Flow Control in Stochastic Networks

François Baccelli

INRIA & ENS

E-mail: Francois.Baccelli@ens.fr

Madison Workshop 2000

Structure of the Course

1. Lecture 1 TCP and its evolution equations;
   joint work with Dohy Hong, Sigcomm '00
2. Lecture 2 The (max,plus) algebra: Spectral properties of matrices and fixed point equations
3. Lecture 3 The (max,plus) algebra: Random media
4. Lecture 4 TCP with cross traffic;
   joint work with Thomas Bonald, Questa (1999)
5. Lecture 5 Beyond (max,plus).

Summary of Lecture 4

1. Part I TCP with Cross Traffic
   - Monotonicity
   - Stochastic Framework
   - Saturation Rule
2. Part II Bounds on Throughput
   - Scaling
   - Examples

Several Connections: FIFO Case, Fixed Window Size

W: window size (fixed), K: number of routers

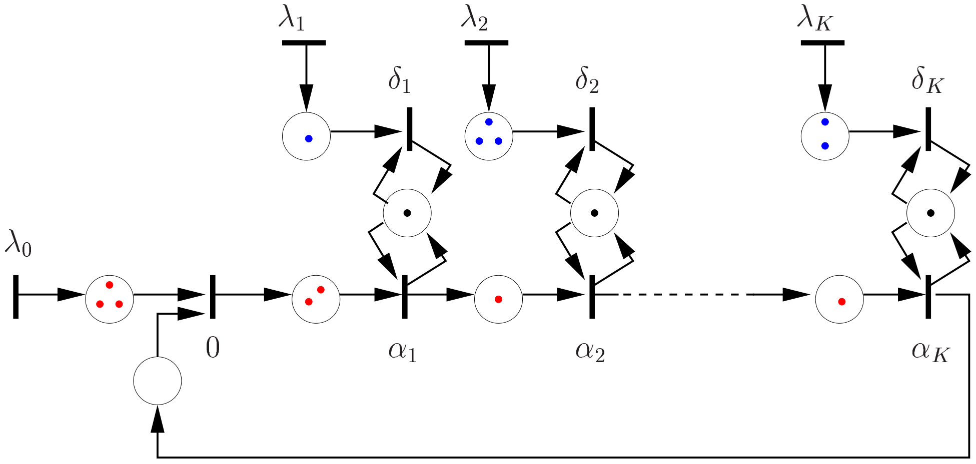
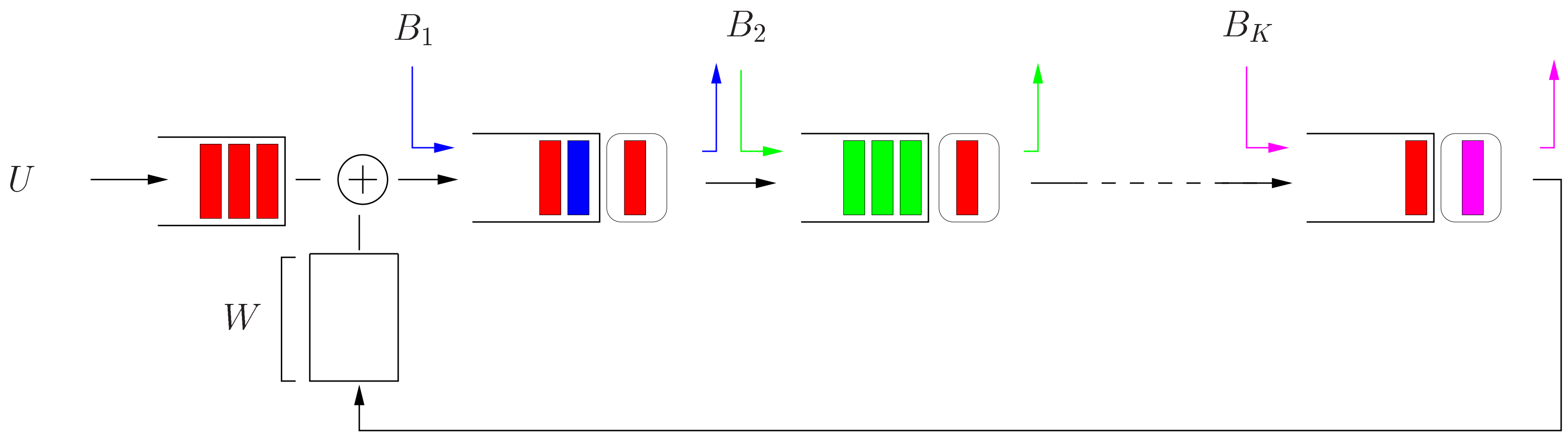


Figure 1: Petri Net Controlled flow Cross flows

Several Connections: FIFO Case, Fixed Window Size (*continued*)

