Heat & Laplace Equations

Problem: design heat sink for power hungry chip.

1. In: active masks; out: temperature.
2. Try & measure is not an option.
3. Simulation has high complexity.
4. Design specific (reconfigurable) hardware for computing heat flow.

Heat Equation

$$\frac{d}{dt}T(x,y,z,t) \quad = \quad c\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)T(x,y,z,t)+p(x,y,z,t)$$

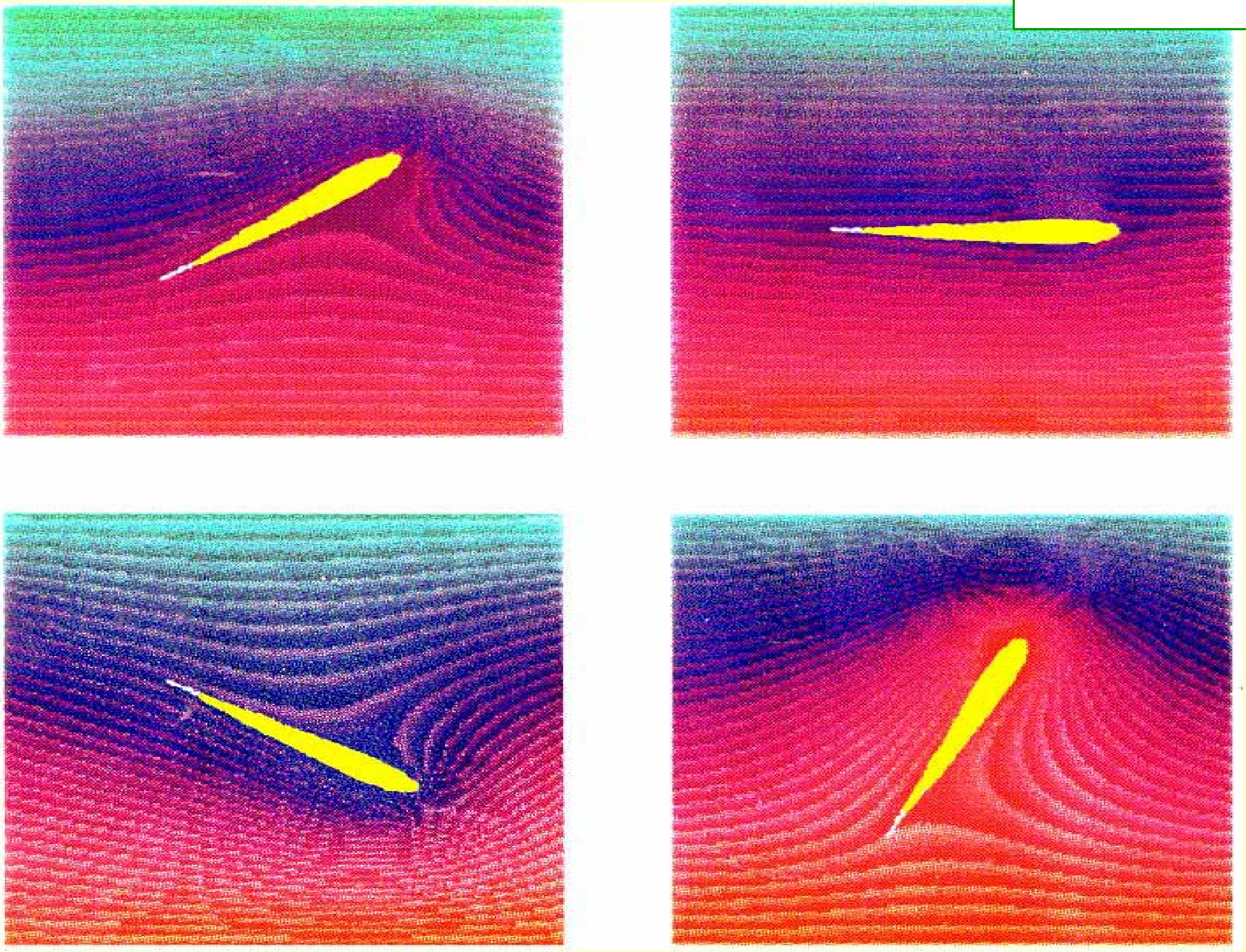
$p(x,y,z,t)$ is the external heat power.

