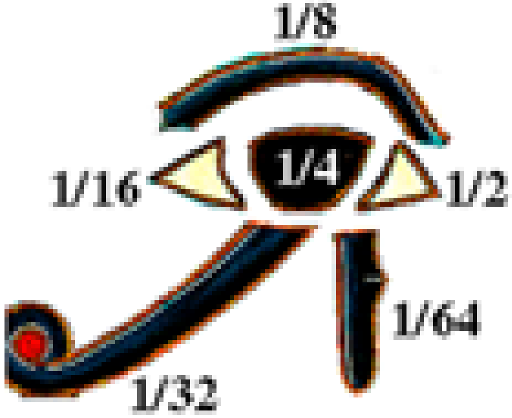
Binary Systems

- *Why Binary?*
  - Binary Physics
  - Binary Information
  - Binary Algebra
- *Automatic System*
  - No error
  - Digital
  - Synchronous
- *Binary Algebra*
  - Boolean Algebra
  - Binary z-Transform
  - Two's Complement Arithmetic



Digital Number $b \in \mathbb{D}$

Digital Signal: $b(t) = \sum_{n \in \mathbb{N}} b_n \partial(t-n)$

Sequence: $(b_{\mathbb{N}}) = b_0 b_1 b_2 \cdots$

Subset: $\{b\} = \{n \in \mathbb{N} : b_n = 1\}$

z-series: $b(z) = \int_0^\infty b(t) z^t dt = \sum_{n \in \mathbb{N}} b_n z^n$

2-adic integer: $b(2) = \int_0^\infty b(t) 2^t dt = \sum_{n \in \mathbb{N}} b_n 2^n$

Real: $b(\frac{1}{2}) = \int_0^\infty b(t) 2^{-t} dt = \sum_{n \in \mathbb{N}} \frac{b_n}{2^n}$

Examples

$$\begin{array}{lclcl} 0 & = & (0) & = & 0000\cdots \\ 1 & = & 1(0) & = & 1000\cdots \end{array}$$

$$u = (1) \quad \{u\} = \mathbb{N} \quad u(z) = \frac{1}{1-z} \qquad \begin{array}{l} u(2) = 1+2+4+8+\cdots = -1 \\ u(\frac{1}{2}) = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2 \end{array}$$

$$v = (10) \quad \{v\} = 2\mathbb{N} \quad v(z) = \frac{1}{1-z^2} \qquad \begin{array}{l} v(2) = 1+4+16+\cdots = -\frac{1}{3} \\ v(\frac{1}{2}) = 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3} \end{array}$$

$$\begin{array}{rcl} h & = & 0110100010000000010\cdots \\ \{h\} & = & \{2^{\mathbb{N}} : \mathrm{N} \in \mathbb{N}\} \end{array}$$

$$h(z) = \sum z^{2^n} = \begin{array}{l} z + h(z^2) \\ z + h^2(z) \end{array}$$

is 2-algebraic.

$$h(\tfrac{1}{2}) = 0.816421509\cdots$$

is transcendental (over $\mathbb{Q}$)

Infinite Binary Number $\quad d \in \mathbb{D}$

$$d = [\![d_{0\cdots} \quad \{d\} \quad d(z) \quad d(2)]\!]$$

1. Bit sequence: $d_{0\cdots} = d_0 \cdot d_{1\cdots}$
2. Integer set: $\{d\} = \{N : d_N = 1\}$
3. Polynomial: $d(z) = \sum d_N z^N$
4. Integer: $d(2) = \sum d_N 2^N$

Limit

$$d_{0\cdots N-1} = d \cap (2^N - 1)$$

$$\{d_{0\cdots N-1}\} \subseteq \{d_{0\cdots N}\}$$

$$\{d\} = \bigcup \{d_{0\cdots N}\}$$

Norm

$$\|0\| = 0, \; \|1+2x\| = 1, \; \|2x\| = \frac{1}{2}\|x\|$$

Distance

$$d(a,b) = \|a-b\| = \|a \oplus b\|$$

$$\|d - d_{0\cdots n-1}\| < 2^{-n}$$

Digital Algebra $\mathbb{D} \simeq \mathbb{N} \to \mathbb{B} \simeq 2^{\mathbb{N}} \simeq \mathbb{Z}_z \simeq \mathbb{Z}_2$

