Stochastic Variance-Reduced Optimization for Machine Learning

Parts 2: Weakening the Assumptions

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Outline

Linear of Convergence of Gradient-Based Methods

- We've seen a variety of results of the form:

$$\text{Smoothness} \quad + \quad \text{Strong-Convexity} \quad \Rightarrow \quad \text{Linear Convergence}$$

  - Error on iteration t is $O(\rho^t)$, or we need $O(\log(1/\epsilon))$ iterations.

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- How much can we relax strong-convexity?

Smoothness $+$??? ~~Strong-Convexity~~ $\Rightarrow$ Linear Convergence

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- A special case of the Łojasiewicz' inequality [1963].
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- Using the PL inequality we can show

Smoothness $+$ **PL Inequality** ~~Strong-Convexity~~ $\Rightarrow$ Linear Convergence

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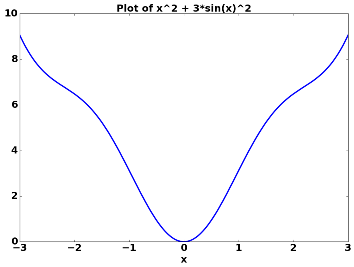
- However, it implies invexity.
  - For smooth f, invexity $\leftrightarrow$ all stationary points are global optimum.
- Example of invex but non-convex function satisfying PL:

$$f(x) = x^2 + 3\sin^2(x).$$



- Gradient descent converges linearly on this non-convex problem.

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- Proofs are often more complicated under these conditions.
- Are they more general?

Relationships Between Conditions

For a function f with a Lipschitz-continuous gradient, we have:

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- QG is the weakest condition but allows non-global local minima.
- PL $\equiv$ EB are most general conditions giving global min.

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- It has now been used to analyze:
  - Classic stochastic gradient methods [Karimi et al., 2016]:
    - $O(1/k)$ without strong-convexity using basic method.
  - Coordinate descent methods [Karimi et al, 2016].
  - Frank-Wolfe [Garber & Hazan, 2015].

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  - Frank-Wolfe [Garber & Hazan, 2015].
  - Variance-reduced stochastic gradient (like SAGA and SVRG) [Reddi et al., 2016].
    - Linear convergence without strong-convexity.

Relevant Problems for Proximal-PL

- Proximal-PL is a generalization for non-smooth composite problems.
  - Reddi et al. [2016] analyze proximal-SVRG and proximal-SAGA.

- Proximal-PL is satisfied when:
  - f is strongly-convex.
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- Another important problem class: principal component analysis (PCA)
  - Non-convex and doesn't satisfy PL, but we can find global optimum.
  - But it satisfies PL on Riemannian manifold [Zhang et al., 2016].
  - New faster method based on SVRG [Shamir, 2015, Garber & Hazan, 2016].

- But can we say anything about general non-convex functions?
- What if all we know is ∇f is Lipschitz and f is bounded below?

Non-Convex Rates for Gradient Descent

- For strongly-convex functions, GD satisfies

$$\|x_t - x_*\|^2 = O(\rho^t).$$

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- For non-convex and bounded below functions, GD still satisfies

$$\min_{k \leq t} \|\nabla f(x^k)\|^2 = O(1/t),$$

a convergence rate in terms of getting to a critical point [Nesterov, 2003].

Non-Convex Rates for Stochastic Gradient

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- For variance-reduced methods, Reddi et al. [2016] show we get faster rate,

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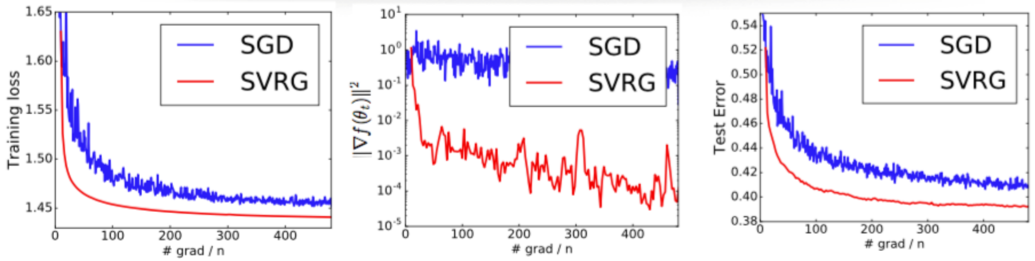
Non-Convex Rates for Stochastic Gradient



CIFAR10 dataset; 2-layer NN

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- We cannot show analogous results for classic proximal stochastic methods.
    - All existing proximal+stochastic results require noise to go to zero.
    - Open problem that needs to be resolved: are analogous results possible?