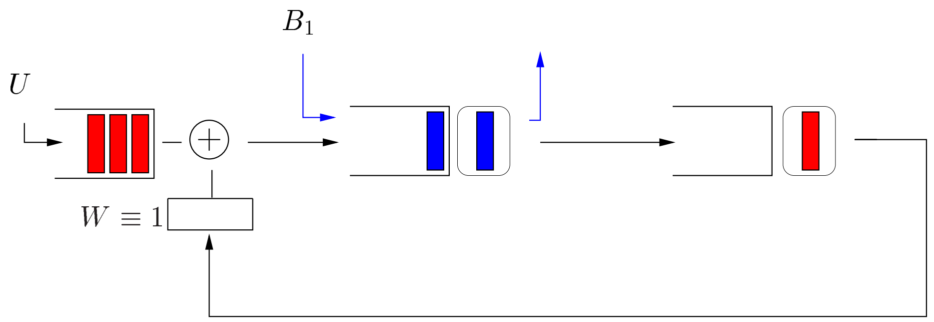
Queueing Network Model



Monotonicity, Homogeneity

Monotonicity Let U_1 and U_2 be two arrival point processes with the same finite number of points. Then, $U_1 \leq U_2 \quad \Rightarrow \quad Y_1 \leq Y_2$.
Remark 1 *no* homogeneity, the non–expansiveness is *not* satisfied.
Remark 2 *no* monotonicity w.r.t. the cross flows.

Monotonicity, Homogeneity (*continued*)



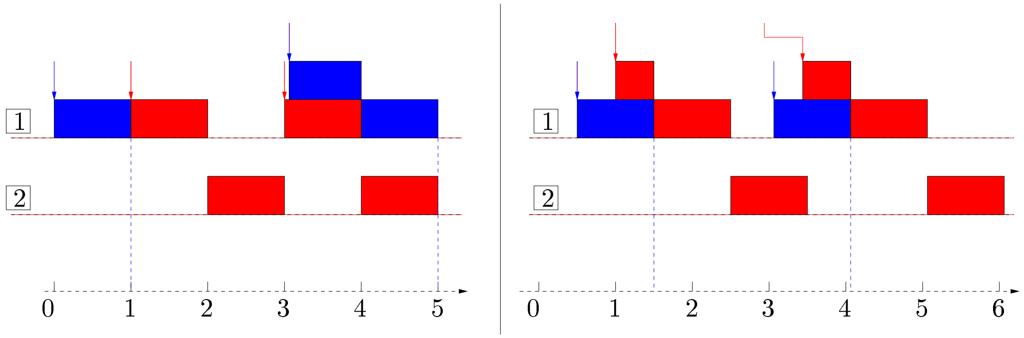
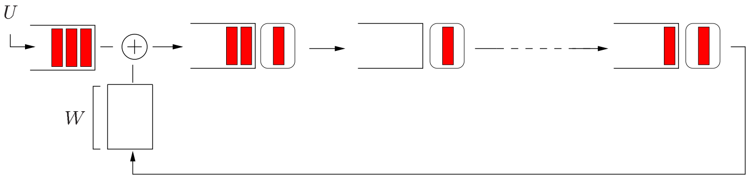


Figure 2: Two routers, one cross flow, $W = 1$

Monotonicity, Homogeneity (*continued*)

Model without cross traffic



This system is (max,+)–linear.
Let λ_0 be the *intensity* of the arrival process U. There exists a constant μ, such that

- If $\lambda_0 < \mu$, the controlled connection is stable;
- If $\lambda_0 > \mu$, the controlled connection is unstable.

In this sense, μ is the *maximum throughput* of the connection.

Monotonicity, Homogeneity (*continued*)

Case of Deterministic Service Times

Denote by $\sigma_i = \mu_i^{-1}$ the (deterministic) service time at station i. Then,

$$\mu = \min\left(\frac{W}{\sum_i \sigma_i}, \min_i \mu_i\right).$$

Case with Random Service Times

Lyapunov exponent.

QUESTIONS

In the presence of cross traffic,

– Existence of a maximum throughput?

– Saturation rule?

– Evaluation of this maximum throughput?

STOCHASTIC FRAMEWORK

Controlled flow:

Let (Ω_U, P_U) be a probability space, equipped with a *stationary* and *ergodic* flow $\{\theta_U(t)\}$.

$$U \text{ is } \{\theta_U(t)\}\text{-compatible.}$$

Cross–flows:

Let (Ω_B, P_B) be a probability space, equipped with a *stationary* and *ergodic* flow $\{\theta_B(t)\}$.

$$B_1, \ldots, B_N \text{ are } \{\theta_B(t)\}\text{-compatible.}$$

Probability space:

Let (Ω, P) be the product space:

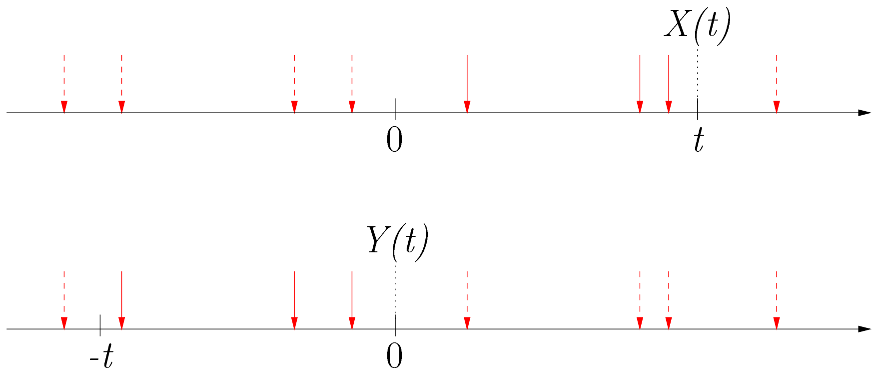
$$\Omega = \Omega_U \times \Omega_B \quad \text{and} \quad P = P_U P_B.$$

We define a flow $\{\theta(t)\}_{t \in \mathbb{R}}$ on this space by

$$\forall t \in \mathbb{R}, \quad \theta(t) = \theta_U(t) \times \theta_B(t).$$

Stationary Regime

For any $t \geq 0$, define $X(t)$ (# of controlled packets) and $Y(t)$ as follows:



By monotonicity, we can define

$$Z = \lim_{t \to +\infty} Y(t).$$

Lemma 2 (Stationarity) *The process $\{X(t)\}$ tends to Z for weak convergence.*

Lemma 3 (Ergodicity) *The random variable Z is a.s. finite or a.s. infinite if the product of the two shifts is ergodic.*

Extension of Saturation Rule: Deterministic service times for controlled customers

$\bar{X}(s,t)$, $s \leq t$, number of controlled customers entering the network in the interval $[s,t)$, when the input is saturated at time s.

Sub-additivity

$$\forall s \leq t \leq u, \quad \bar{X}(s,u) \leq \bar{X}(s,t) + \bar{X}(t,u).$$

By Kingman's sub-additive theorem, under appropriate ergocicity and integrability assumptions,

$$\forall s \in I\!R, \quad \frac{\bar{X}(s,t)}{t} \xrightarrow{p.s.} \lambda \quad \text{when} \quad t \to +\infty.$$

λ_0 intensity of controlled flow:

- If $\lambda_0 < \lambda$, the system is stable (backward scheme)
- If $\lambda_0 > \lambda$, the system is unstable.

Bounds on Maximum Throughput – Deterministic Service Times

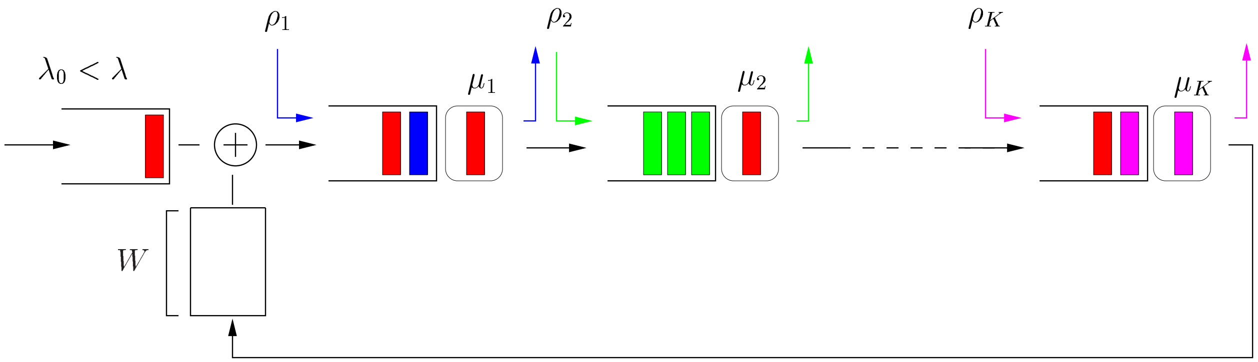


Figure 3: K routers, window: W, service time $\sigma_i = \mu_i^{-1}$ in router i

Upper Bounds: $\gamma = \max(\frac{\sum_i \sigma_i}{W}, \max_i \sigma_i) \quad \mu = \gamma^{-1}$

$$\lambda \leq \mu \quad \text{and} \quad \lambda \leq \min_i \sigma_i^{-1}(1 - \rho_i).$$

Lower Bound:

$$\lambda \geq \mu\ P, \quad P = P\{\text{system without controlled flow empty}\}.$$

Numerical Example

Consider the case of a controlled connection which goes through $K = 10$ routers. The window is $W = 10$. The size of the packets is 1.

