Flow Control in Stochastic Networks

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Structure of the Course

1. Lecture 1 TCP and its evolution equations;
   joint work with Dohy Hong, Sigcomm '00
2. Lecture 2 The (max,plus) algebra: Spectral properties of matrices and fixed point equations
3. Lecture 3 The (max,plus) algebra: Random media
4. Lecture 4 TCP with cross traffic;
   joint work with Thomas Bonald, Questa (1999)
5. Lecture 5 Beyond (max,plus).

Summary of Lectures 3

1. Part I Lyapunov Exponents
   - TCP's throughput as a Lyapunov exponent
   - Representations
   - Expansions
2. Part II Open Systems
   - Representations
   - Computational aspects
3. Part III Algebraic Simulation
   - Sequential
   - Parallel

Random Service Times

- Assumptions
  - Random service times on all routers
  - Sequence $\{\sigma_i(n),\ i = 1, \ldots, K\}_n$ i.i.d.
- Then for all loss detection models:
  - the sequence of matrices $\{A_i(n),\ i = 1, \ldots, w^*\}_n$ is i.i.d.
  - the sequence $\{(W(n), \overline{Z}(n))\}$ forms a Markov chain.

Throughput and Lyapunov Exponents

- The mean throughput λ of the controlled connection is the *long term averaging of the instantaneous throughput*:

$$\lambda = \lim_{n\to+\infty} \frac{n}{y_K(n)} = \lim_{n\to+\infty} \frac{n}{\Sigma_{p=1}^{n}(y_K(p) - y_K(p-1))} \quad a.s.$$

- Using bounds, one shows that if this a.s. limit exists, then $\lim_{n\to+\infty} \frac{n}{y_i(n)} = \lambda$ a.s. for all i.

- The max-plus Lyapunov exponent γ of the sequence of matrices is defined as the following limit (when $\exists$):

$$\gamma = \lim_{n\to+\infty} \frac{\max_{1\le l,k\le K.w^*} \left(A_{v_{n-1}}(n) \otimes \cdots \otimes A_{v_0}(1)\right)_{l,k}}{n}.$$

- Key observation : $\lambda = \gamma^{-1}$.

EXISTENCE OF LYAPUNOV EXPONENT

- For all matrices A : $|A|_\oplus = \oplus_{ij} A_{ij}$.
- For all square matrices A and B : $|A \otimes B|_\oplus \leq |A|_\oplus \otimes |B|_\oplus$

For all stationary and ergodic sequences of square matrices $\{I\!P_n\}$,

$|I\!P_1 \otimes \ldots \otimes I\!P_n \otimes \ldots \otimes I\!P_p|_\oplus \leq |I\!P_1 \otimes \ldots \otimes I\!P_n|_\oplus \otimes |I\!P_{n+1} \otimes \ldots \otimes I\!P_p|_\oplus$

From Kingman's Subadditive Ergodic Theorem, if $|A_0|_\oplus$ is nonnegative and integrable, then there exists a real number γ such that

$$\frac{|I\!P_1 \otimes \ldots \otimes I\!P_n|_\oplus}{n} \to \gamma \quad a.s. \text{ and in } L_1(P).$$

RELATION BETWEEN LYAPUNOV EXPONENT AND EIGENVALUES

- Case with i.i.d. matrices (e.g. fixed window size with independent processing times)
- By convexity arguments, for all n

$$E|\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_n|_\oplus \geq |E(\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_n)|_\oplus$$

Therefore, if the matrix $E(\mathbb{P}_1 \otimes \ldots \otimes \mathbb{P}_n)$ is irreducible, for n large enough, and if we denote $\rho(n)$ its eigenvalue, then

$$\gamma = \lim_n \frac{\rho(n)}{n} \quad \text{and} \quad \gamma \geq \frac{\rho(n)}{n} \quad \text{for} \quad n = 2^k.$$

RELATION BETWEEN LYAPUNOV EXPONENT AND EIGENVALUES (*continued*)

- The bound is tight as n goes to ∞, but it is hard to compute;
- The maximal cycle mean does not give the right answer in the stochastic case;
- Role of variance as in G/G/1 queues

TCP THROUGHPUT

A sufficient condition for the the limit defining γ to exist (in the almost sure sense) is that the sequence $\{A_{v_{n-1}}(n)\}$ converges (with shift coupling) to some stationary and ergodic sequence.