Outline

Non-IID Setting

- We discussed stochastic minimization problems

$$\underset{x}{\operatorname{argmin}}\ \mathbb{E}[f_i(x)],$$

where we have the ability to generate IID samples $f_i(x)$.

- Using IID samples is justified by the law of large numbers.

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- What if we can't get IID samples?

- Classic non-IID sampling scheme [Bertsekas & Tsitsiklis, 1996]:
    - Samples follow a Markov chain with stationary distribution of $\mathbb{E}[f_i(x)]$.
    - Obtain standard guarantees if Markov chain mixes fast enough [Duchi et al., 2012].

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- What about general non-IID sampling schemes?
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- Can we say anything in this case?
- Optimization error can be arbitrarily bad, but we can bound regret...

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the total error compared to the best x^* we could have chosen for first t functions.

- The x^* is not the solution to the problem, it's just the best we could have done.
- The x^* depends on t, the "solution" is changing over time.

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- Consider applying stochastic gradient, treating the f_t as the samples.
  - For convex functions, has a regret of $O(\sqrt{t})$ [Zinkevich, 2003].
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    - These are optimal.
- Key idea: x^* isn't moving faster than stochastic gradient is converging.

Online Convex Optimization

- AdaGrad is a very-popular online method [Duchi et al., 2011]:
  - Improves on constants in regret bounds using diagonal-scaleing

$$x^{t+1} = x^t - \alpha_t D_t^{-1} \nabla f_t(x^t),$$

with diagonal entries $(D_t)_{ii} = \delta + \sqrt{\sum_{k=1}^{t} \nabla_i f_k(x^k)}$.

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- Adam is a generalization that is incredibly-popular for deep learning.
  [Kingma & Ba, 2015]
  - Though trend is returning to variations on accelerated stochastic gradient.
- Online learning remains active area and many variations exist:
  - Bandit methods only receive evaluation $f_t(x^t)$ rather than function f_t.
  - Main application: internet advertising and recommender systems.

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Graph-Structured Optimization

- Another structure arising in machine learning is graph-structured problems,

$$\operatorname*{argmin}_x \sum_{(i,j)\in E} f_{ij}(x_i, x_j) + \sum_{i=1}^{n} f_i(x_i).$$

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- Includes quadratic functions,

$$x^T Ax + b^T x = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j + \sum_{i=1}^n b_i x_i,$$

  and other models like label propagation for semi-supervised learning.
  - The graph is sparsity pattern of A.

Coordinate Descent for Graph-Structured Optimization

- Coordinate descent seems well-suited to this problem structure:

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- To update x_i, we only need to consider f_i and the f_{ij} for each neighbour.
- With random selection of coordinates, expected iteration cost is $O(|E|/n)$.
    - This is n-times faster than GD iteration which cost $O(|E|)$.

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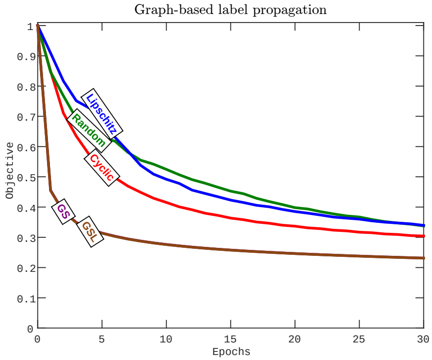
- But for many problems randomized coordinate descent doesn't work well...



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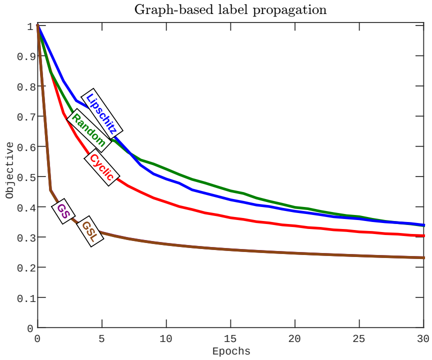

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- But is plotting "epochs" cheating because Gauss-Southwell is more expensive?

