A Stochastic Geometry Model of CDMA Coverage

François Baccelli, Bartek Błaszczyszyn

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Structure of the Talk

Poisson processes, Voronoi Tessellations, Delaunay Graphs

Planar Poisson point processes Π of intensity measure μ

– Numbers of points of Π in disjoint sets are independent;

– $\Pi(S)$ number of points of Π in the subset S of the plane:

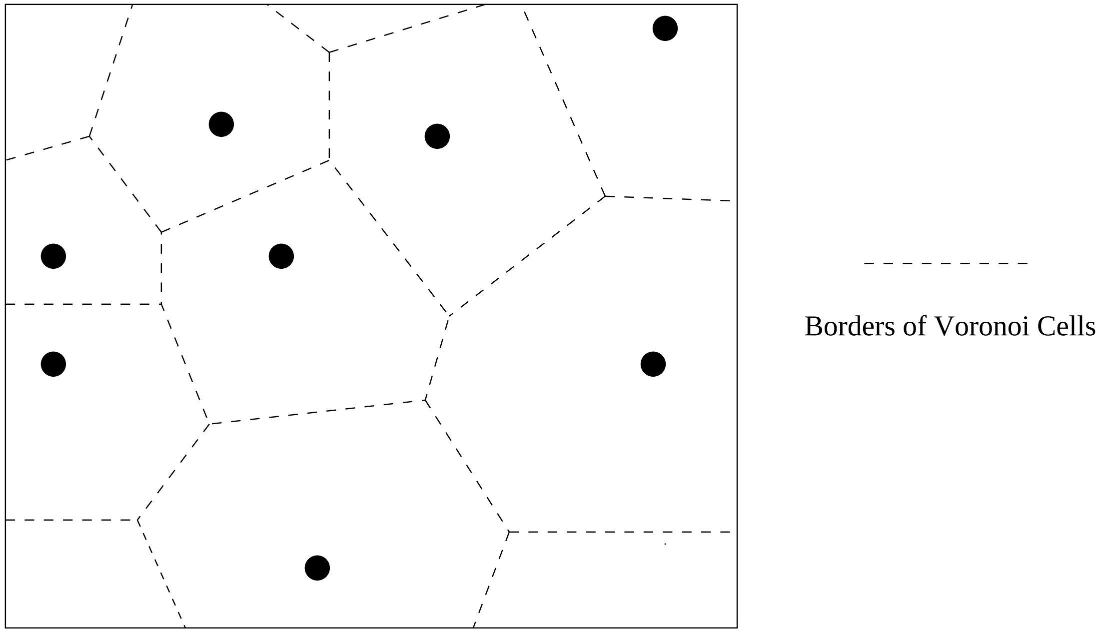
$$P(\Pi(S) = k) = e^{-\mu(S)} \frac{(\mu(S))^k}{k!}$$

with $\mu(.)$ measure on the plane with finite mass on finite sets.

– Homogeneous if $\mu(S) = \lambda|S|$, with $|S|$ Lebesgue measure of S.

Voronoi Tessellations, Delaunay Graphs

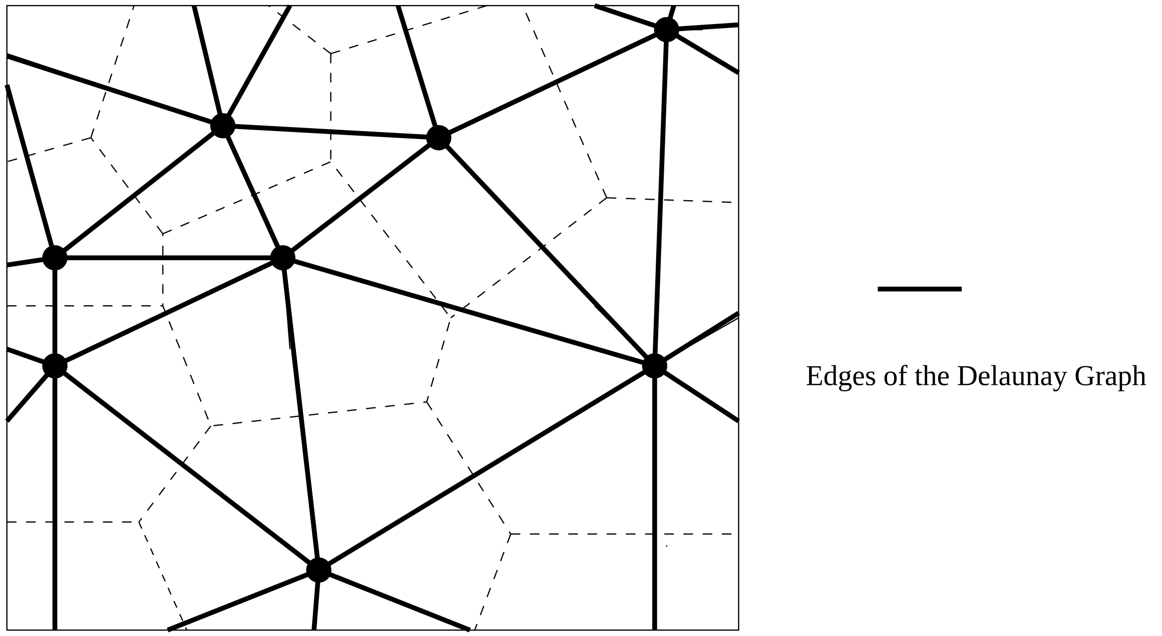
For X_0 a point of Π, $y \in V_{X_0}$ if $||y - X_0|| \leq ||y - X_i||, \quad \forall i \neq 0$



