# Finite Binary Number $d \in \mathbb{D}_n$

$$d = [\![ d_{0\cdots n-1} \quad \{d\} \quad d(z) \quad d(2) ]\!]$$

1. Bit sequence: $d_{0\cdots n-1} = d_0 \cdots d_{n-1}$
2. Integer set: $\{d\} = \{k : d_k = 1\}$
3. Polynomial: $d(z) = \sum d_k z^k$
4. Integer: $d(2) = \sum d_k 2^k$

# 4 bits

| $b_0b_1b_2b_3$ | $\{i : 1=b_i\}$ | $b(z)$ | $\sum_{k<4} b_k 2^k$ |
|---|---|---|---|
| 0000 | | 0 | 0 |
| 1000 | 0 | 1 | 1 |
| 0100 | 1 | $z$ | 2 |
| 1100 | 0,1 | $1+z$ | 3 |
| 0010 | 2 | $z^2$ | 4 |
| 1010 | 0,2 | $1+z^2$ | 5 |
| 0110 | 1,2 | $z+z^2$ | 6 |
| 1110 | 0,1,2 | $1+z+z^2$ | 7 |
| 0001 | 3 | $z^3$ | 8 |
| 1001 | 0,3 | $1+z^3$ | 9 |
| 0101 | 1,3 | $z+z^3$ | 10 |
| 1101 | 0,1,3 | $1+z+z^3$ | 11 |
| 0011 | 2,3 | $z^2+z^3$ | 12 |
| 1011 | 0,2,3 | $1+z^2+z^3$ | 13 |
| 0111 | 1,2,3 | $z+z^2+z^3$ | 14 |
| 1111 | 0,1,2,3 | $1+z+z^2+z^3$ | 15 |

# Boolean Algebra

$$\mathbb{D}_n \text{ is isomorphic to } \left\langle 2^n \quad \neg \quad \cap \quad \cup \right\rangle.$$

Every finite Boolean Algebra is isomorphic to $D_n$ for some $n$.

***Duality***

$$\begin{array}{rcl} \neg(a \cup b) & = & \neg a \cap \neg b \\ \neg(a \cap b) & = & \neg a \cup \neg b \end{array}$$

$$\begin{array}{rcl} a \cap b & = & b \cap a \\ a \cap (b \cap c) & = & (a \cap b) \cap c \\ a \cap \neg 0 & = & a \\ a \cap \neg a & = & 0 \\ a \cap (b \cup c) & = & (a \cap b) \cup (a \cap c) \end{array}$$

$$\begin{array}{rcl} a \cup b & = & b \cup a \\ a \cup (b \cup c) & = & (a \cup b) \cup c \\ a \cup 0 & = & a \\ a \cup \neg a & = & \neg 0 \\ a \cup (b \cap c) & = & (a \cup b) \cap (a \cup c) \end{array}$$

# Boolean Ring

$$\boxed{a \oplus b \;=\; (a \cap \neg b) \cup (\neg a \cap b)}$$

$$\boxed{\begin{array}{l} 0 = a \oplus a \\ a = a \cap a \end{array}}$$

$\mathbb{D}_n$ is isomorphic to $\langle \mathbb{F}_2^n \quad \oplus \quad \cap \rangle$.