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- Gauss-Southwell greedy rule for picking a coordinate to update:

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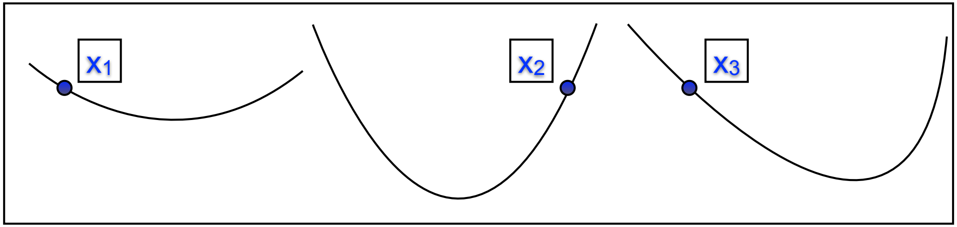

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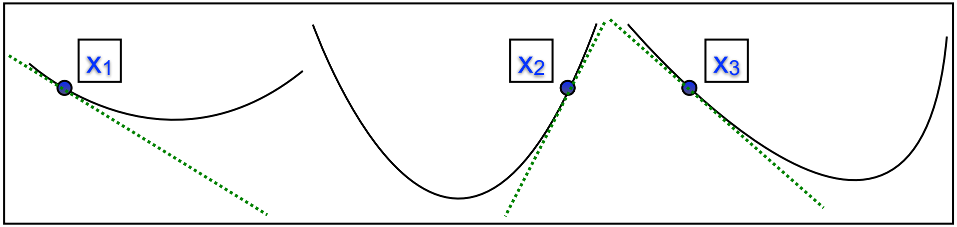

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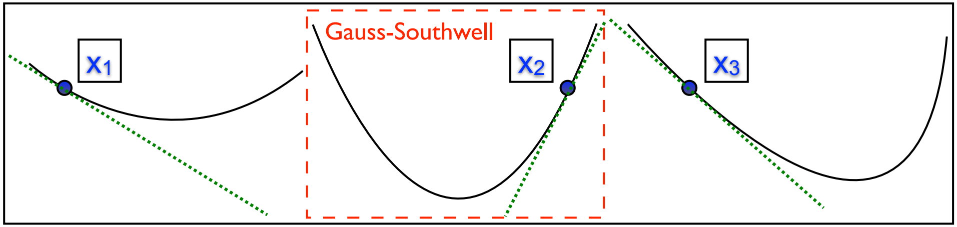

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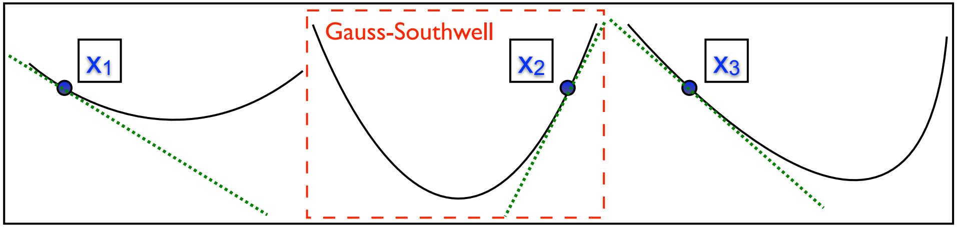

- Looks expensive because computing the gradient costs $O(|E|)$.

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  - Facebook graph (max is 7000, average is ≈ 200).

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    - Facebook graph (max is 7000, average is ≈ 200).
- Here we can efficiently track the gradient and it's max [Meshi et al., 2012].

Cost of Greedy Coordinate Descnet

- Gauss-Southwell cost depends on graph structure.
  - Same is true of Lipschitz sampling.
- Consider problems where maximum degree and average degree are similar:
  - Lattice graphs (max is 4, average is ≈ 4).
  - Complete graphs (max and average degrees are $n-1$).
  - Facebook graph (max is 7000, average is ≈ 200).
- Here we can efficiently track the gradient and it's max [Meshi et al., 2012].
  - Updating x_i, it only changes $|\nabla_j f(x^k)|$ for i and its neighbours.
  - We can use a max-heap to track the maximum.

Convergence Rate of Greedy Coordinate Descent

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- Constant μ_1 satisfies

$$\underbrace{\frac{\mu}{n}}_{\text{random}} \leq \underbrace{\mu_1}_{\text{greedy}} \leq \underbrace{\mu}_{\text{gradient}},$$

so we should expect more progress under Gauss-Southwell.