Spatial analogue of the intervals of renewal theory

Voronoi Tessellations, Delaunay Graphs (*continued*)

Edges connecting the nuclei of adjacent cells



Boolean Model – Germ Grain Model

A Boolean model Ξ_{BM} is the union

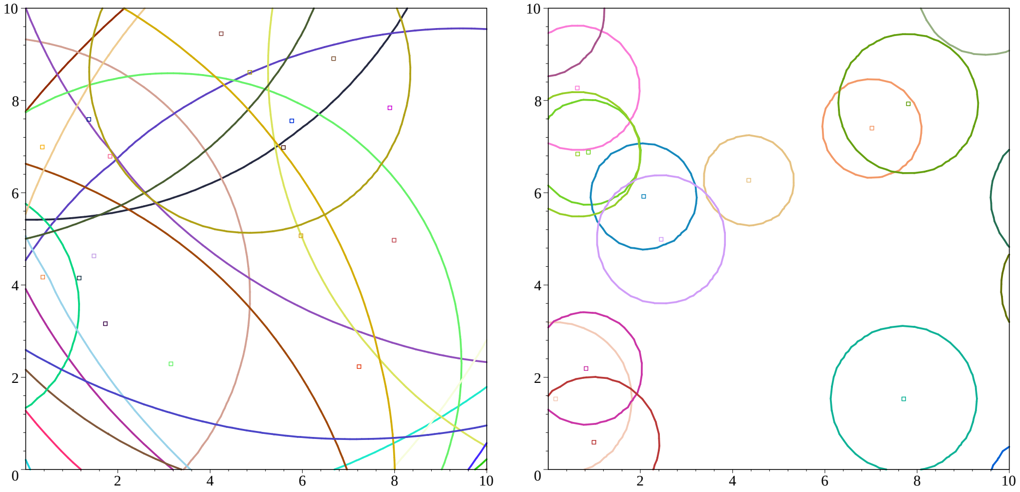
$$\Xi_{\mathrm{BM}} = \bigcup_i G_i \oplus X_i$$

- $\Psi = \{(X_i, G_i)\}$ is an independently marked Poisson p.p.,
- $\{X_i\}$ points of the Poisson p.p. on $I\!R^d$,
- $\{G_i\}$ sequence of i.i.d random compact sets
- and $G \oplus x = \{y + x : y \in G\}$.

Boolean Model – Germ Grain Model (*continued*)

Example

- $\{S_i\}$ sequence of integrable i.i.d. non-negative random variables, independent of the PPP;
- $b(u, x)$: closed ball centered in u and with radius $|x|$: $G_i = b(0, S_i)$.



Results on the Boolean Model

- The expected number of grains (sets $G_i \oplus X_i$) hitting a given bounded set B is

$$\mathbb{E}[\#\{(X_i, G_i) \in \Psi : B \cap G_i \oplus X_i \neq \emptyset\}] = \mathbb{E}[\mu(B \oplus \check{G})]\,,$$

  G is a generic (typical) random variable of the sequence $\{G_i\}$
  $B \oplus \check{G} = \{x + y : x \in B, -y \in G\}$.

- Spatial analogue of the infinite server queue.

Results on the Boolean Model (*continued*)

- Example: homogeneous case with balls:
  - Ξ is a random closed set if the radius of balls have moments of order d (2 in the plane);
  - The distribution of the number of balls covering a given location is Poisson.

CDMA Coverage Model

SINGLE CELL

- $\Phi = \{X_i, (S_i, T_i)\}$ marked point process
- $\{X_i\}$ points of the p.p. on $I\!R^d$,
- (S_i, T_i) mark of point X_i: $(S_i, T_i) \in I\!R^+ \times (0,1)$

Cell attached to point X_0: SIR w.r.t. antenna 0 at location y

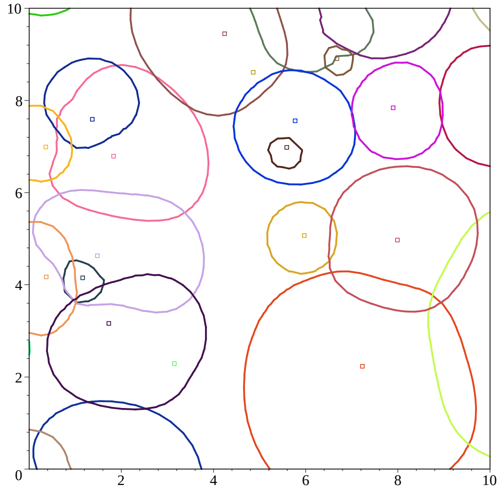
$$C_0(\Phi, W) = C(X_0, S_0, T_0, \Phi \setminus (X_0, S_0, T_0), W) = \{y : \frac{S_0 l(y - X_0)}{W + I_\phi(y)} \geq T_0\}$$