Examples

– Fixed window size case

– Case with Periodic or Markov window size (e.g. G1 or G4)

– Case where v_n is regenerative w.r.t. point 1

REPRESENTATION FORMULAS

- $\{\mathbb{P}_n\}$ stationary and ergodic, fixed support, irreducible $X_{n+1} = \mathbb{P}_n X_n$.
- Projective Space: $u \sim v$ in $\mathbb{R}_{max}^K$ if for some $\lambda \in \mathbb{R}$ $u^i = v^i + \lambda$: $\overline{u}$ equivalence class of u

THEOREM J. Mairesse 97, JAP
Suppose there is an integer N such that within the interior of the support of $\mathbb{P}_{N-1} \otimes \cdots \otimes \mathbb{P}_0$, there exists an irreducible matrix $\mathcal{P}$ which is COC 1 CYC 1. Then $\overline{X}_n$ converges in total variation to a unique stationary distribution, independently of the initial condition.

IDEA OF PROOF & REPRESENTATION FORMULA: Backward coupling

Example: $I\!P_n \in \{A, A(1), \ldots, A(m)\}$ with A COC 1–CYC 1.
Backward Scheme $X^{0,0} = X, \quad X^{-1,0} = A_{-1}X, \quad X^{-2,0} = A_{-1}A_{-2}X, \ldots$

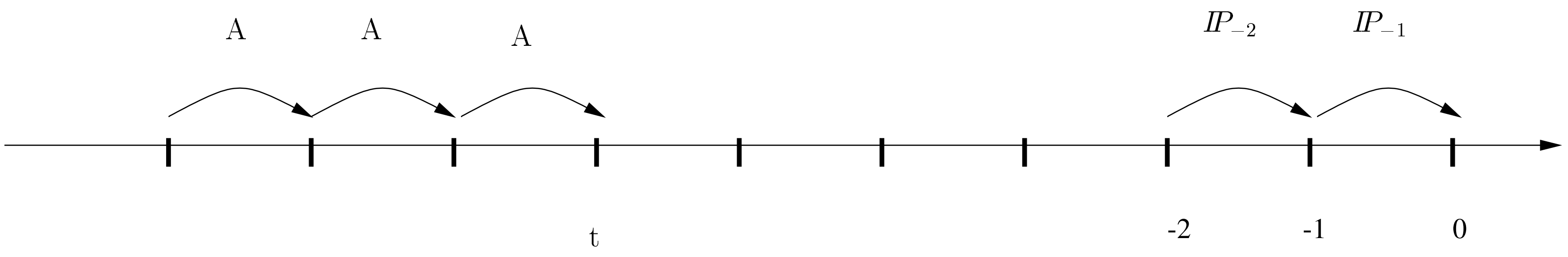

Figure 1: c=3, t: first backward time that A comes 3 times in a row

$$\text{(REP2)} \qquad \overline{X}^{-n,0} \to_{n\to\infty} \overline{X}^{-\infty,0} = \overline{X}^{-t,0}.$$

- Direct extension to Markov dependent case (in place of i.i.d).

IID MATRICES: REGENERATIVE FORMULAS–TWO MATRIX CASE

- $\mathcal{C}_n$: set of all sequences of n elements of the set $\{0,1\}$ such that there are never more than $c-1$ consecutive 0's. Let $p = p_1 = P(\mathbb{P}_1 = A(1))$, $p_0 = 1 - p_1$, $A(0) = A$.