$\langle \mathbb{D}, \neg, \cap, \cup \rangle$ is a Boolean Algebra

$\langle \mathbb{D}, \oplus, \cap \rangle$ is a Boolean Ring

$\langle \mathbb{D}, \subset, \cap, \cup \rangle$ is a Set Lattice

$\langle \mathbb{D}, \oplus, \otimes \rangle$ is an Integral Domain.

$$\boxed{\frac{1}{1 \oplus zx} = \bigoplus z^N a^N}$$

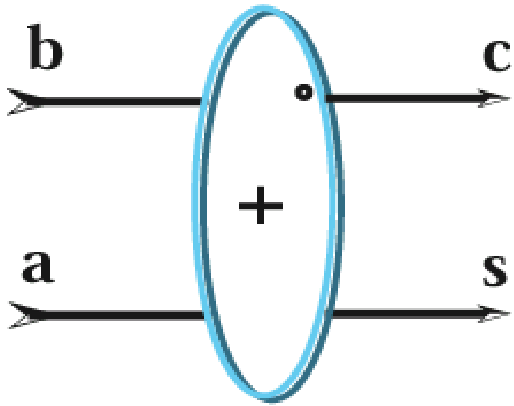
$\langle \mathbb{D}, +, -, \times \rangle$ is an Integral Domain.

$$\boxed{\frac{1}{1-2x} = \sum 2^N a^N}$$

$\langle \mathbb{D}, +_{\mathbb{R}}, \times_{\mathbb{R}} \rangle$ is an Integral Domain.

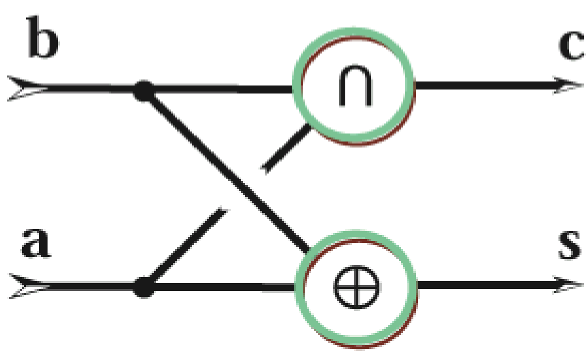
Half Adder

$$a_N + b_N = s_N + 2c_N$$



$$a = \sum a_N 2^N \quad b = \sum b_N 2^N$$
$$s = \sum s_N 2^N \quad c = \sum c_N 2^N$$

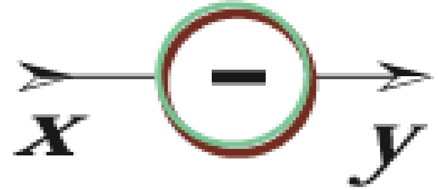
$$\boxed{a + b = s + 2c}$$



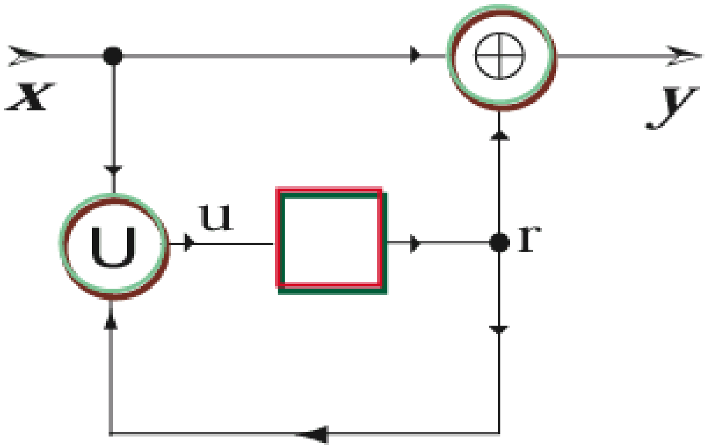
$$\begin{aligned} s_N &= a_N + b_N - 2a_N b_N \\ c_N &= a_N b_N \end{aligned}$$