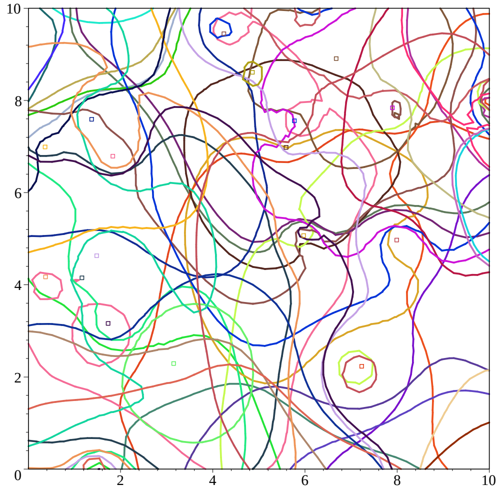
- $I_\phi(y) = \Sigma_{i \neq 0} S_i l(y - X_i)$
- $W \geq 0$ *external noise*,
- $l(\cdot)$ *attenuation function*

THE COVERAGE PROCESS

Given: $\Phi = \{X_i, (S_i, T_i))\}$ — marked Poisson point process on $I\!R^d$,

$$\Xi = \Xi(\Phi; W) = \bigcup_{i \in I\!N} C_i(\Phi, W).$$





QUESTIONS

- All addressed under non homogeneous Poisson (intensity μ), independent marking assumptions.

1. Is the model well-defined? $\rightarrow$ Part 1 Under some moment conditions for W and S, each particular cell $C_i = C_i(\Phi, , W)$ as well as the union Ξ are random closed sets.
2. What can we calculate? $\rightarrow$ Part 2
   Probability for a typical cell to cover one or more locations (1−outage probability)
   Distribution of the number of cells covering given location (hand-off number).
3. Relation to known models $\rightarrow$ Part 3 For some limit values of parameters, Ξ converges to a Boolean model or the Poisson-Voronoi tessellation.

PART 1: RANDOM CLOSED SET CONDITIONS FOR A SINGLE CELL

THEOREM

Suppose $l(\cdot)$ is continuous and for each $y \in \mathbb{R}^d$ there exists a ball $b(y, \delta_y)$ such that

$$\int_{\mathbb{R}^d} \sup_{z \in b(y,\delta_y)} l(z - x)\, \mu(dx) < \infty\,.$$

If $\mathbb{E}[S] < \infty$ then with probability one the function

$$I_\Phi(y) = \sum_i S_i l(y - X_i)$$

is everywhere continuous with respect to y.

PART 1: RANDOM CLOSED SET CONDITIONS FOR A SINGLE CELL (*continued*)

COROLLARY

Each particular cell

$$C_i = \{y : \frac{S_i l(y - X_i)}{W + I_\Phi(y)} \geq T_i\}$$

is a random closed set.

RANDOM CLOSED SET CONDITIONS FOR Ξ

Assumptions: homogeneous PPP; conditions of the previous theorem

THEOREM For all bouded sets B, $\mathbb{E}[\#\{C_i : B \cap C_i \neq \emptyset\}] < \infty$ and Ξ is a random closed set if one of these conditions is satisfied:

(a) finite range of attenuation function: $l(z) = 0 \quad \text{for } |z| > R^*$.

(b) presence of noise W:

$$l(z) \le a(1 + |z|)^{-\alpha} \quad \text{for some } a, \alpha > 0$$

and

$$\mathbb{E}[(S/T)^{d/\alpha}] < \infty, \quad \mathbb{E}[W^{-d/\alpha}] < \infty$$

(c) no noise W: $l(z)$ as above, $\mathbb{E}[S^{-d/\alpha}] < \infty$ and for ach $R > 0$,

$$\int_{\mathbb{R}^d} e^{-\lambda b_d |x|^d} (\underline{l}(|x| + R))^{-d/\alpha}\, dx < \infty\,, \quad \text{where} \quad \underline{l}(r) = \inf_{|z| \le r} l(z).$$

IDEA OF PROOF

$$C_0 = \{y : \frac{s_0 l(y - x_0)}{w + I_\phi(y)} \geq t_0\}$$

$$\subset \{y : l(y - x_0) \geq \frac{w}{s_0/t_0}\}$$

$$\subset \{y : |y - x_0| \leq (\frac{a s_0/t_0}{w})^{1/\alpha}\} = b(x_0, (\frac{a s_0/t_0}{w})^{1/\alpha}) \; .$$

$$C_i \subset b(X_i, \rho_i) \, , \quad \text{with} \quad \rho_i = (\frac{a S_i/T_i}{W})^{1/\alpha} \, .$$

PART 2: PROBABILITY FOR A TYPICAL CELL TO COVER A POINT

- Φ marked Poisson point process representing antennas in $\mathbb{R}^d$,
- $(x,(S,T))$ additional antenna located at fixed location x with random mark (S,T) distributed as any mark of Φ, independent of it,
- y location (of a customer) in $\mathbb{R}^d$.

Probability of covering y by the cell attached to the given additional antenna:

$$
\begin{aligned}
p_x(y) &= \mathbb{P}(\, y \in C(x,S,T;\Phi,W)\,) \\
&= \mathbb{P}(\,\frac{S}{T} l(y-x) - W - I_\Phi(y) > 0\,)\,.
\end{aligned}
$$

PART 2: PROBABILITY FOR A TYPICAL CELL TO COVER A POINT (*continued*)

THEOREM

The covering probability $p_x(y)$ can be obtained via an integral formula from the Laplace transforms of (S, T), W and $I_\Phi(y)$; the Laplace transform of $I_\Phi(y)$ is

$$\mathbb{E}[\exp(-\xi I_\Phi(y))] = \exp[-\int_{\mathbb{R}^d}(1 - L_S(\xi l(y-z)))\, \mu(dz)]\,,$$

where $L_S(\xi) = \mathbb{E}[e^{-\xi S}]$ is the Laplace transform of S and μ is the intensity of Poisson point process.