$$\gamma(p) = \frac{1}{\mathbb{E}[T_1]} \sum_{n=c}^{\infty} p^c \sum_{i_0, i_1, i_{n-c-1} \in \mathcal{C}_{n-c}} p_{i_0} \cdots p_{i_{n-c-1}} \left[A^c . A(i_{n-c-1}) \cdots A(i_0) X_o - X_o\right],$$

with $\pi(n) = P(T_1 = n)$ and

$$\Phi(z) = \mathbb{E}\left[z^{T_1}\right] = \sum \pi(n) z^n = \frac{(1-p)^c z^c \left(1 + \frac{pz}{1-z}\right)}{1 + p(1-p)^c \left(\frac{z^{c+1}}{1-z}\right)}.$$

IID MATRICES: LIGHT TRAFFIC EXPANSIONS

The simplest possible case: $\{A_n\}$ i.i.d. $A_n \in \{A, A'\}$, $P(A_n = A') = p$

– A COC 1, CYC 1, irr., with eigenvector X_o and coupling time c;

– $\gamma(p)$: top and bottom Lyapunov exponent of $\{A_n\}$.

From (REP2), $\gamma(p)$ analytic in p in 0, with radius of $\geq \frac{1}{2c}$

$$\frac{1}{l!}\frac{d^l}{dp^l}\left[\Gamma(p)\right]_{p=0} = \begin{bmatrix} \pi(l) \\ \vdots \\ \pi(l) \end{bmatrix} = (-1)^l \left\{ \binom{c}{l-1} X_o + \binom{c+1}{l} \Gamma(0) \right\} + \sum_{k=1}^{l}$$

$$\left\{ (-1)^{l-k} \sum_{j_1,\ldots,j_{k-1}=0}^{c-1} \binom{2c+j_1+..+j_{k-1}}{l-k} A^c \otimes A' \otimes A^{j_1} \otimes A' \cdots A^{j_{k-1}} \otimes A' \otimes X_o \right\}.$$

For $p < \delta < \frac{1}{2c}$, error term in Taylor expansion decreases like $(2c\delta)^n$
FB & D. Hong 98 $\rightarrow$ Annals Appl. Prob.

FIRST COEFFICIENTS

$$
\begin{aligned}
\begin{bmatrix} \pi(0) \\ \vdots \\ \pi(0) \end{bmatrix} &= \begin{bmatrix} \gamma(0) \\ \vdots \\ \gamma(0) \end{bmatrix} = \Gamma(0), \\
\begin{bmatrix} \pi(1) \\ \vdots \\ \pi(1) \end{bmatrix} &= A^c \otimes A' \otimes X_o - X_o - (c+1)\Gamma(0), \\
\begin{bmatrix} \pi(2) \\ \vdots \\ \pi(2) \end{bmatrix} &= \sum_{j=0}^{c-1} \{A^c \otimes A' \otimes A^j \otimes A' \otimes X_o\} \\
&\quad -2cA^c \otimes A' \otimes X_o + cX_o + \binom{c+1}{2} \Gamma(0).
\end{aligned}
$$

CASES COVERED

– Fixed support matrices, with random coefficients: e.g. fixed window size with random service times.

– Random support matrices: e.g. fluctuating window size with deterministc service times.

EXAMPLE

$$A = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}, \quad A' = \begin{pmatrix} \sigma & \varepsilon & \sigma' & \varepsilon \\ \sigma & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \sigma' & \varepsilon \\ \varepsilon & \varepsilon & \sigma' & \varepsilon \end{pmatrix}.$$

1. For $\sigma' < 2\sigma$, there are two regimes :

   – if $2\sigma' < 3\sigma$, $\gamma(p) = \sigma$

   – if $2\sigma' > 3\sigma$:

$$\begin{aligned} \gamma(p) \;&=\; \sigma + (2\sigma' - 3\sigma)p + (5\sigma - 3\sigma')p^2 + (4\sigma' - 6\sigma)p^3 + (7\sigma - 5\sigma')p^4 \\ &\quad + (9\sigma' - 13\sigma)p^5 + (26\sigma - 17\sigma')p^6 + \cdots \end{aligned}$$

2. For $\sigma' > 2\sigma$, A becomes 2-periodic.

MARKOV DEPENDENT MATRICES/REGENERATIVE APPROACH: TCP EXAMPLES – Tahoe (G1)-(G4)

– (G1) accounts for the losses due to an excessive send rate of the reference flow:

$$S(n) = S = E(\sum_{i=1}^{K} \sigma_i(1)), \quad \sigma^*(n) = \sigma^* = \max E(\sigma^*), \quad b^* = \left\lfloor \frac{\mathrm{RTT}}{\sigma^*} \right\rfloor + 1.$$

– (G4) accounts for the losses and timeouts due to the variations of cross traffic.