$$\boxed{\begin{aligned} s &= a \oplus b \\ c &= a \cap b \end{aligned}}$$

Circuit y=-x



$$\boxed{y = -x}$$



$$r = z\ u$$

$$y = x \oplus r$$

$$u = x \cup r$$

$$r_N = u_{N-1}$$

$$u_{-1} = 0$$

$$y_N = x_N \oplus r_N$$

$$u_N = x_N \cup r_N$$

$$x = \sum x_N 2^N \qquad r = \sum r_N 2^N$$

$$y = \sum y_N 2^N \qquad u = \sum u_N 2^N$$

$$y = x + r - 2p$$

$$r = 2(x + r) - 2p$$

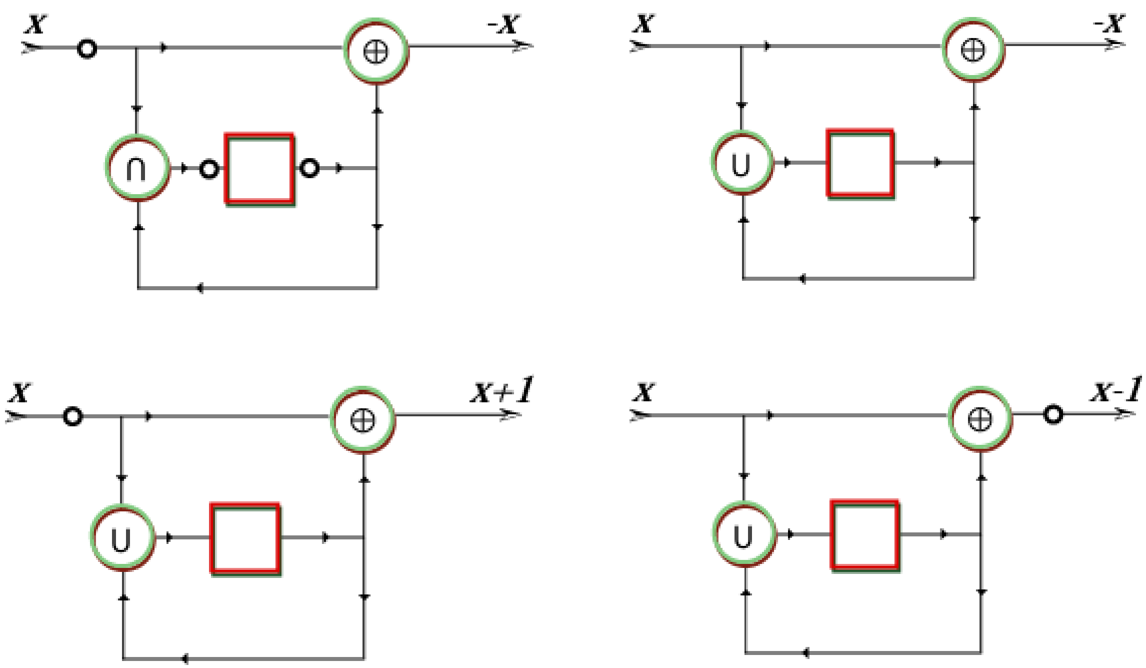
$$y - r = -x - r$$

$$y_N = x_N + r_N - 2x_N r_N$$

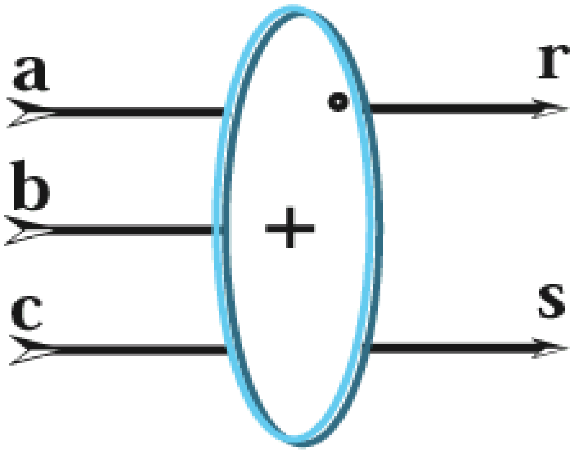
$$r_N = x_{N-1} + r_{N-1} - x_{N-1} r_{N-1}$$