PROBABILITY FOR A TYPICAL CELL TO COVER TWO POINTS

- (y_1, y_2) two locations to be covered by a given cell $C(x, S, T; \Phi, W)$
- From the joint Laplace transform of $(I_\Phi(y_1), I_\Phi(y_2))$, which is given by

$$
\begin{aligned}
&\mathbb{E}[\exp(-\xi_1 I_\Phi(y_1) - \xi_2 I_\Phi(y_2))] = \\
&\exp[-\int_{\mathbb{R}^d}(1 - L_S(\xi_1 l(y_1 - z) + \xi_2 l(y_2 - z)))\mu(dz)]\,.
\end{aligned}
$$

OVERLAPPING CELLS

- Given: n cells $C(x_i, s_i, t_i; \phi, w)$, $i = 1, \ldots, n$

THEOREM The inequality

$$\sum_{i=1}^{n} t_i < 1$$

is a necessary condition for $\cap_{i=1}^{n} C(x_i, s_i, t_i; \phi, w) \neq \emptyset$.

PROOF The set of inequalities

$$\frac{s_i l(y - x_i)}{w + \Sigma_{i=1}^{n} s_i l(y - x_i)} \geq t_i \quad (i = 1, \ldots, n)$$

implies

$$1 - \frac{w}{w + \Sigma_{i=1}^{n} s_i l(y - x_i)} \geq \sum_{i=1}^{n} t_i$$

and for $w > 0$ the LHS is strictly less than 1.

OVERLAPPING CELLS (*continued*)

COROLLARY If the distribution of T is such that

$$T \geq \tau \qquad \text{a.s, for some } \tau > 0,$$

then the number K_y of cells of Ξ covering any location y is a.s. bounded

$$K_y < 1/\tau.$$

- Spatial analogue of s server queue: no location can be covered by $s = 1/\tau$ or more cells, no matter how close they are and how strong their signals are.

MOMENT EXPANSION OF THE NUMBER OF CELLS K_y COVERING y

THEOREM

The factorial moment of K_y : $\mathbb{E}[K_y^{(n)}] = \mathbb{E}[K_y(K_y-1)\dots(K_y-n+1)_+]$ is given by:

$$\int_{(\mathbb{R}^d)^n} \mathbb{P}\Big(y \in \bigcap_{k=1}^{n} C\big(x_k, S_k, T_k; \Phi + \sum_{\substack{i=1\\ i\neq k}}^{n} \varepsilon_{(x_i,(S_i,T_i))}, W\big)\Big)\mu(dx_1)\dots\mu(dx_n)\ .$$

MOMENT EXPANSION OF THE NUMBER OF CELLS K_y COVERING y *(continued)*

In particular, for a homogeneous Poisson point process with intensity λ

$$\mathbb{E}[K_0] = \lambda \mathbb{E}[\nu_d(C(0,S,T;\Phi,W))]\,,$$

where $\nu_d(C(\ldots))$ is the d-dimensional volume of the typical cell.
The *volume fraction* p (fraction of the space covered by Ξ) is given by

$$p = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \mathbb{E}[(K_0)^{(k)}]\,.$$

If $T \geq \tau$ a.s. then the summation is over $0 \leq k < 1/\tau$.

Proof

$$K_x^{(n)} = \int_{(\mathbb{R}^d \times \mathbb{R}^+ \times (0,1))^n} \prod_{k=1}^{n} \mathbf{1}\left(x \in C(x_k, s_k, t_k; \Phi - \varepsilon_{x_k,(s_k,t_k)}), W\right)$$
$$\times \Phi^{(n)}\left(d\left((x_1, \ldots, x_n), ((s_1, \ldots, s_n), (t_1, \ldots, t_n))\right)\right) \ ,$$

with $\Phi^{(n)}$ the n-th factorial power of Φ.

- Apply the refined Campbell theorem to the expectation of this integral and the fact that the reduced Palm distribution of the Poisson p.p. is equal to the original distribution.

CONVERGENCE TO A BOOLEAN MODEL

- Ξ^γ the coverage process generated by the Poisson point process
$$\Phi^\gamma = \{(X_i, (S_i, T_i/\gamma))\}$$
in the presence of the noise γW ($W > 0$ a.s.)

- Basic observation the sets
$$C_i(\gamma) \; = \; \{y : S_i \, l(y - X_i) \geq W T_i + I_\Phi(y) \, T_i/\gamma\}$$
are increasing when $\gamma \to \infty$.

CONVERGENCE TO A BOOLEAN MODEL (*continued*)

THEOREM

$$\Xi^\gamma \quad \xrightarrow{\gamma\to\infty} \quad \tilde{\Xi} \quad \text{a.s}$$

where

$$\tilde{\Xi} = \tilde{\Xi}(\Phi; W) = \bigcup_{(X_i,(S_i,T_i))\in\Phi} \tilde{C}(X_i, S_i, T_i; W)$$