$$\boxed{\frac{dT}{dt} \quad = \quad c\nabla^2 T + P}$$

- ***Thermodynamics***
- ***Electromagnetic fields***
- ***RC circuits***
- ***Chemical kinetics***
- ***Hydrodynamics***
- ***Neutron diffusion***

Laplace Equation

$$\nabla^2 T = 0$$



Solve in 3D:

$$\frac{dT}{dt} = c\nabla^2 T + P$$

Linear:

$$P(x,y,z,t) = \sum_{(k,l,m,n)\in D} q\partial(x-k)\partial(y-l)\partial(z-m)\partial(t-n),$$

$$T(x,y,z,t) = \sum_{(k,l,m,n)\in D} qg_3(x-k,y-l,z-m,t-n).$$

$$\boxed{\frac{dg_3}{dt} = c\nabla^2 g_3(x,y,z,t)+\partial(x)\partial(y)\partial(z)\partial(t)}$$

Isotropic:

$$g_3(x,y,z,t) = g(x,2ct)g(y,2ct)g(z,2ct)$$

$$\boxed{\left(\frac{d}{dt}-c\frac{d}{2dx^2}\right)g(x,t) = \partial(x)\partial(t)}$$

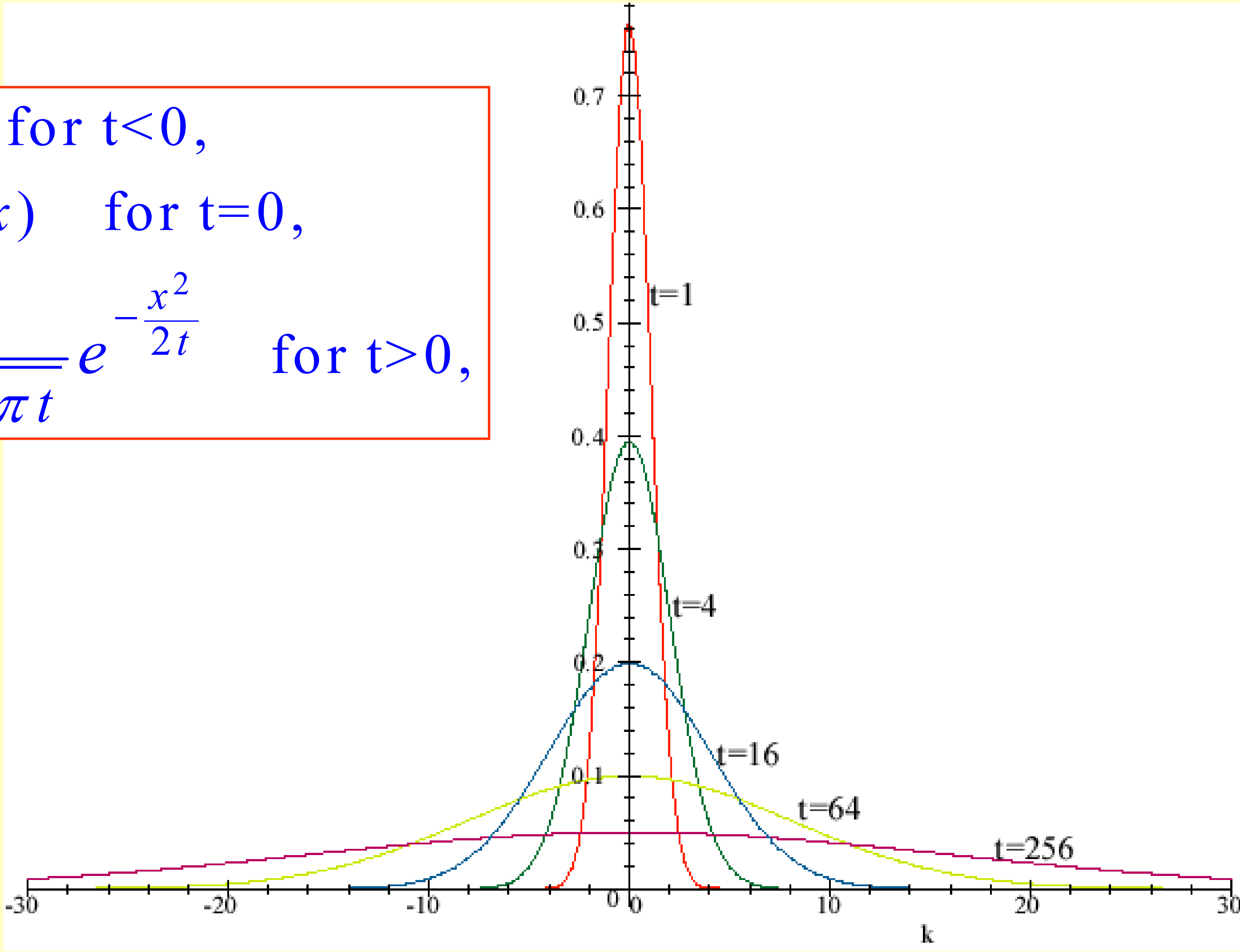
Solve in 1D:

$$\left(\frac{d}{dt}-c\frac{d}{2dx^2}\right)g(x,t) = \partial(x)\partial(t)$$

$$g(x,t) = 0 \quad \text{for } t<0,$$

$$g(x,0) = \partial(x) \quad \text{for } t=0,$$

$$g(x,t) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}} \quad \text{for } t>0,$$



Discrete Model

$$\frac{dT}{dt} = c\frac{d^2T}{dt^2}$$

$$q(k,n) = \frac{1}{\Delta x \Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_{(k-\frac{1}{2})\Delta x}^{(k+\frac{1}{2})\Delta x} T(x,t)\,dx\,dt$$

$$q(k,n+1) = \frac{1}{\Delta x \Delta t}\int_{n\Delta t}^{(n+1)\Delta t}\int_{(k-\frac{1}{2})\Delta x}^{(k+\frac{1}{2})\Delta x} T(x,t+\Delta t)\,dx\,dt$$

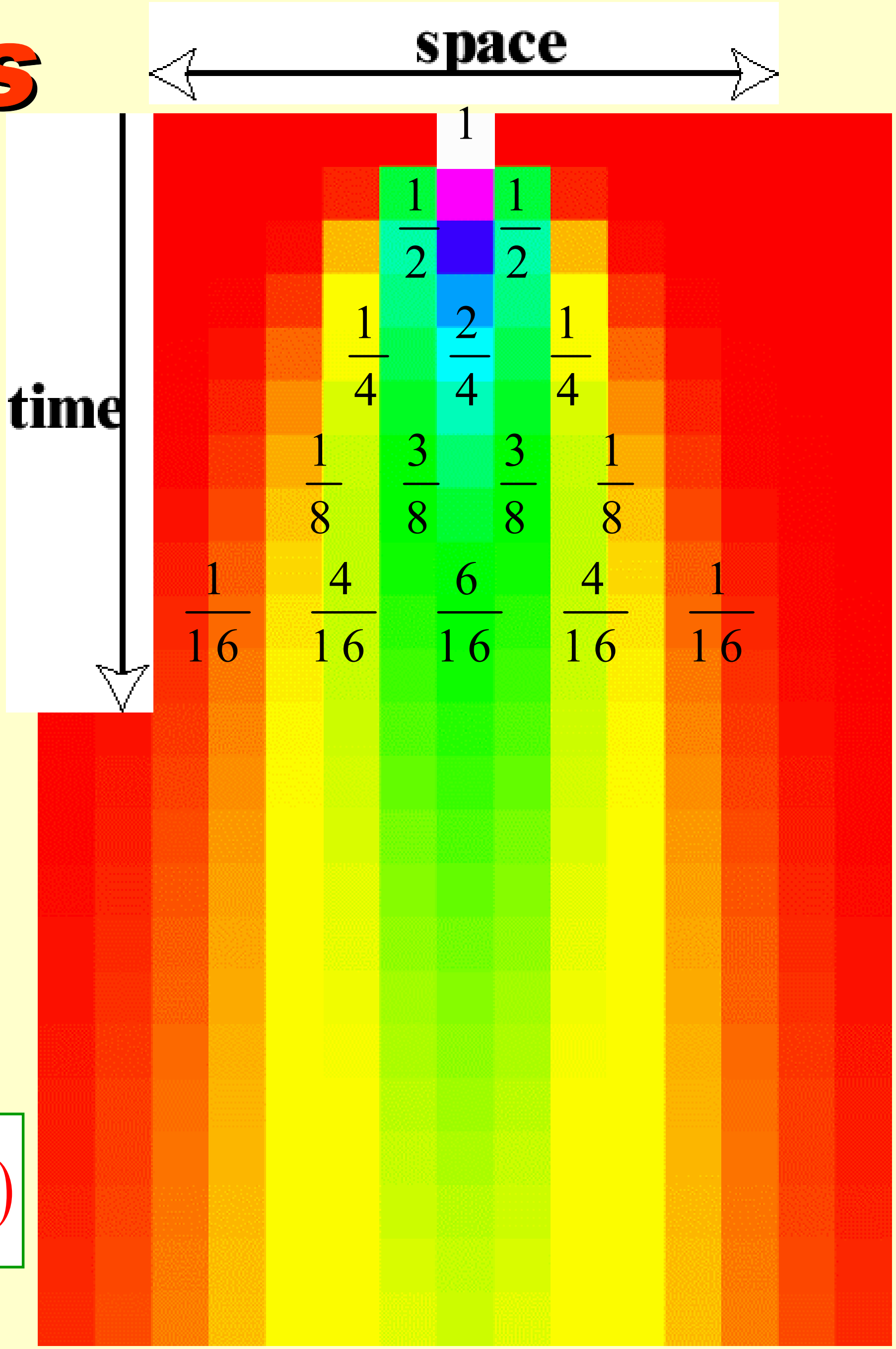
$$q(k,n+1) \approx q(k,n) + c\frac{\Delta t}{\Delta x^2}\big(q(k+1,n) - 2q(k,n) + q(k-1,n)\big)$$