$$\mu = \min_i \mu_i \stackrel{\triangle}{=} 1.$$

If the traffic intensities of the cross flows are the same $(\rho_1 = \ldots = \rho_K = \rho)$,

$$\lambda_{\max} = \min\left\{\mu, \min_i \mu_i(1-\rho_i)\right\} = 1 - \rho,$$

and

$$\lambda_{\min} = \mu \prod_i (1-\rho_i) = (1-\rho)^{10}.$$

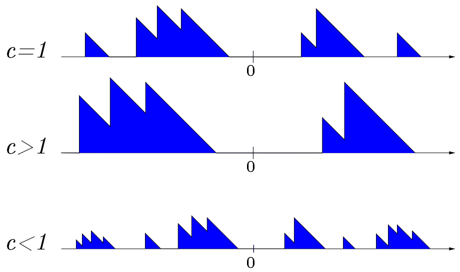
ρ	0.1	0.2	0.5	0.8
$\lambda_{\max}$	0.9	0.8	0.5	0.2
$\lambda_{\min}$	0.3	0.1	10^{-3}	10^{-7}

Tightness of the Bounds

$$\underbrace{\mu\ P}_{\lambda_{\min}} \leq \lambda \leq \underbrace{\min\left\{\mu, \min_i \mu_i(1-\rho_i)\right\}}_{\lambda_{\max}}.$$

Time–Space Scaling For any $c > 0$, let $\lambda(c)$ be the maximum throughput obtained when the cross flows are scaled by c. Then,

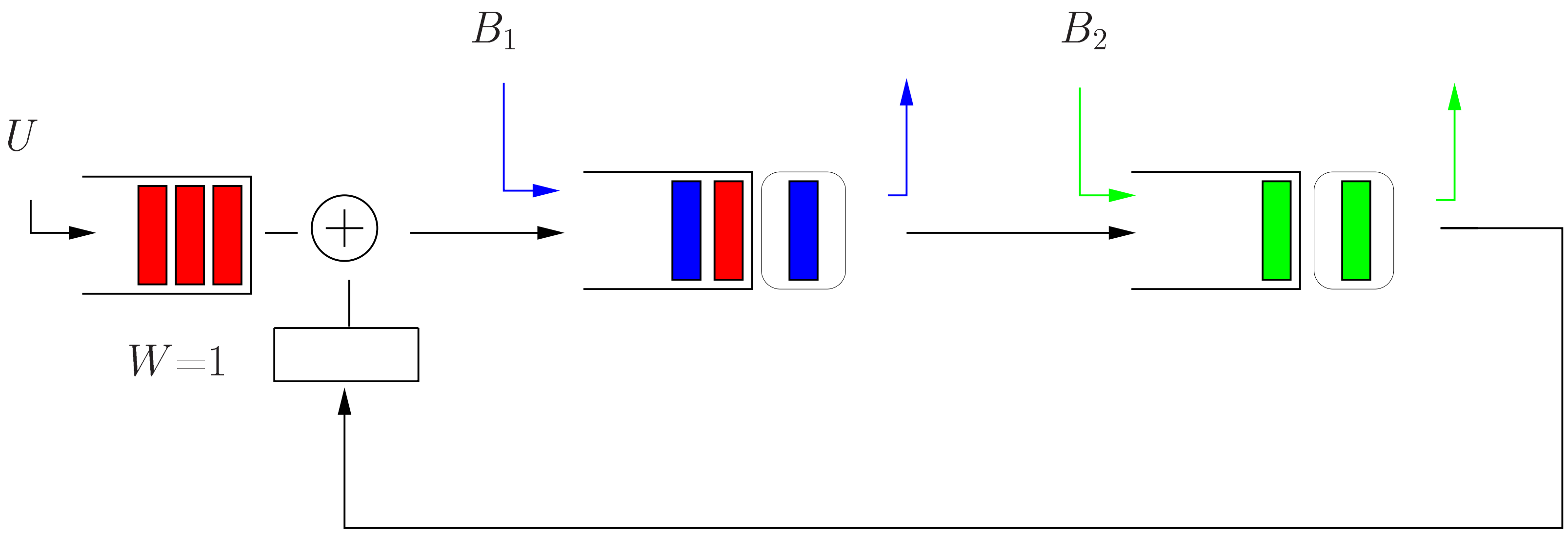
$$\lim_{c\to+\infty} \lambda(c) = \lambda_{\min} \quad \text{and} \quad \lim_{c\to 0} \lambda(c) = \lambda_{\max}.$$



EXAMPLES

Consider the simplest non–trivial case:

$$K = 2, \quad W = 1.$$



EXAMPLES (*continued*)

Controlled packets all of size 1:

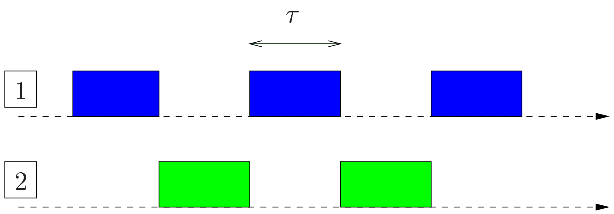
$$\mu_1 = \mu_2 = 1 \quad \text{and} \quad \mu = \frac{1}{2}$$

– Ex 1: B_1, B_2 periodic, with interarrival times 2τ, service time τ and $B_2 = B_1 + \tau$;

– Ex 2: B_1, B_2 as above, but stationary and mutually independent;

– Ex 3: B_1 periodic, $B_2 = \emptyset$.