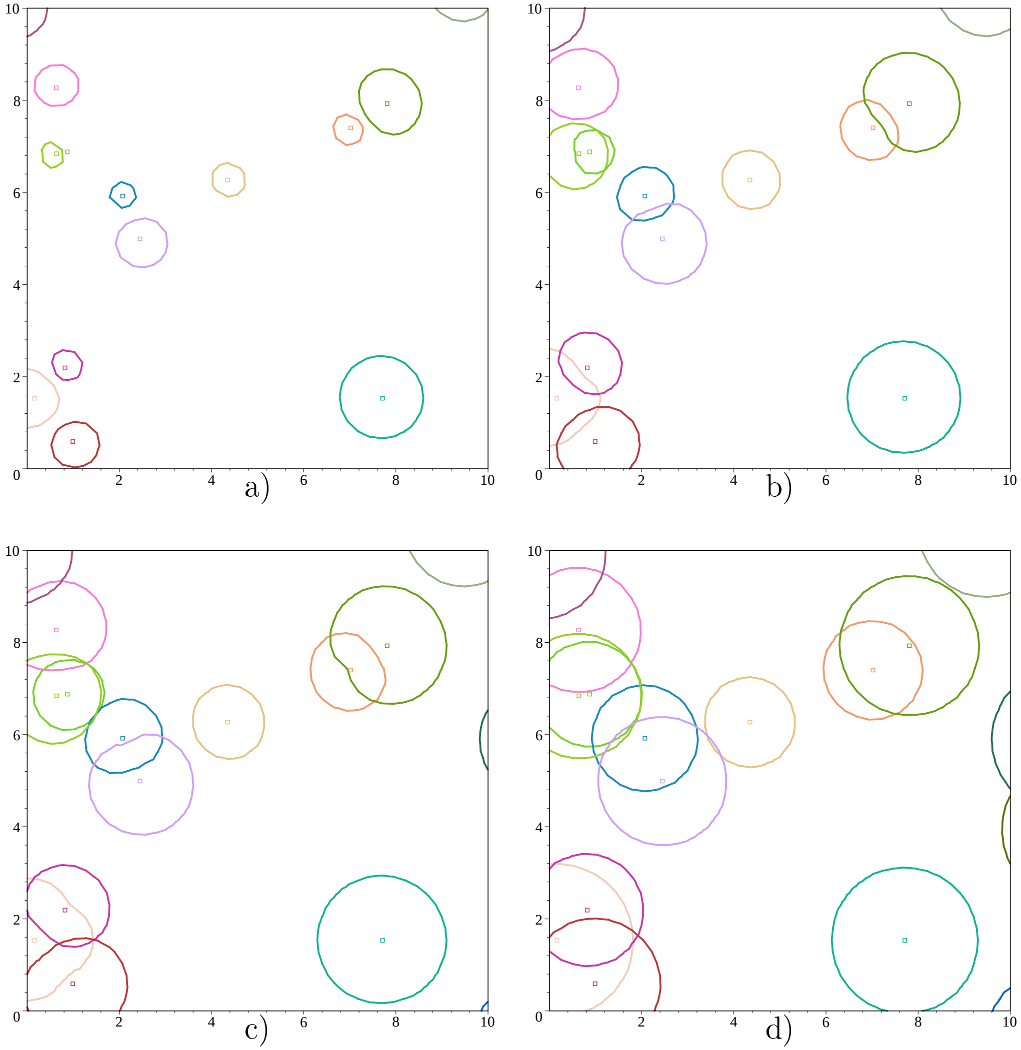
is a conditional Boolean model with cells

$$\tilde{C}(X_i, S_i, T_i; w) = \{y \in I\!R^d : S_i\, l(y - X_i) \geq wT_i\}$$

independent given $W = w$.