Serial Operators

$$
\begin{aligned}
-x &= 1+\neg x \\
x+1 &= -\neg x \\
x-1 &= \neg - x
\end{aligned}
$$



Full Adder

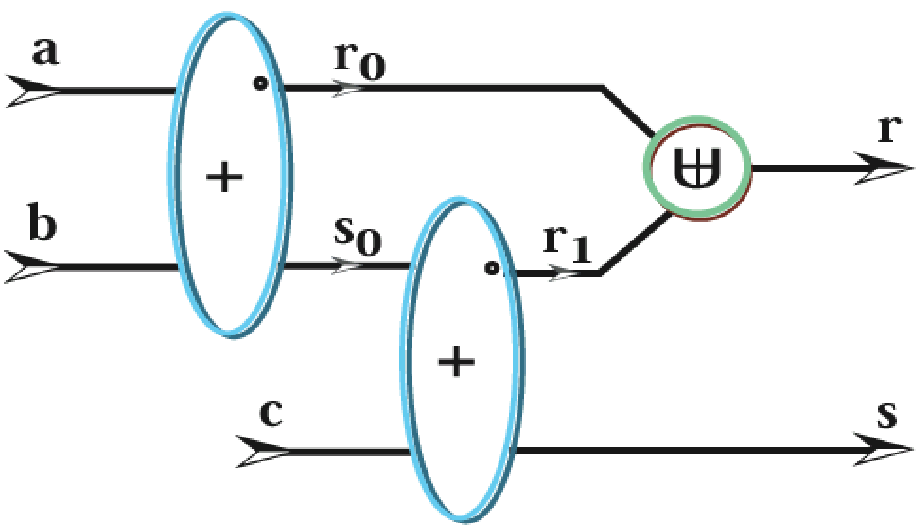


$$a_N + b_N + c_N = S_N = s_N + 2r_N$$

$$s_N = a_N \oplus b_N \oplus c_N = S_N \div 2$$

$$r_N = a_N b_N \oplus b_N c_N \oplus c_N a_N = S_N \dot{|} 2$$

$$\boxed{a + b + c = s + 2r}$$



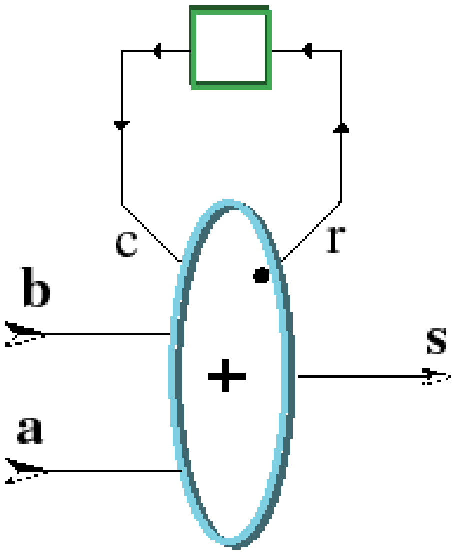
$$a + b = s0 + 2r0$$

$$c + s0 = s + 2r1$$

$$r0 + r1 = r$$

$$r0 \cap r1 = 0$$

Serial Adder



$\boxed{s = a+b}$

$$a+b+c = s+2r$$
$$c = 2r$$