Example 1



$$\lambda_{\min} = 0 \quad \text{and} \quad \lambda_{\max} = \min(\mu, \frac{1}{2}) = \frac{1}{2}$$

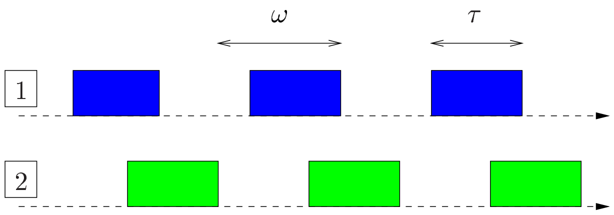
For any $\tau \geq 1$, we have

$$\lambda = \frac{1}{2\tau}.$$

In particular,

$$\lim_{\tau \to +\infty} \lambda = 0.$$

Example 2



$$\lambda_{\min} = \mu \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \quad \text{and} \quad \lambda_{\max} = \frac{1}{2}$$

Here the maximum throughput is a random variable, which depends on the *phase* ω.

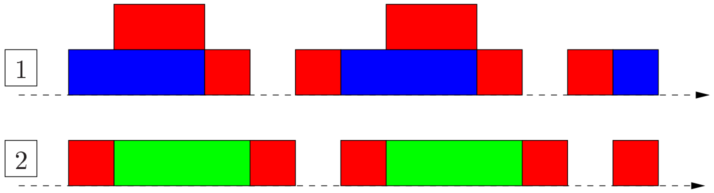
If $\tau = 2n-1$, where n is a positive integer, there is a *spectrum* of n possible values for λ, namely

$$\lambda \in \{1, \dots, n\} \times \frac{1}{2\tau}.$$

Remark: In this non–ergodic case, $\lambda_{\min}$ does not provide a lower bound for the maximum throughput but for its *expectation*.

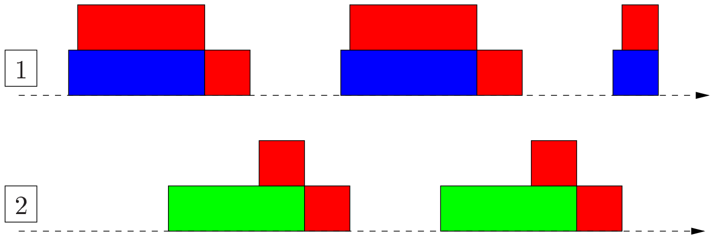
Example 2: Case $\tau = 3$

Cross–flows “in phase”:



$$\bar{\lambda} = \frac{1}{\tau}.$$

Cross–flows “out of phase”:



$$\bar{\lambda} = \frac{1}{2\tau}.$$

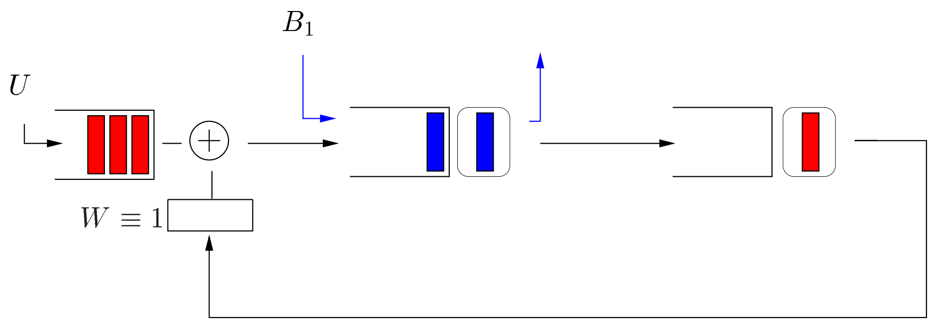
Example 3

The simplest non–trivial case:

$$K = 2, \quad W = 1.$$

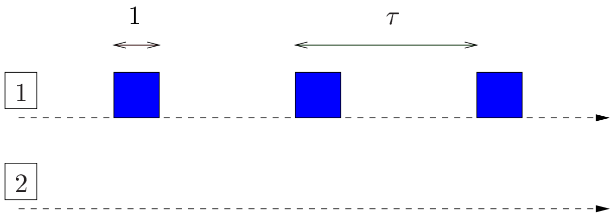
$$\mu_1 = \mu_2 = 1 \quad \text{and} \quad \mu = \frac{1}{2}$$

B_1 periodic, $B_2 = \emptyset$.