$$\boxed{\Delta x^2 = 2c\,\Delta t}$$

$$\boxed{q(k,n+1) = \frac{1}{2}\big(q(k-1,n) + q(k+1,n)\big)}$$

Finite Elements

$$\frac{dT}{dt} = c\frac{d^2T}{dt^2}$$



$$\Delta x^2 = 2c\,\Delta t$$

$$q(k,n+1) = \frac{1}{2}\left(q(k-1,n)+q(k+1,n)\right)$$

Discrete Solution

$$\frac{dT}{dt} = c\frac{d^2T}{dt^2}$$

$$\Delta x^2 = 2c\,\Delta t$$

$$q(k,0) = \partial_0^k$$

$$q(k,n+1) = \frac{1}{2}\big(q(k-1,n)+q(k+1,n)\big)$$

$$Q(z,t) = \sum_{k\in\mathbb{Z},n\in\mathbb{N}} q(k,n)z^{k/2}\,t^n$$

$$Q(z,t) = 1+\frac{t}{2}\left(z^{1/2}+z^{-1/2}\right)Q(z,t)$$

$$Q(z,t) = \frac{1}{1-tH(z)}$$

$$H(z) = \frac{1}{2}\left(\sqrt{z}+\frac{1}{\sqrt{z}}\right)$$

$$H^2(z) = \frac{1}{4}\left(z+2+\frac{1}{z}\right)$$

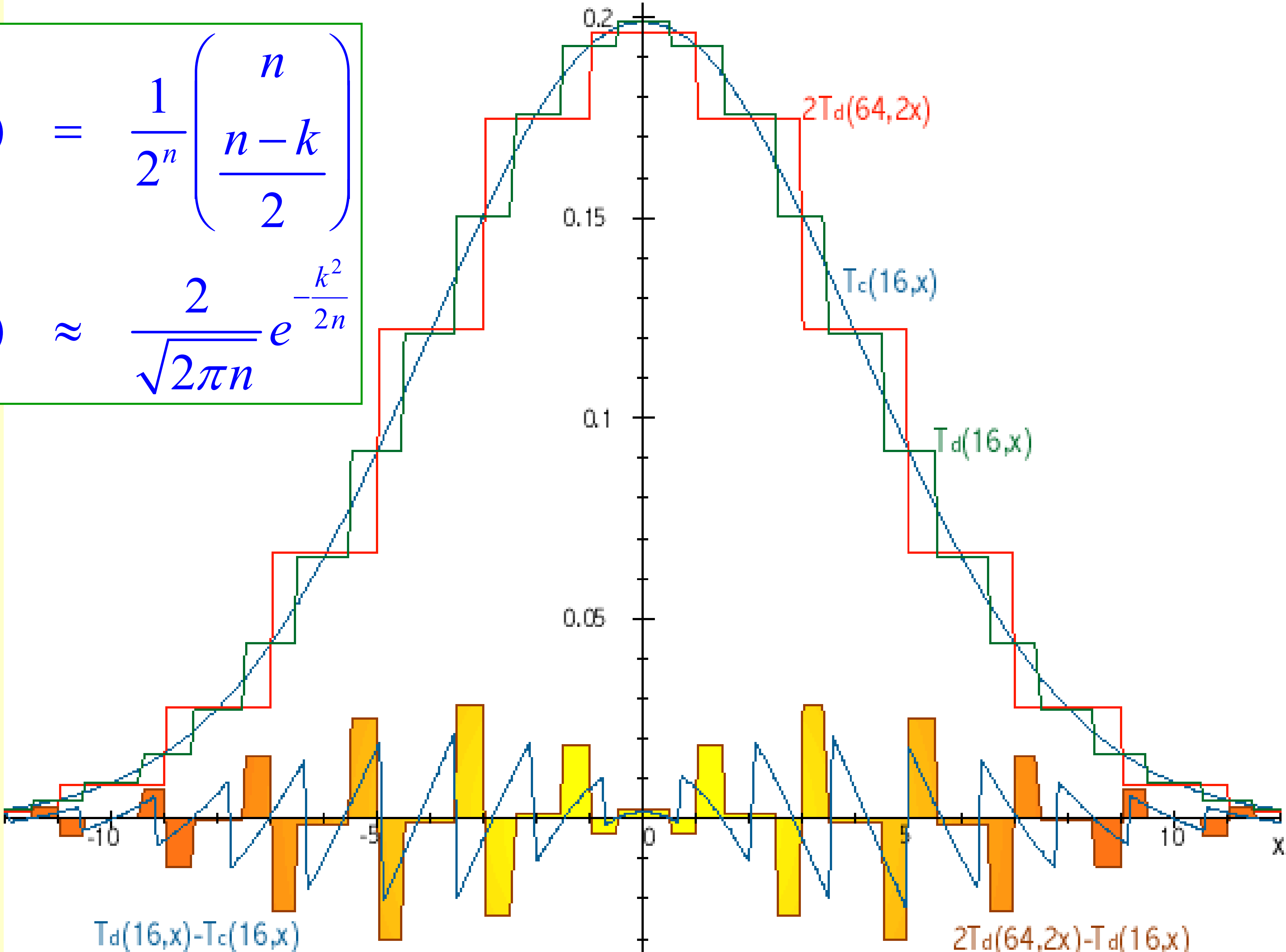
$$q(k,n) = \frac{1}{2^n}\binom{n}{\frac{n-k}{2}}$$

