under the OWP
  Granted his new game-based methodology
- [Fujisaki-Okamoto-Pointcheval-Stern – Crypto '01]
  FOPS proved the security for IND-CCA2, under the PD-OWP
  Using the game-based methodology