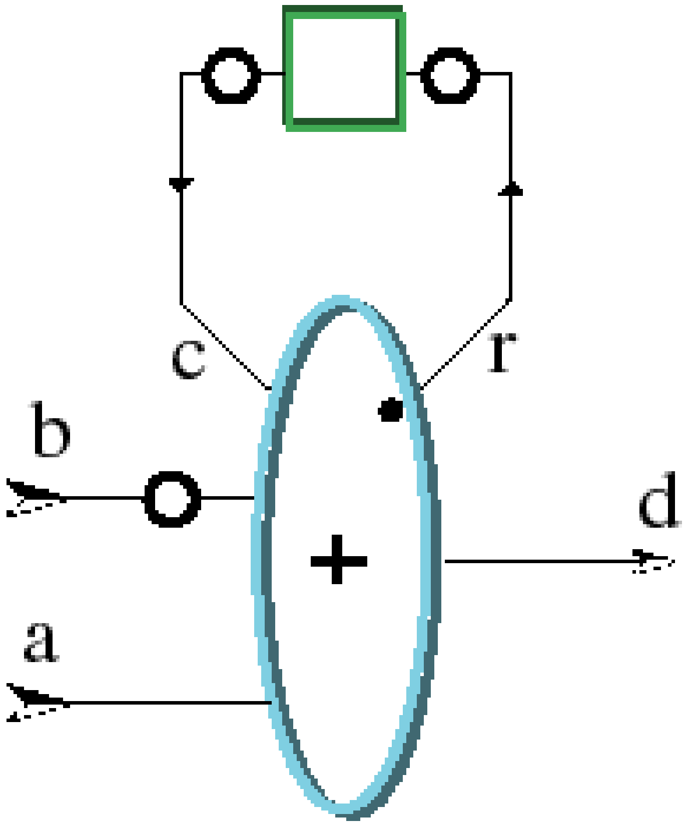
$$\begin{aligned} c_0 &= 0 \\ c_{n+1} &= r_n \\ \sum_{n \in N} c_n 2^n &= 2\sum_{n \in N} r_n 2^n \end{aligned}$$

$$\begin{aligned} a_n + b_n + c_n &= s_n + 2r_n \\ \sum_{n \in N} a_n 2^n + \sum_{n \in N} b_n 2^n + \sum_{n \in N} c_n 2^n &= \sum_{n \in N} s_n 2^n + 2\sum_{n \in N} r_n 2^n \\ a + b + c &= s + 2r \end{aligned}$$

Serial Subtract

$$\begin{aligned} -\boldsymbol{b} &= \mathbf{1}+\neg\boldsymbol{b} \\ \boldsymbol{b}+\neg\boldsymbol{b} &= -\mathbf{1} \\ \boldsymbol{b}-\mathbf{1} &= \neg-\boldsymbol{b} \end{aligned}$$