MARKOV DEPENDENT MATRICES/REGENERATIVE APPROACH: TCP EXAMPLES – Tahoe (G1)-(G4) (*continued*)

For $n \leq \frac{(b^*-1)b^*}{2} + 1$, let

$$\widetilde{B}(n) = \widehat{A}_{v_n}(n+1) \otimes \widetilde{A}_{v_{n-1}}(n) \otimes \cdots \otimes \widetilde{A}_{v_1}(2) \otimes Z(0),$$

– $v_1, v_2, \ldots,$ is the Tahoe sequence $(1, 2, 2, 3, 3, 3, \ldots, b^* - 1, b^* - 1, b^*)$.

– the matrices $\{\widetilde{A}_i(n),\ i = 1, \ldots, K\}$ are i.i.d. and defined as $\{A_i,\ i = 1, \ldots, K\}$, but when using the i.i.d. random variables $\widetilde{\sigma}(k)$ in place of $\sigma(k)$, where $\widetilde{\sigma}(k)$ is a random vector with the law of $\sigma(k)$ conditional on the property that $\sigma(k) \notin \mathcal{B}$

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Example: Tahoe (G1)-(G4)

The throughput of TCP Tahoe without slow start is:

$$\lambda = \frac{\Sigma_{k=1}^{b^*} \Sigma_{i=0}^{k-1} p_{ki} \left(\frac{(k-1)k}{2} + i + 1\right)}{\Sigma_{k=1}^{b^*} \Sigma_{i=0}^{k-1} p_{ki}\ E\left[\left(\widetilde{B}\left(\frac{(k-1)k}{2} + i + 1\right)\right)_K\right]},$$

where

$$p_{ki} = \pi^{\frac{(k-1)k}{2}+i}(1-\pi), \text{ with } \pi = P(\sigma(n) \in \mathcal{B}).$$

- The stochastic process $\{W(n), \overline{Z}(n)\}$ is a regenerative process where the regeneration times are the epochs when the window size is equal to 1.
- The lengths T_l, $l \geq 1$, of the successive regeneration cycles are i.i.d.
- p_{ki} denote the probability that T_1 is equal to $\frac{k(k-1)}{2}+i+1$, $0 \leq i < k \leq b^*$.

Example: Tahoe (G1)-(G3)

Assumptions

– $RTO(n) = RTO$;

– The service times can take a finite number of rational values

- Then the variables $\{y_K(n) - y_K(n - w_{n-1}), n \in I\!N\}$ can only take a finite number of values in a set Ψ, and this sequence is regenerative.

- The joint law of regeneration cycle T_1 and the dater $\overline{Z}(n)$ can be computed by the following recursion:

$$
\begin{aligned}
& P(T_1 > n, \overline{Z}(n) = \overline{z}) = P(\cap_{k=1}^{n}\{y_K(k) \text{ - } y_K(k\text{-}w_{k\text{-}1}) < \mathrm{RTO}\}, \overline{Z}(n) = \overline{z}) \\
= \quad & \sum_{\overline{z}' \in \Psi} P(\cap_{k=1}^{n}\{y_K(k) \text{ - } y_K(k\text{-}w_{k\text{-}1}) < \mathrm{RTO}\}, \overline{Z}(n-1) = \overline{z}', \overline{Z}(n) = \overline{z}) \\
= \quad & \sum_{\overline{z}' \in \Psi} P(\overline{Z}(n) = \overline{z}, y_K(n)\text{-}y_K(w_{n\text{-}1}) < \mathrm{RTO} \mid \overline{Z}(n-1) = \overline{z}') \\
& \qquad\qquad P(T_1 > n\text{-}1, \overline{Z}(n-1) = \overline{z}').
\end{aligned}
$$