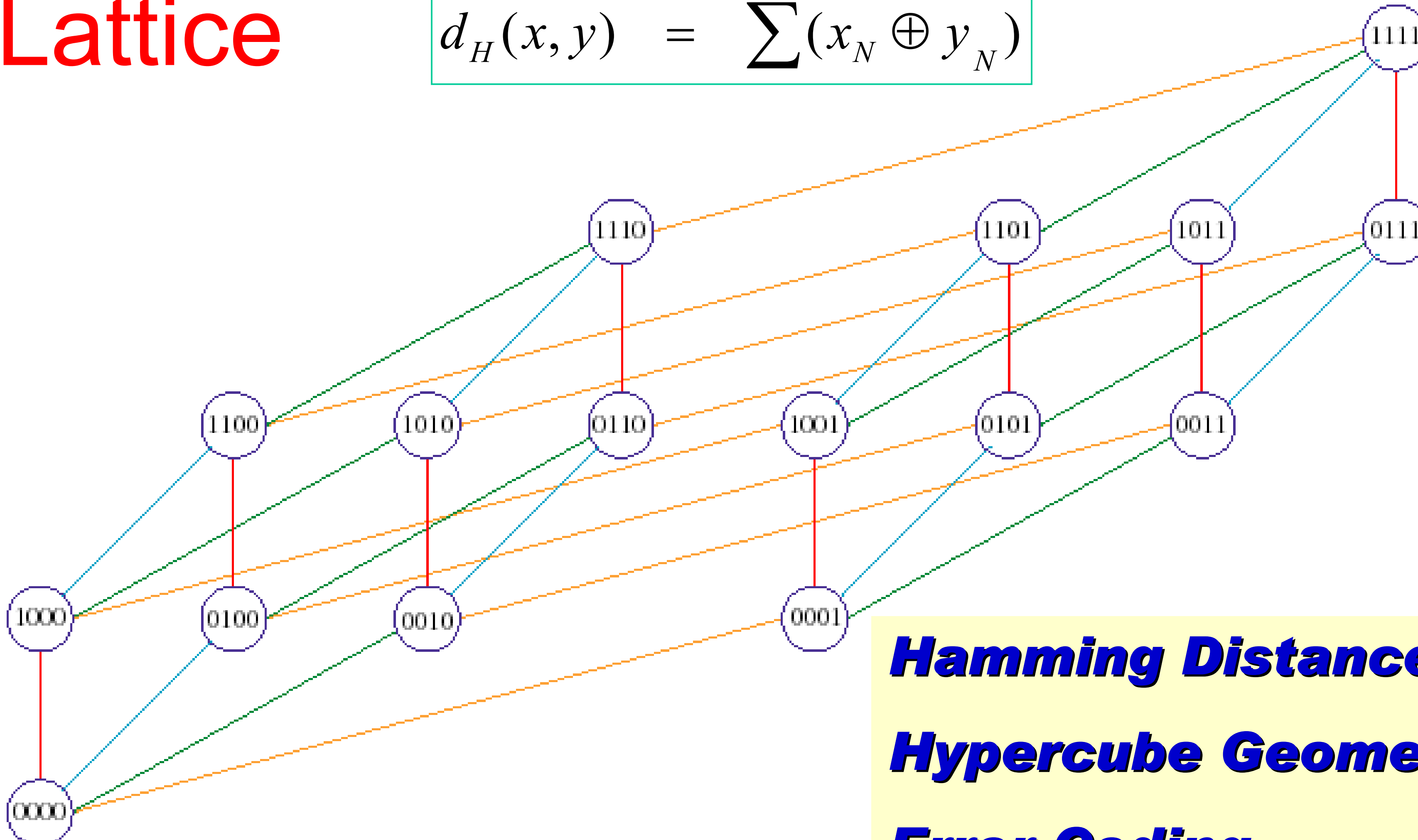
$$\begin{array}{rcl} a \cap b & = & b \cap a \\ a \cap (b \cap c) & = & (a \cap b) \cap c \\ a \cap \neg 0 & = & a \\ a \cap 0 & = & 0 \\ a \cap (b \oplus c) & = & (a \cap b) \oplus (a \cap c) \end{array} \qquad \begin{array}{rcl} a \oplus b & = & b \oplus a \\ a \oplus (b \oplus c) & = & (a \oplus b) \oplus c \\ a \oplus 0 & = & a \\ a \oplus a & = & 0 \\ a \cap a & = & a \end{array}$$

Every finite Boolean Ring is isomorphic to $D_n$ for some $n$.

# Boolean Lattice

$$x \subseteq y \quad \Leftrightarrow \quad \forall N : x_N \Rightarrow y_N \quad \Leftrightarrow \quad \varnothing = x \cap \neg y$$

$$d_H(x, y) \;=\; \sum (x_N \oplus y_N)$$




***Hamming Distance***

***Hypercube Geometry***

***Error Coding***

# Polynomial Ring

$$\boxed{a \otimes b \triangleq a(z) \times b(z) \ (\mathrm{mod}\ z^n)(\mathrm{mod}\ 2)}$$

$\mathbb{D}_n$ is isomorphic to $\left\langle \mathbb{F}_2[z] / z^n \mathbb{F}_2[z] \quad \oplus \quad \otimes \right\rangle$.