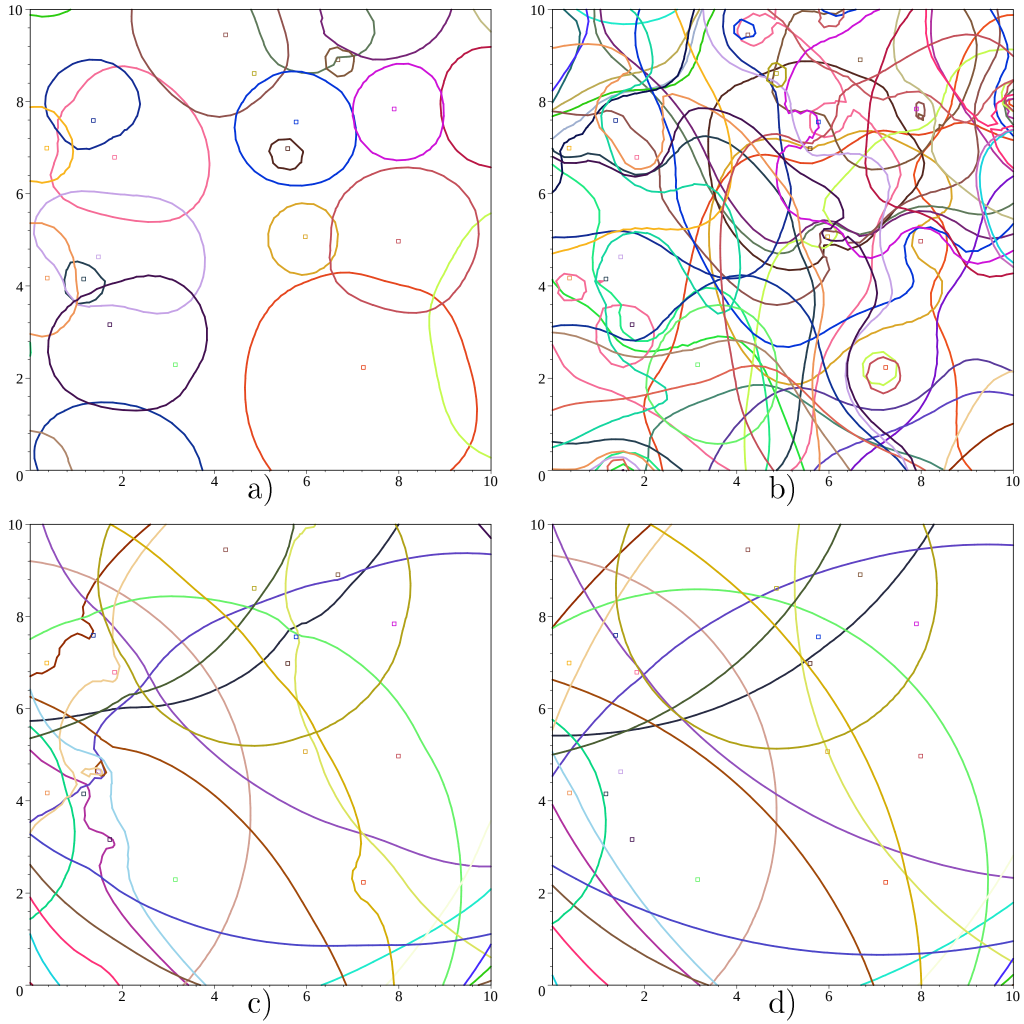
EXAMPLE 1

- Simulation of the coverage process $\Xi(\Phi; W)$ "on its way" to a Boolean model
- There are 60 points on the square $[-5, 15]^2$ (making $\mu = 0.15$) with S uniformly distributed on $[0, 2]$
- The observation window is $[0, 10]^2$,
- $l(y) = (1 + |y|)^{-3}$,
- a) $T = 0.4$, $W = 0.25$, b) $T = 0.2$, $W = 0.5$, c) $T = 0.1$, $W = 1$; d) $T = 0.0001$, $W = 1000$
- In the limiting case, each cell is a disk with independent radius distributed as $(S/(WT))^{1/3} - 1 = (10S)^{1/3} - 1$ with the mean $\approx$ 1.035.


a)
b)
c)
d)

EXAMPLE 2

- a) $T = 0.2$, $W = (0.1)^3$, b) $T = 0.2 \cdot 10^{-2}$, $W = (0.1)^2$, c) $T = 0.4 \cdot 10^{-4}$, $W = 5$; d) $T = 0.2 \cdot 10^{-5}$, $W = 100$
- In the limiting case, each cell is a disk with independent radius distributed as $(S/(WT))^{1/3} - 1 = (5000S)^{1/3} - 1$ with mean $\approx$ 16.15.



10
8
6
4
2
0
2
4
6
8
10
a)
b)
c)
d)

CONVERGENCE TO THE POISSON VORONOI TESSELLATION

- $\Xi^n = \cup_i C_i^n$ coverage process generated by the Poisson point process with attenuation function
$$l_n(z) = (1 + |z|)^{-n}$$
and with $W = 0$ a.s.