$$\boxed{d = a\text{-}b}$$



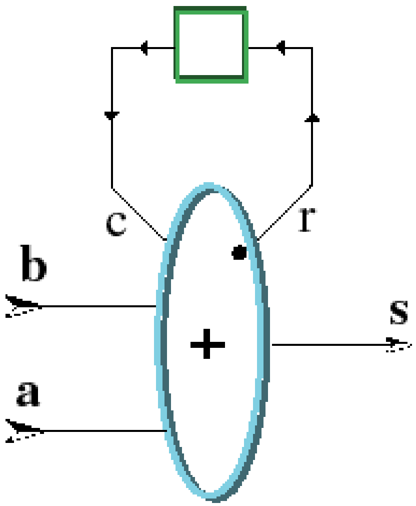
$$\neg b = -1-b$$

$$a+\neg b+c = d+2r$$

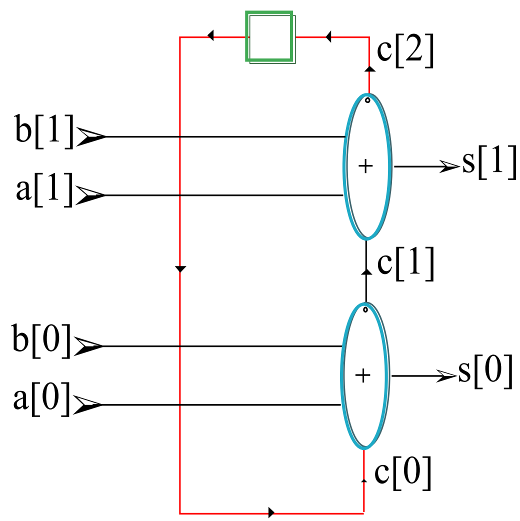
$$c = 1+2r$$

Time/Space Tradeoffs

$z=2$

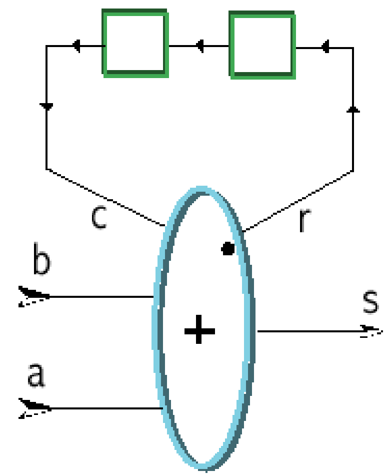


$z=4$



$z=\sqrt{2}$

$z^2=2$



$$\boxed{\begin{aligned} a+b+c &= s+2r \\ c &= 2r \\ s &= a+b \end{aligned}}$$

$$\begin{aligned} a_0+2a_1+b_0+2b_1+c_0 &= s_0+2s_1+4c_2 \\ c_0 &= 4c_2 \\ s &= a+b \end{aligned}$$

$$\begin{aligned} a &= u+v\sqrt{2} \\ b &= w+x\sqrt{2} \\ s &= (u+w)+(v+x)\sqrt{2} \end{aligned}$$

Binary Rings

$$\mathbb{F}_2 \subset \mathbb{D}_N \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{P} \subset \mathbb{Z}_{(2)} \subset \mathbb{A}_2 \subset \mathbb{D}_c \subset \mathbb{D}$$

$$d \in \mathbb{N} \quad d = d_0 \cdots d_{l-1}(0)$$

$$d \in \mathbb{Z} \quad d = d_0 \cdots d_{l-1}(d_l)$$

$$d \in \mathbb{P} \quad d = (d_0 \cdots d_{p-1})$$

$$d \in \mathbb{Z}_{(2)} \quad d = d_0 \cdots d_{i-1}(d_i \cdots d_{i+p-1})$$

$$\boxed{\mathbb{Z}_{(2)} = \mathbb{D} \cap \mathbb{Q}}$$

$\neg$	$\cup$	$\cap$	$\oplus$
$-$	$+$	$\times$	$\frac{1}{1-2x}$
z	z^{-}	$\otimes$	$\frac{1}{1 \oplus zx}$
$\uparrow$	$\odot$	$\downarrow$	$\downarrow z^{-}$

Binary Fields

$$\frac{1}{2} \notin \mathbb{D} \qquad \mathbb{D}_\bullet = \left\langle \frac{1}{2} \quad \mathbb{D} \right\rangle$$

$$d \in \mathbb{D}_\bullet \Leftrightarrow d = 2^v m : v \in \mathbb{Z},\ m \in \mathbb{D}$$

$\langle \mathbb{D}_\bullet\ +, -, \times, / \rangle$ is a Field, isomorphic to the 2-adic numbers $\mathbb{Q}_2$

$\langle \mathbb{D}_\bullet\ \oplus, \otimes, / \rangle$ is a Field, isomorphic to the binary Laurent series

Boolean operations: $\langle \mathbb{D}_\bullet\ \ \cup, \cap, \oplus \rangle$

Binary Rational

Let $n, d \in \mathbb{N}$ with $d = 1 + 2d'$ odd.