$$\begin{array}{rclrcl}
a \otimes b & = & b \otimes a & a \oplus b & = & b \oplus a \\
a \otimes (b \otimes c) & = & (a \otimes b) \otimes c & a \oplus (b \oplus c) & = & (a \oplus b) \oplus c \\
a \otimes 1 & = & a & a \oplus 0 & = & a \\
a \otimes 0 & = & 0 & a \oplus a & = & 0 \\
a \otimes (b \oplus c) & = & (a \otimes b) \oplus (a \otimes c) & (a \oplus b)^2 & = & a^2 \oplus b^2
\end{array}$$

$$\boxed{\frac{1}{1+za} = 1 \oplus za \cdots \oplus z^{n-1} a^{n-1} \ (\mathrm{mod}\ z^n)}$$

# Integer Ring

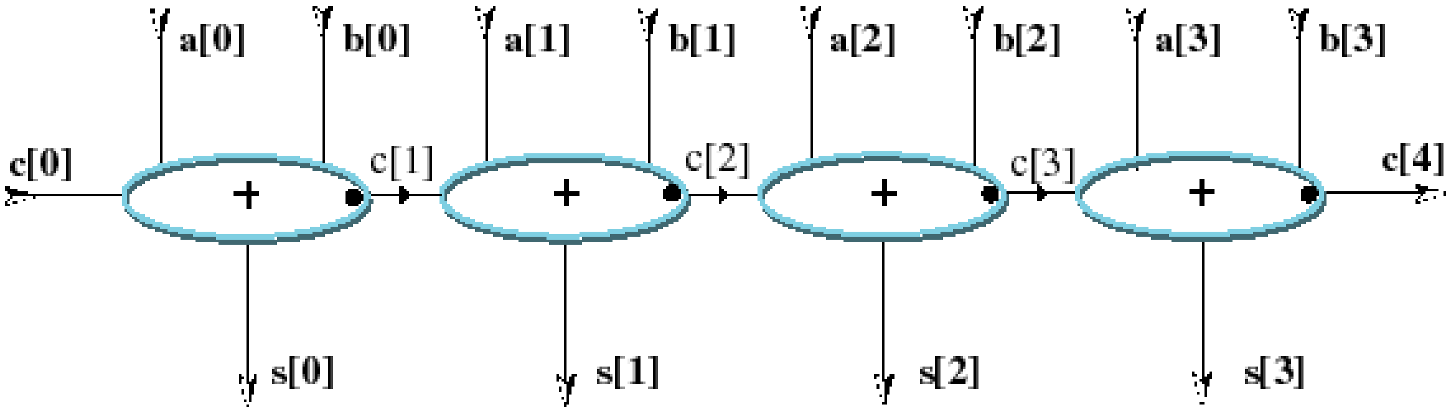
$$a+b \triangleq a(2)+b(2) \pmod{2^n}$$
$$a-b \triangleq a(2)-b(2) \pmod{2^n}$$
$$a\times b \triangleq a(2)\times b(2) \pmod{2^n}$$

$\mathbb{D}_n$ is isomorphic to $\langle \mathbb{Z}/2^n\mathbb{Z} \quad \oplus \quad \otimes \rangle$.

$$\begin{array}{rcl}
a\times b & = & b\times a \\
a\times(b\times c) & = & (a\times b)\times c \\
a\times 1 & = & a \\
a\times 0 & = & 0 \\
a\times(b+c) & = & (a\times b)+(a\times c)
\end{array}
\qquad
\begin{array}{rcl}
a+b & = & b+a \\
a+(b+c) & = & (a+b)+c \\
a+0 & = & a \\
a-a & = & 0 \\
-a & = & -1\times a
\end{array}$$

$$\frac{1}{1-2a} = 1+2a+\cdots+2^{n-1}a^{n-1} \pmod{2^n}$$

# Parallel Adder




$$a_0 + b_0 + c_0 = s_0 + 2c_1$$

$$a_1 + b_1 + c_1 = s_1 + 2c_2$$

$$\ldots$$

$$\boxed{c_0 + \sum_{k<N} a_k 2^k + \sum_{k<N} b_k 2^k = \sum_{k<N} s_k 2^k + c_N 2^N}$$

# Finite Binary Algebra

$\langle \mathbb{D}_n, \neg, \cap, \cup \rangle$ is a Boolean Algebra

$\langle \mathbb{D}_n, \oplus, \cap \rangle$ is a Boolean Ring

$\langle \mathbb{D}_n, \subseteq, \cap, \cup \rangle$ is a Boolean Lattice

$\langle \mathbb{D}_n, \oplus, \otimes \rangle$ is an Integral Domain.