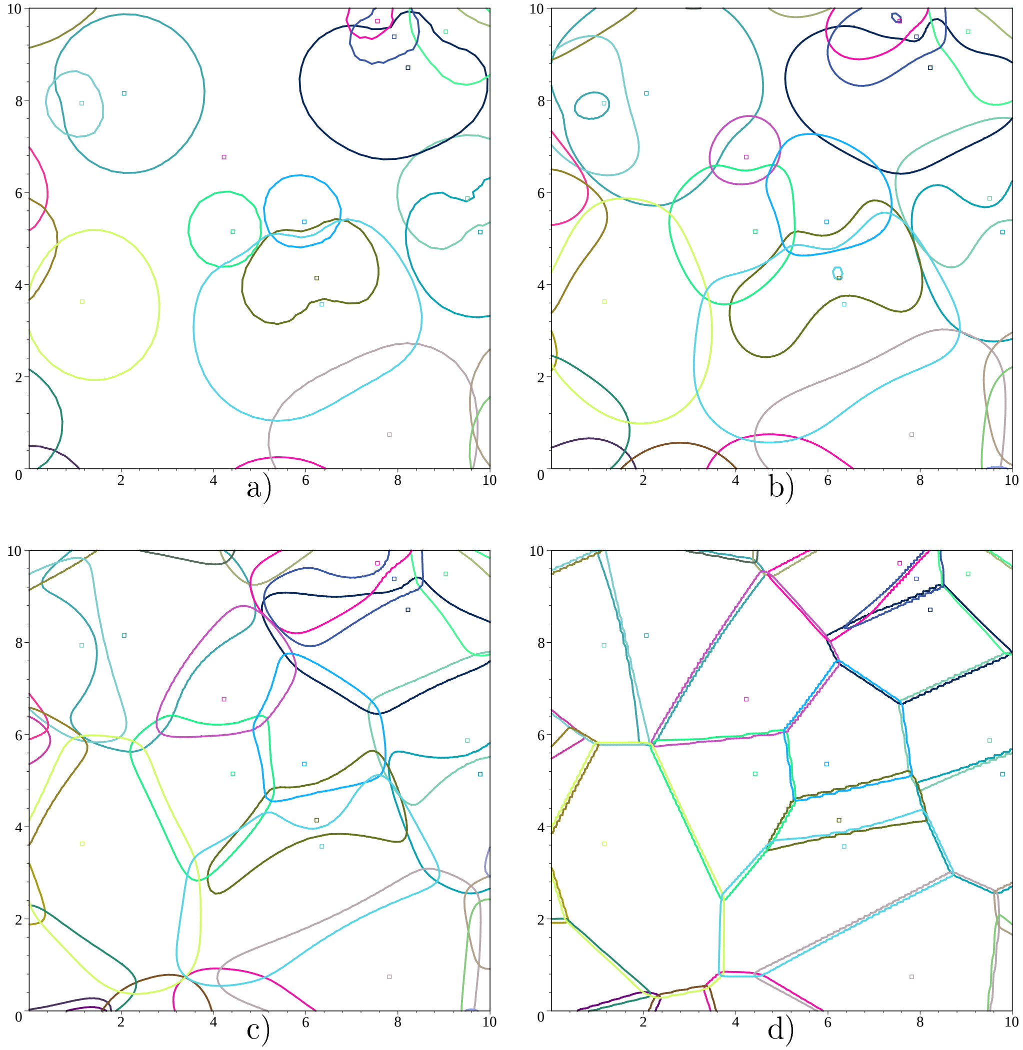
THEOREM

$$C_i^n \quad \overset{n\to\infty}{\longrightarrow} \quad V_i \quad \text{a.s}$$

where
$$V_i = \{y \in I\!R^d : |y - X_i| \leq \inf_{X_k \in \Phi - \varepsilon_{X_i}} |y - X_k|\}$$
is the Voronoi cell generated by the point X_i of Φ.
Convergence holds on the space of closed sets.
We also have convergence of the volume of the typical cell.

EXAMPLE 3

- Simulation of the coverage process Ξ tending to Voronoi tessellation of the plane
- Same window as above
- $W = 0$ and $T = 0.2$ (allowing for intersections).
- $l(y) = (1 + |y|)^{-\alpha}$ with: a) $\alpha = 3$, b) $\alpha = 5$, c) $\alpha = 12$, d) $\alpha = 100$.
- The effect of overlapping is still visible.


a)
b)
c)
d)

RELATION TO THE JOHNSON-MEHL MODEL

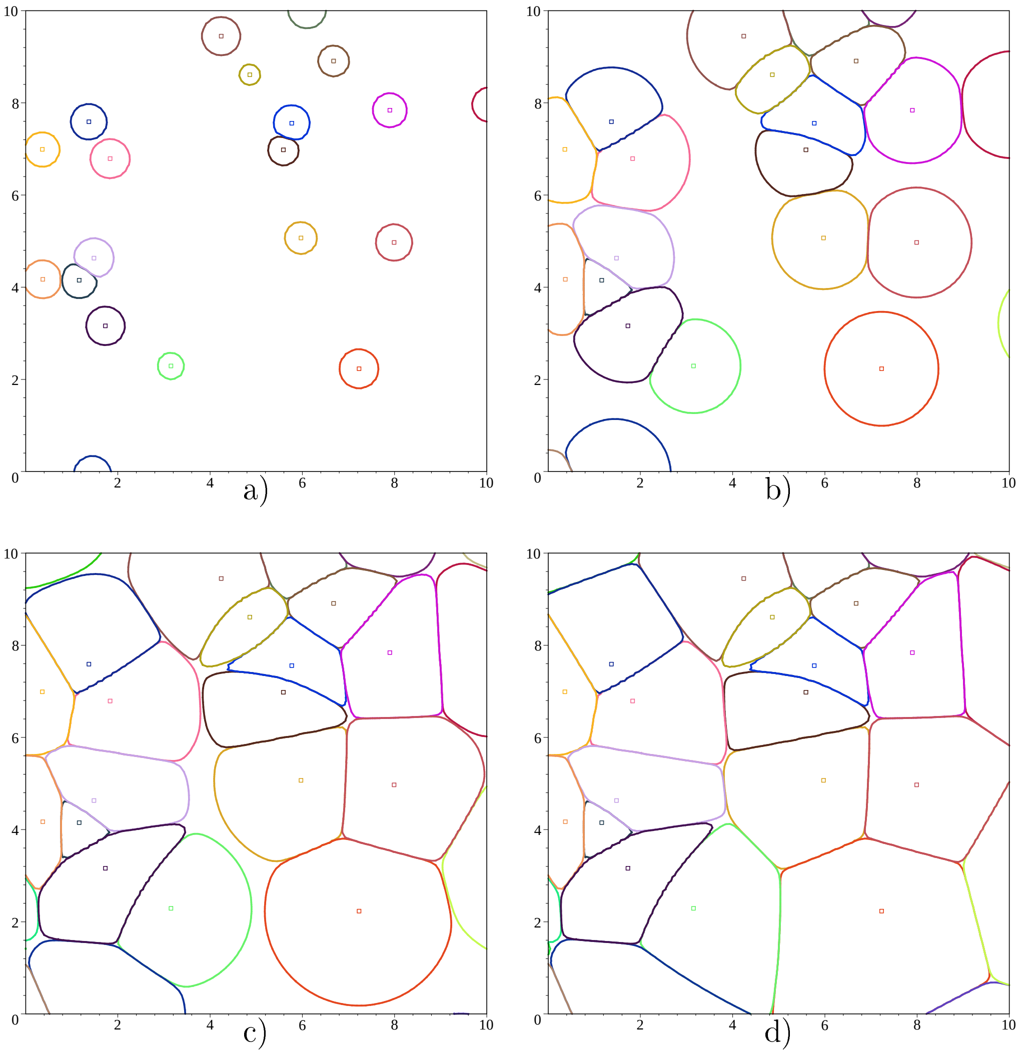
COROLLARY If $W = (R+1)^n$ then

$$C_i^n \xrightarrow{n\to\infty} V_i \cap b(X_i, R)$$

where $b(x, r)$ is the ball centered at x with radius r.