Example 3 (*continued*)

Assumption on cross traffic:



The traffic intensity of the cross–flow is:

$$\rho = \frac{1}{\tau}.$$

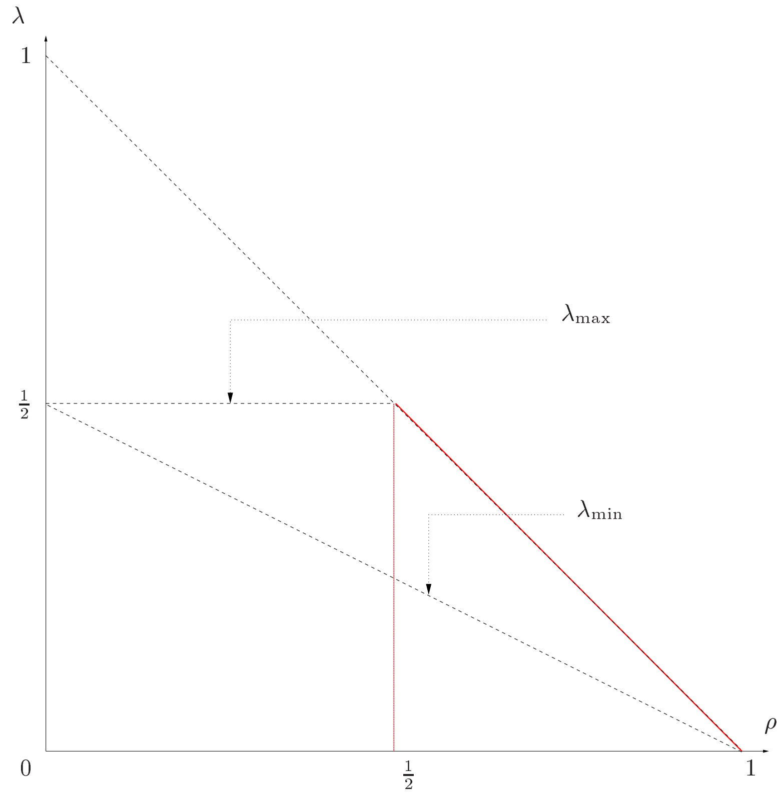
We have

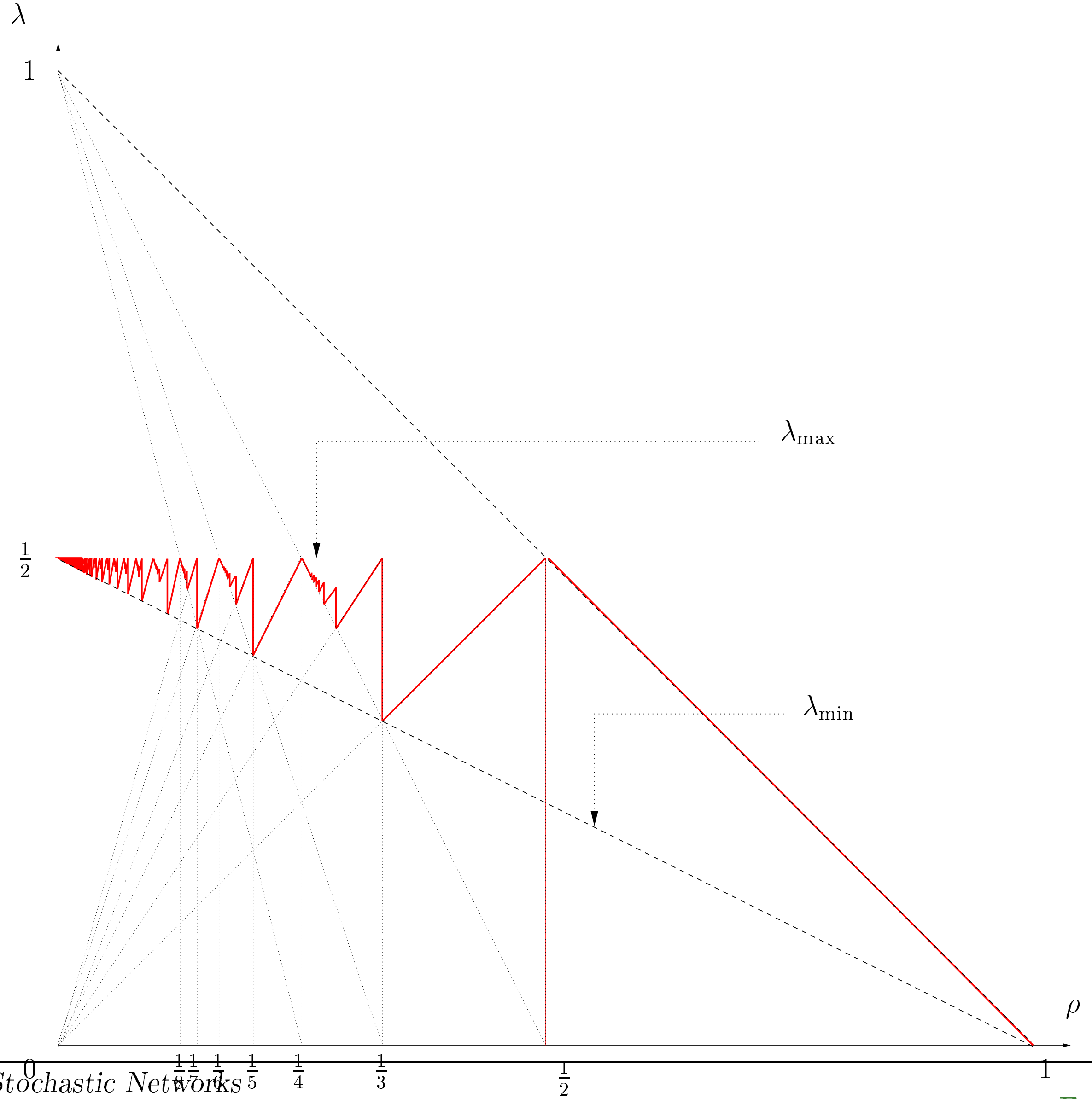
$$\lambda_{\min} = \frac{1}{2}(1-\rho)$$