$$q(k,n) \approx \frac{2}{\sqrt{2\pi n}}e^{-\frac{k^2}{2n}}$$

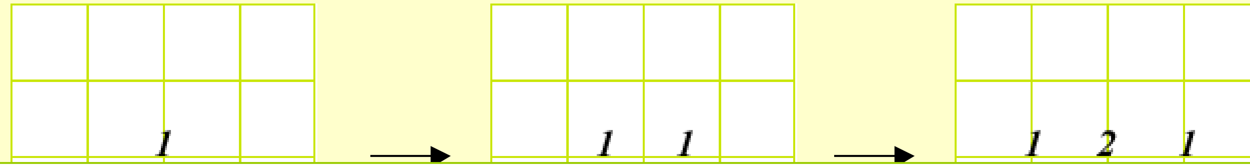
Numerical Errors

$$q(k,n) = \frac{1}{2^n}\binom{n}{\frac{n-k}{2}}$$

$$q(k,n) \approx \frac{2}{\sqrt{2\pi n}} e^{-\frac{k^2}{2n}}$$



Finite Elements: 2D



$$Q(x,y,t) = \frac{1}{1-tH_2(x,y)} = \sum_{n\in\mathbb{N}} H_2^{\,n}(x,y)t^n$$

$$H_2(x,y) = \frac{1}{4}\left(\sqrt{xy}+\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}+\frac{1}{\sqrt{xy}}\right)$$

$$H_2(x,y) = H(x)H(y) = H(y)H(x)$$

$$H_2^{\,n}(x,y) = H^n(x)H^n(y) = H^n(y)H^n(x)$$

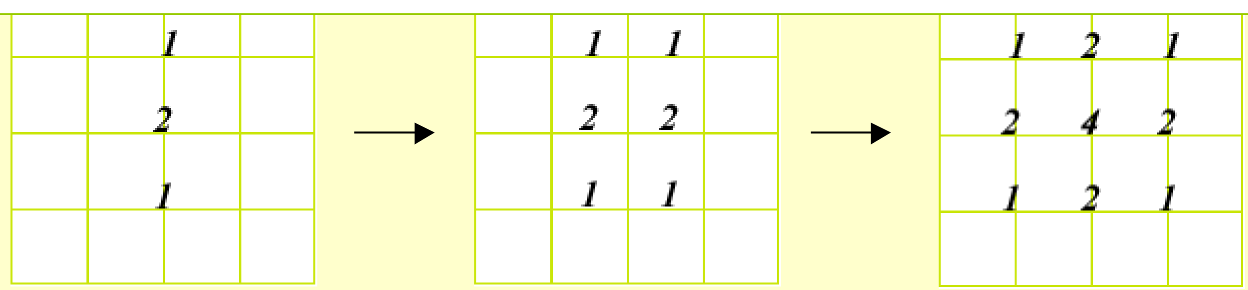


Image Convolution

Image convolution: 2D

$$T_{n+2}^{00} = \frac{1}{16}\begin{pmatrix} T_n^{--} + 2T_n^{0-} + T_n^{+-} \\ +2T_n^{-0} + 4T_n^{00} + 2T_n^{+0} \\ +T_n^{-+} + 2T_n^{0+} + T_n^{++} \end{pmatrix}$$

3 load, 1 store, 8 add, 6 shift = 18 operations + 10 registers

Separable convolution: 1Dx1D

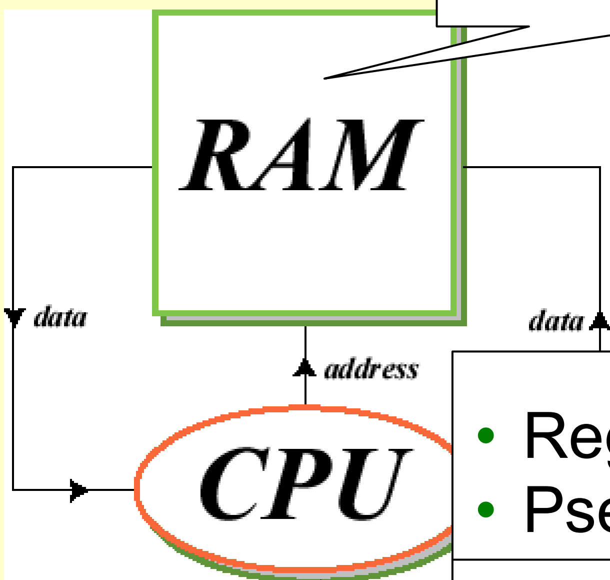
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