$\langle \mathbb{D}_n, +, -, \times \rangle$ is an Integral Domain.

# Sums & Products modulo 8

| $\cup$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | **0** | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | **1** | 1 | 3 | 3 | 5 | 5 | 7 | 7 |
| 2 | **2** | 3 | 2 | 3 | 6 | 7 | 6 | 7 |
| 3 | **3** | 3 | 3 | 3 | 7 | 7 | 7 | 7 |
| 4 | **4** | 5 | 6 | 7 | 4 | 5 | 6 | 7 |
| 5 | **5** | 5 | 7 | 7 | 5 | 5 | 7 | 7 |
| 6 | **6** | 7 | 6 | 7 | 6 | 7 | 6 | 7 |
| 7 | **7** | 7 | 7 | 7 | 7 | 7 | 7 | 7 |

| $\oplus$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | **0** | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | **1** | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 2 | **2** | 3 | 0 | 1 | 6 | 7 | 4 | 5 |
| 3 | **3** | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 4 | **4** | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 5 | **5** | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 6 | **6** | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 7 | **7** | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | **0** | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 1 | **1** | 2 | 3 | 4 | 5 | 6 | 7 | 0 |
| 2 | **2** | 3 | 4 | 5 | 6 | 7 | 0 | 1 |
| 3 | **3** | 4 | 5 | 6 | 7 | 0 | 1 | 2 |
| 4 | **4** | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 5 | **5** | 6 | 7 | 0 | 1 | 2 | 3 | 4 |
| 6 | **6** | 7 | 0 | 1 | 2 | 3 | 4 | 5 |
| 7 | **7** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

| $\cap$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 | 0 | 0 | 2 | 2 | 0 | 0 | 2 | 2 |
| 3 | 0 | 1 | 2 | 3 | 0 | 1 | 2 | 3 |
| 4 | 0 | 0 | 0 | 0 | 4 | 4 | 4 | 4 |
| 5 | 0 | 1 | 0 | 1 | 4 | 5 | 4 | 5 |
| 6 | 0 | 0 | 2 | 2 | 4 | 4 | 6 | 6 |
| 7 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

| $\otimes$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 |
| 3 | 0 | 3 | 6 | 5 | 4 | 7 | 2 | 1 |
| 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 |
| 5 | 0 | 5 | 2 | 7 | 4 | 1 | 6 | 3 |
| 6 | 0 | 6 | 4 | 2 | 0 | 6 | 4 | 2 |
| 7 | 0 | 7 | 6 | 1 | 4 | 3 | 2 | 5 |

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 0 | 2 | 4 | 6 | 0 | 2 | 4 | 6 |
| 3 | 0 | 3 | 6 | 1 | 4 | 7 | 2 | 5 |
| 4 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 4 |
| 5 | 0 | 5 | 2 | 7 | 4 | 1 | 6 | 3 |
| 6 | 0 | 6 | 4 | 2 | 0 | 6 | 4 | 2 |
| 7 | 0 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |

# Hybrid Formulas

Sums

$$\begin{aligned} a \cup b &= a \oplus b \oplus (a \cap b) \\ a + b &= (a \oplus b) + 2(a \cap b) \\ a \oplus b &= (a \cup b) - (a \cap b) \end{aligned}$$

Disjoint sum

$$a \cap b = 0 \quad \Leftrightarrow \quad a \uplus b = a \cup b = a \oplus b = a + b$$

Two's complement

$$\begin{aligned} 2^n - 1 &= \neg 0 \\ 1 + \neg 0 &= 0 \\ b + \neg b &= -1 \end{aligned}$$

$$\begin{aligned} -b &= 1 + \neg b \\ b + 1 &= -\neg b \\ b - 1 &= \neg - b \end{aligned}$$

Products

$$a \otimes b \leq a \times b$$

$$a \times b < 2a \otimes b$$

$$\forall a \in \mathbb{B} : a \otimes b = a \times b = b \cap -a$$

$$2^n a = 2^n \otimes a = 2^n \times a$$