and

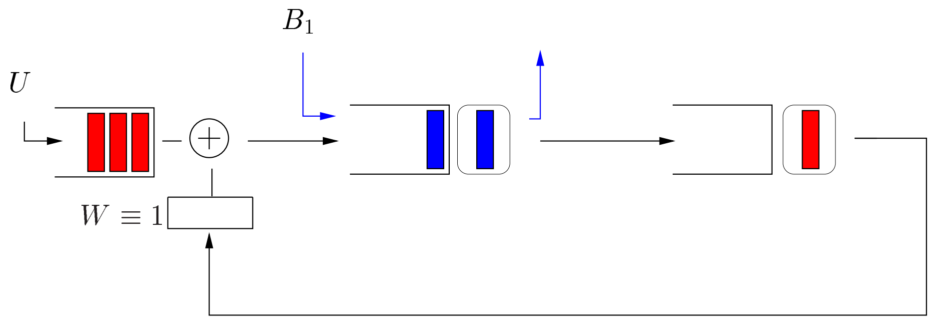
$$\lambda_{\max} = \min(\frac{1}{2}, 1-\rho).$$

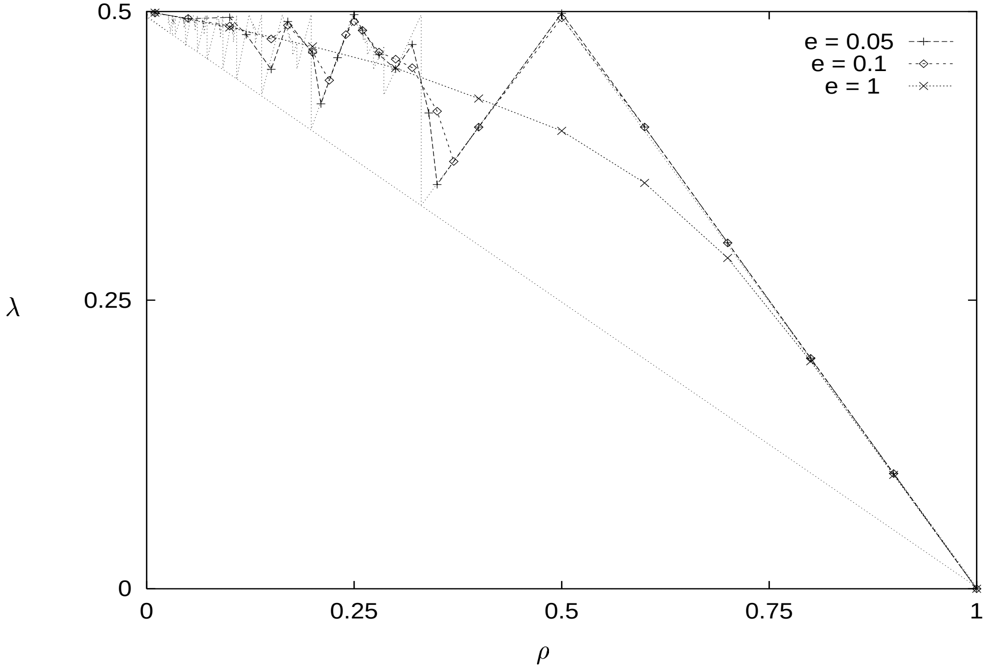
Stability Region



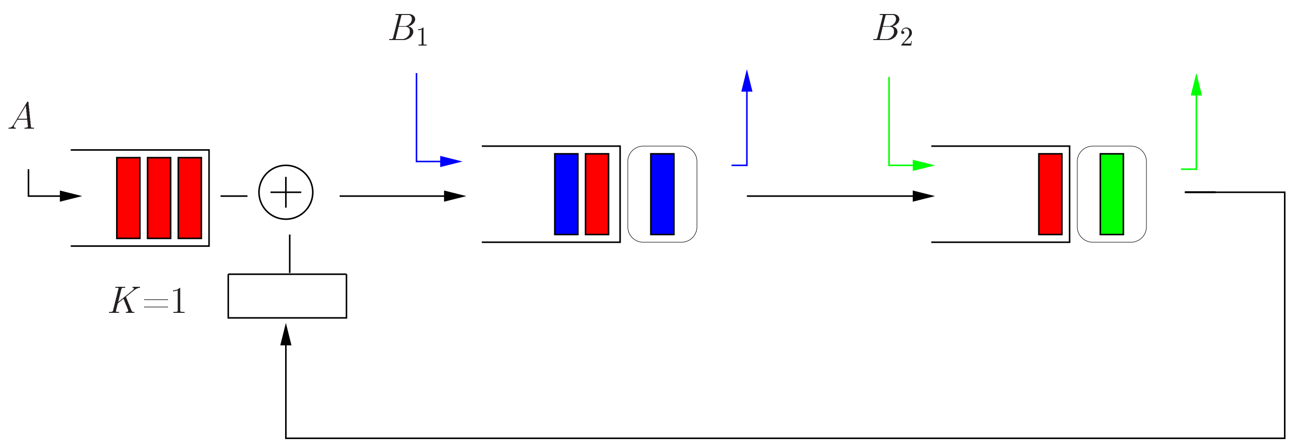

λ
1
$\frac{1}{2}$
$\lambda_{\max}$
$\lambda_{\min}$
ρ
0
$\frac{1}{8}$ $\frac{1}{7}$ $\frac{1}{6}$ $\frac{1}{5}$ $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{2}$ 1

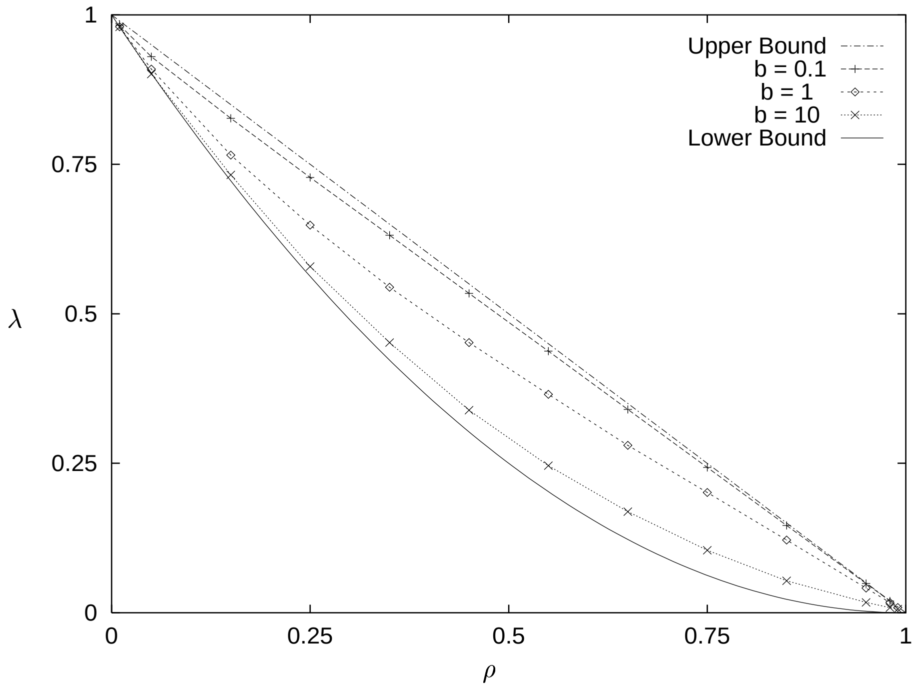
Effect of Randomization




0.5
0.25
0
λ
0
0.25
0.5
0.75
1
ρ
e = 0.05
e = 0.1
e = 1

Effect of Burstiness




1
0.75
0.5
0.25
0
λ
0
0.25
0.5
0.75
1
ρ
Upper Bound
b = 0.1
b = 1
b = 10
Lower Bound

Conclusion

The efficiency of the window flow control is very *sensitive* to the statistical characteristics of the cross–flows:

– Periodic flows *vs* random flows;

– Bursty flows *vs* fluid flows.

This is due to the fact that the model is not *monotone* with respect to the cross–flows, as soon as $K > 1$.
These results can be easily extended to more general models (e.g. presence of propagation delays, multicast connection,...).