2x(1 load, 1 store, 2 add, 2 shift) = 12 operations + 3 registers

System Architecture

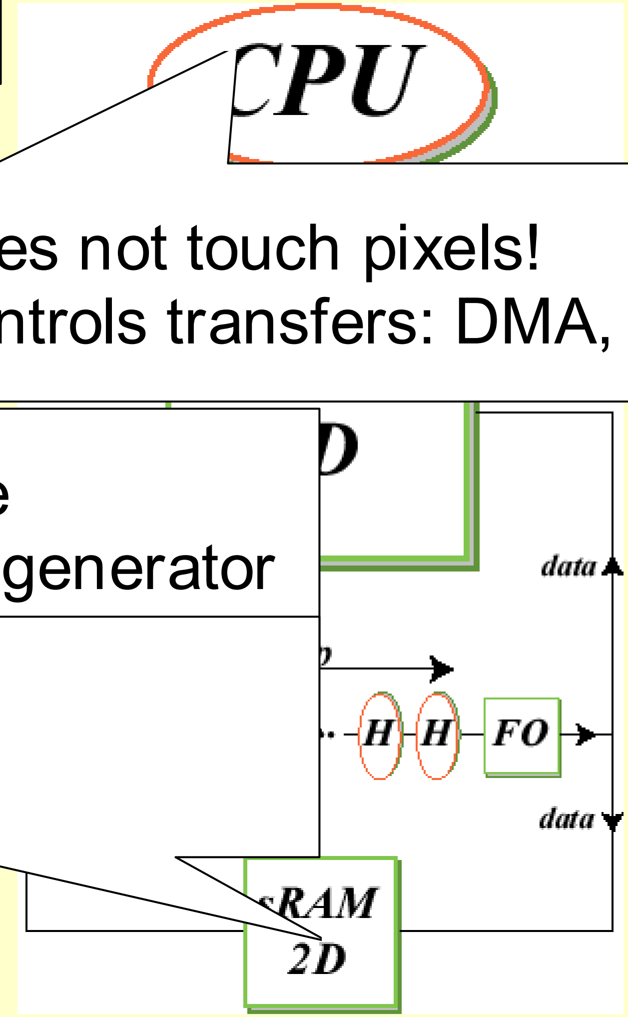
Software

- Big: V = xyz
- Update: in-place



- Bandwid
- At best: 2dV cycles/Δt

Hardware



- Does not touch pixels!
- Controls transfers: DMA, I/O

- Regular pipe-line
- Pseudo-random generator

- Heat sources
- Heat sinks
- Heat mirrors

Limit Conditions

Method of Images

$$\frac{dT}{dt} = c\nabla^2 T + P, \quad (x,y,z) \in \mathfrak{D}$$

$$\frac{dT}{d\bar{o}} = 0, \quad (x,y,z) \in \mathfrak{B}, \quad \bar{o} \perp \mathfrak{B}$$

$$\frac{dT}{dt} = 0, \quad (x,y,z) \notin \mathfrak{D} \cup \mathfrak{B}$$

$$\begin{array}{ccc|ccc|ccc} & 1 & & & 1 & & & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

- Heat mirror

$$\begin{array}{ccc|c|ccc} & -1 & & 0 & & 1 & \\ -1 & -2 & -1 & 0 & 1 & 2 & 1 \end{array}$$

- Heat sink

$$\begin{array}{ccc|c|ccc} & & & 1 & & & \\ & & \frac{1}{4} & \frac{2}{4}+1 & \frac{1}{4} & & \\ & \frac{1}{16} & \frac{8}{16} & \frac{14}{16}+1 & \frac{8}{16} & \frac{1}{16} & \\ \frac{1}{64} & \frac{10}{64} & \frac{31}{64} & \frac{44}{64}+1 & \frac{31}{64} & \frac{10}{64} & \frac{1}{64} \end{array}$$