Gauss-Southwell-Lipschitz Rule

- Nutini et al. [2015] also give a rule with faster rate by incorporating the L_i,

$$i_k = \operatorname*{argmax}_i \frac{|\nabla_i f(x^k)|}{\sqrt{L_i}},$$

  which is called the Gauss-Southwell-Lipschitz rule.
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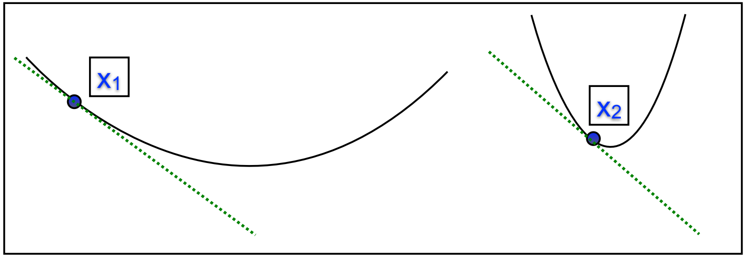

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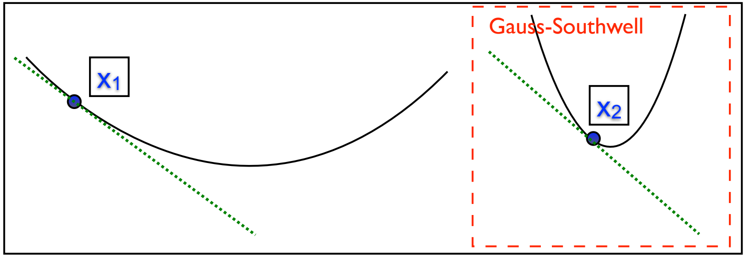

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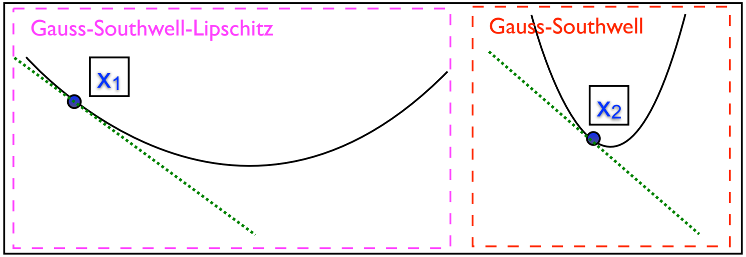

- Greedy rules have lead to new methods for computing leading eigenvectors.
  - Coordinate-wise power methods [Wei et al., 2016, Wang et al., 2017].

Outline

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  - Datasets no longer fit on a single machine.
- Result: we must use parallel and distributed computation.
- Two major new issues:
  - Synchronization: we can't wait for the slowest machine.
  - Communication: we can't transfer all information.

Embarrassing Parallelism in Machine Learning

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- These allow optimal linear speedups.
  - You should always consider this first!

Asynchronous Computation

- For stochastic gradient and SVRG, we can compute batch of gradients in parallel:
$$x^{k+1} = x^k - \alpha_k \frac{1}{|\mathcal{B}|}\sum_{i \in \mathcal{B}} \nabla f_i(x^k),$$
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- You need to decrease step-size in proportion to asynchrony.
- Convergence rate decays elegantly with delay m [Niu et al., 2011].
  - Now exists asynchronous variance-reduced methods.

[Reddi et al., 2015, Leblond et al., 2016, Mania et al., 2016]

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- Only needs to communicate single coordinates.
- Again need to decrease step-size for convergence.
- Speedup is based on dependencies between variables [Richtarik & Takac, 2013].

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- Gradient descent is special case where all neighbours communicate.
- Modified update has fast rate in terms of graph Laplacian [Shi et al., 2014].
- Can also consider communication failures [Agarwal & Duchi, 2011].
- An active area with several other recent distributed methods.

[Jaggi et al., 2014, Shamir et al., 2013, Lee et al. 2015]

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- Convergence rate of gradient norm, and variance-reduction appears.
- Stochastic algorithms have good regret for arbitrary sequences.
- Greedy coordinate descent seems like the right tool for some problems.
- Parallel/distributed methods are the future, but pose new challenges.