Write $q = \frac{n}{d} = \sum \mathrm{q}_{\mathrm{N}} 2^N$ in binary!

$$\boxed{\text{Algorithm: } n_0 = n,\ q_k = n_k \ (\mathrm{mod}\ 2),\ n_{k+1} = \frac{n_k - q_k d}{2}.}$$

$$\frac{2}{7} = 01(110)$$

$$\boxed{\frac{n}{d} = q_0 \cdots q_{i-1}(q_i \cdots q_{i+p-1})}$$

$$\boxed{\begin{gathered} e = \lceil q \rceil \in \mathbb{Z} \\ -1 \le q - e \le 0 \\ i = \ |e| \end{gathered}}$$

Initial part

$$n_k > 0 \Rightarrow n_{k+1} < n_k$$

$$n_k < -d \Rightarrow n_k < n_{k+1}$$

Period

$$-d \le n_k \le 0 \Rightarrow -d \le n_{k+1} \le 0$$

$$\boxed{\begin{gathered} p = \mathrm{order}(2, d) \\ 1 = 2^p \ (\mathrm{mod\ d}) \end{gathered}}$$

$$\boxed{i = 0 \Leftrightarrow -1 \le q \le 0}$$

$$\boxed{n_k = n2^{-k} \ (\mathrm{mod}\ d)}$$

Ultimately Periodic Binary Sequences

$$\frac{22}{7} = 10 + \frac{3\times 16}{-7}$$

$$\frac{22}{7} = 4 - \frac{6}{7}$$

$W(22/7) = {}_2 0\ W(11/7)$
$= {}_2 01\ W(2/7)$
$= {}_2 010\ W(1/7)$
$= {}_2 0101\ W(-3/7)$
$= {}_2 01011\ W(-5/7)$
$= {}_2 010111\ W(-6/7)$
$= {}_2 0101110\ W(-3/7)$

$22/7 = {}_2 0101(110)$

$$a = a_0 \cdots a_{i-1}(a_i \cdots a_{i+p-1}) = \sum a_N 2^N$$

$$a = b + \frac{c2^i}{1-2^p}$$

$$b = \sum_{k<i} a_k 2^k$$

$$c = \sum_{i\le k<i+p} a_k 2^k$$

$$a = \frac{n}{d}$$

$$\frac{n}{d} = \frac{b\left(1-2^p\right)+c2^i}{1-2^p}$$

$$p = ord_2\left(d\right)$$

$$a = e - f$$

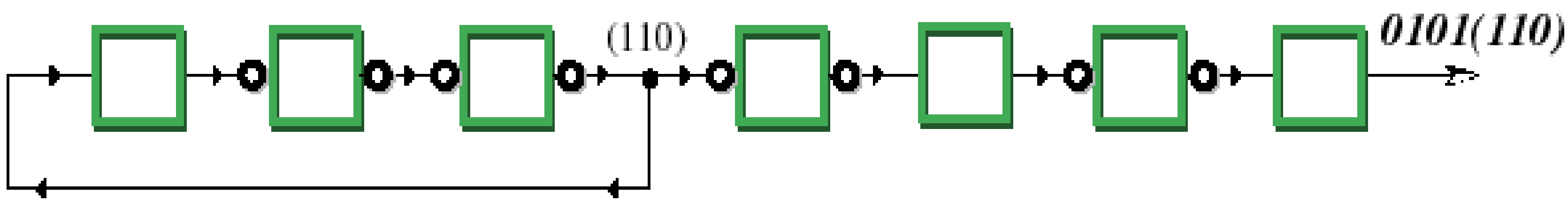
$$e = \lceil a \rceil$$

$$0 \le f < 1$$

Periodic Constant Synthesis

$$\frac{22}{7} = {}_2 0101(110)$$

Unary Encoding: 7 reg + 5 not



s[0]	s[1]	s[2]	s'[0]	s'[1]	s'[2]	o
0	0	0	1	0	0	1
1	0	0	0	1	0	0
0	1	0	1	1	0	1
1	1	0	0	0	1	0
0	0	1	1	0	1	1
1	0	1	0	1	1	1
0	1	1	0	0	1	0
1	1	1	*0*	*1*	*1*	*0*

Binary Encoding

OBDD: 3 reg + 9 mux