- Heat source

Energy Invariant

$$q(k,n+1) = \frac{1}{2}\big(q(k-1,n)+q(k+1,n)\big)$$

$$1 = \int_{-\infty}^{\infty} g_3(x,y,z,t)\,dxdydz$$

$$1 = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2t}}\,dx$$

$$1 = \frac{1}{2^n}\sum_k \binom{n}{\frac{n-k}{2}}$$

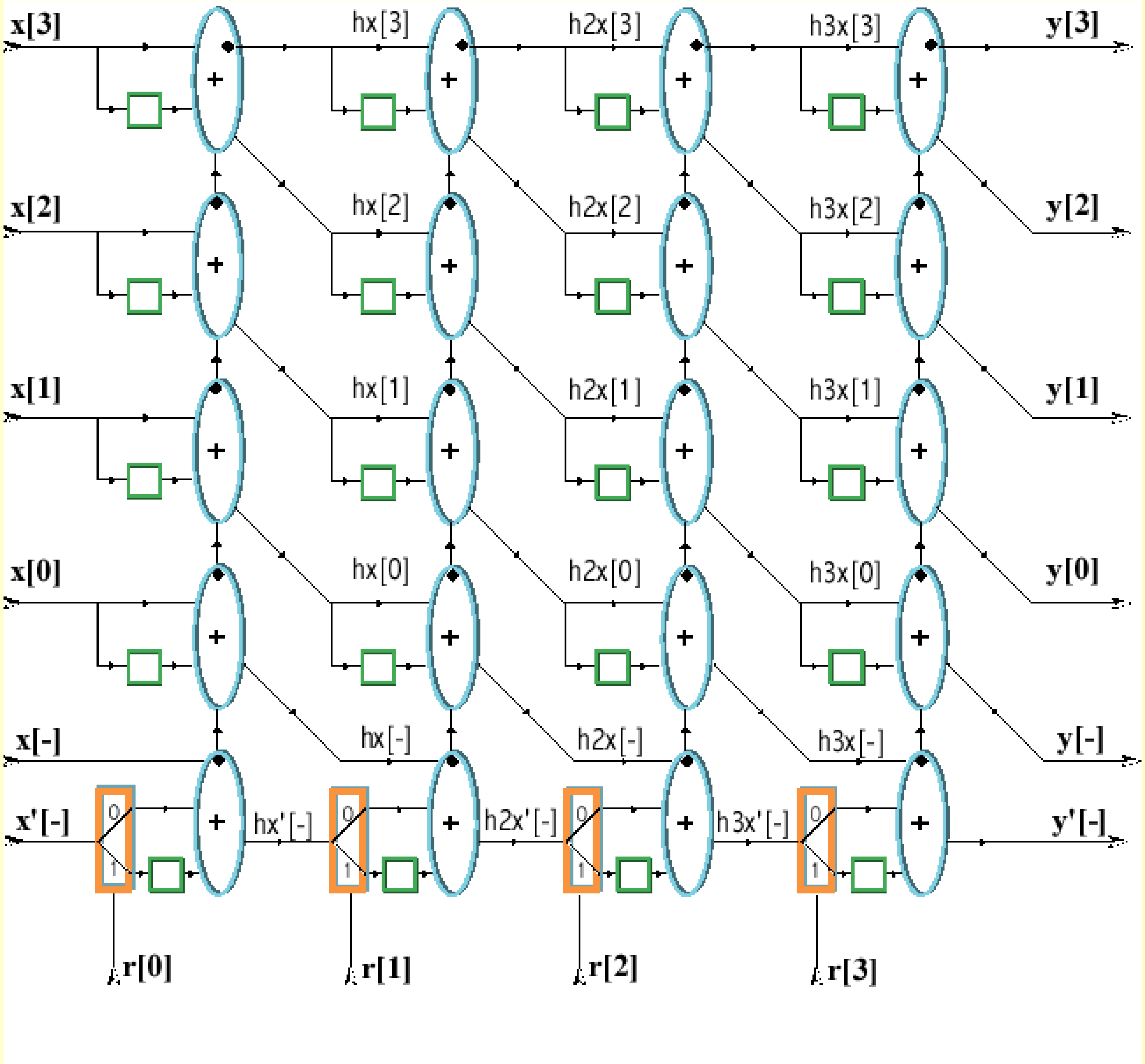
How to split a bit?

- Truncating induces an energy leak, proportional to volume, and time!
- Each deterministic split induces parasitic stable solutions!
- Random walk solves the Heat Equation.
- Solution: Random Split!

Data Path Design

Pipeline

Random split



Reconfigurable Emulator