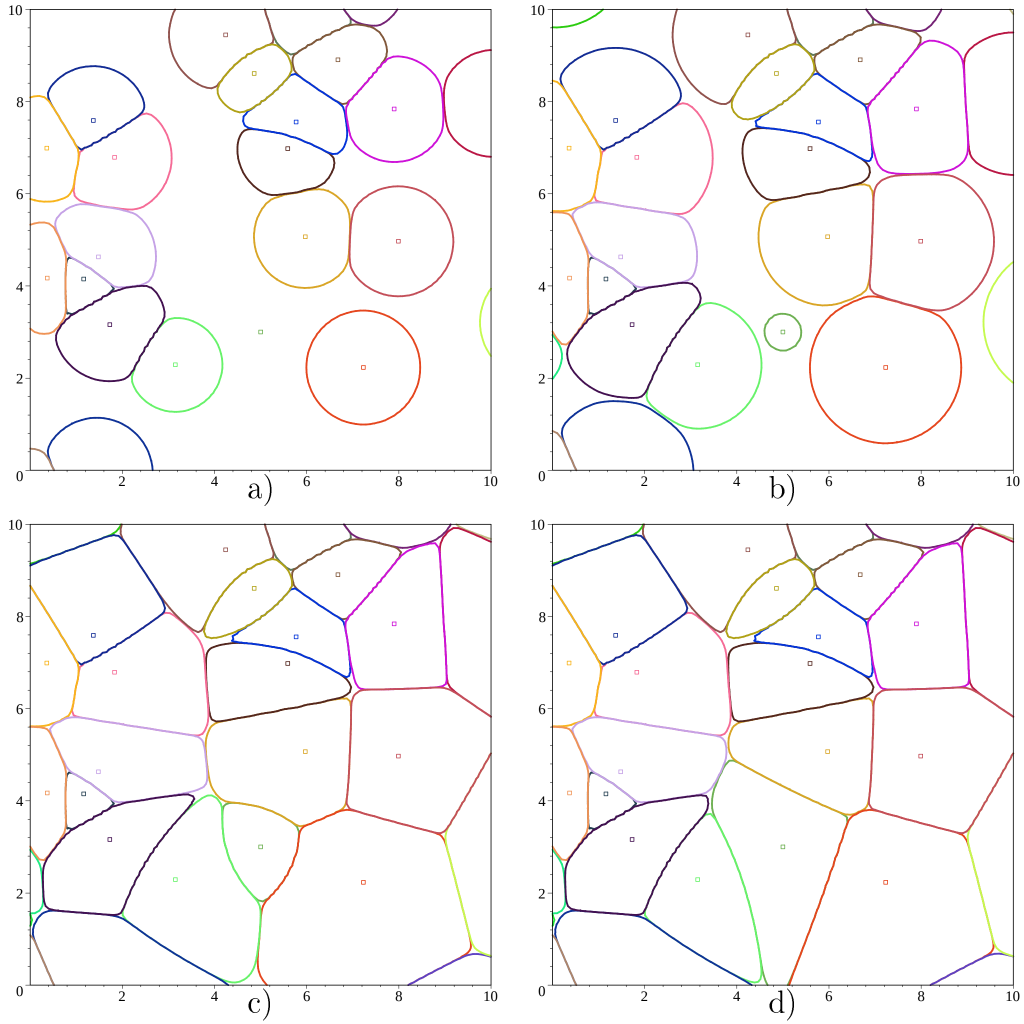
EXAMPLE 4

- Simulation of the coverage process Ξ growing to the tessellation of the plane as in the Johnson-Mehl model;
- Window as above;
- We take $T = 0.5$, thus inhibiting any intersections;
- $l(y) = (1+|y|)^{-30}$, strong enough to give the tessellation covering almost the whole plane when there is no external noise W;
- $W = (1+R)^{-30}$
- a) $R = 0.4$, b) $R = 1.2$, c) $R = 2$, d) $R = \infty$ $(W = 0)$.
- All cells start growing at the same time.


a)
b)
c)
d)

EXAMPLE 5

- Pattern with a "late comer": an extra point is added at $X = (5, 3)$, with $S = ((1 + R - 1.2)/(1 + R))^{30}$;
- a) $R = 1.2$, b) $R = 1.6$, c) $R = 3$, d) $R = \infty$ ($S = 1$, $W = 0$).
- The new cell is always "younger" (its maximal diameter is less) than that of others; however, unlike in the original Johnson-Mehl model, the younger cell can conquer the space already taken by an older cell. Thus on d) we have almost Voronoi tessellation, where no effect of coming late can be detected.


a)
b)
c)
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CONCLUSIONS – SG

- New basic parametric SG model
- Contains well known models of SG as particular cases
- Corresponds to a sharing of space which is a spatial analogue of multi-server queues
- Large set of open questions e.g. infinite components, local interactions, etc.

CONCLUSIONS – CDMA

- Tool for predicting network evolution: e.g. what about coverage when doubling the number of antennas?
- Only a first step
  - Coverage varies with time: dynamical aspects
  - Attenuation should be antenna-centered and random (i.e. marks of the antennas)
  - The neighboring (S_i, T_i) may be seen as random at any given time, but they are the results of joint adaptive schemes, and it would make more sense to have dependencies between the marks of nearby cells
  - Introduction of traffic/capacity in the model