Example: Tahoe (G1)-(G3) (*continued*)

- Formula for throughput:

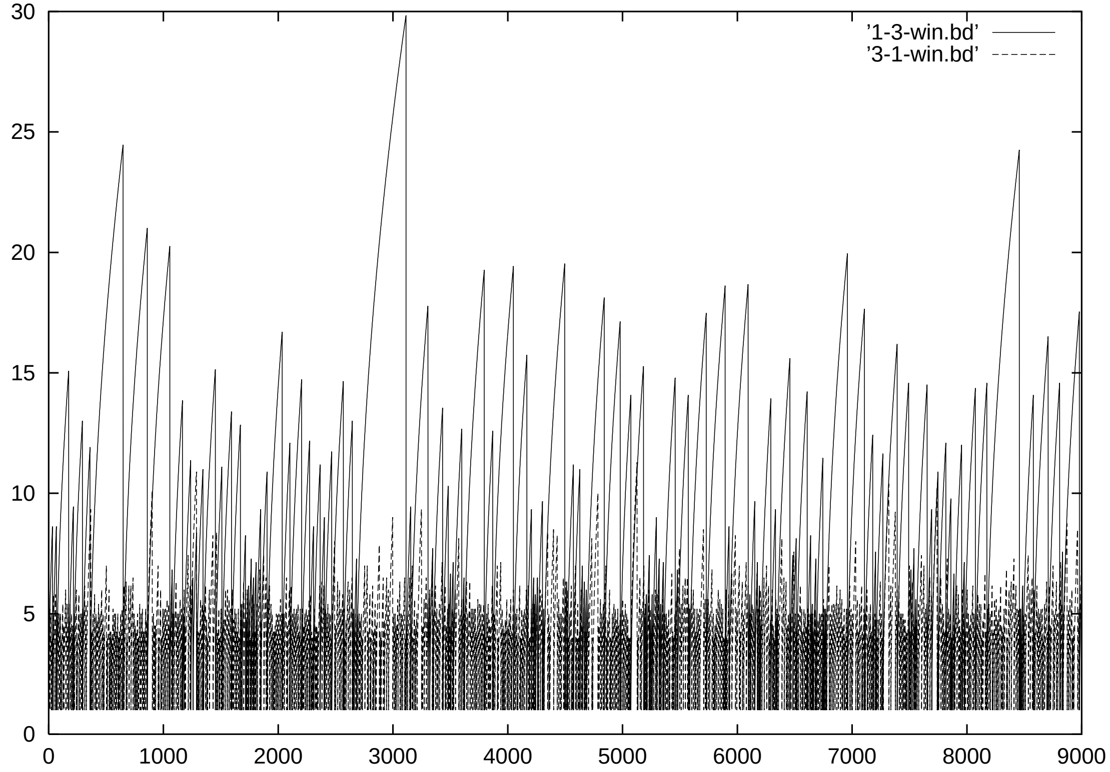
$$\lambda = \frac{\Sigma_{k=0}^{b^*-1} P(T_1 > k)}{\Sigma_{k=0}^{b^*-1} \Sigma_{\overline{z} \in \Psi} P(T_1 > k, \overline{Z}(k) = \overline{z}) \alpha(\overline{z})},$$

Numerical Example

- TCP Tahoe with $K = 3$ routers, under (G2) with $\beta_1 = \beta_3 = \infty$ and $\beta_2 = 3$. The random variables $\sigma_i(n)$ are i.i.d. multinomial with
  - $\sigma_1(n)$ is equal to $\{1, 10, 20\}$ with probability $0.1, 0.2, 0.7$;
  - $\sigma_2(n)$ is equal to $\{13, 15, 17\}$ with probability $0.25, 0.5, 0.25$;
  - $\sigma_3(n)$ is equal to $\{1, 10, 20\}$ with probability $0.7, 0.2, 0.1$.

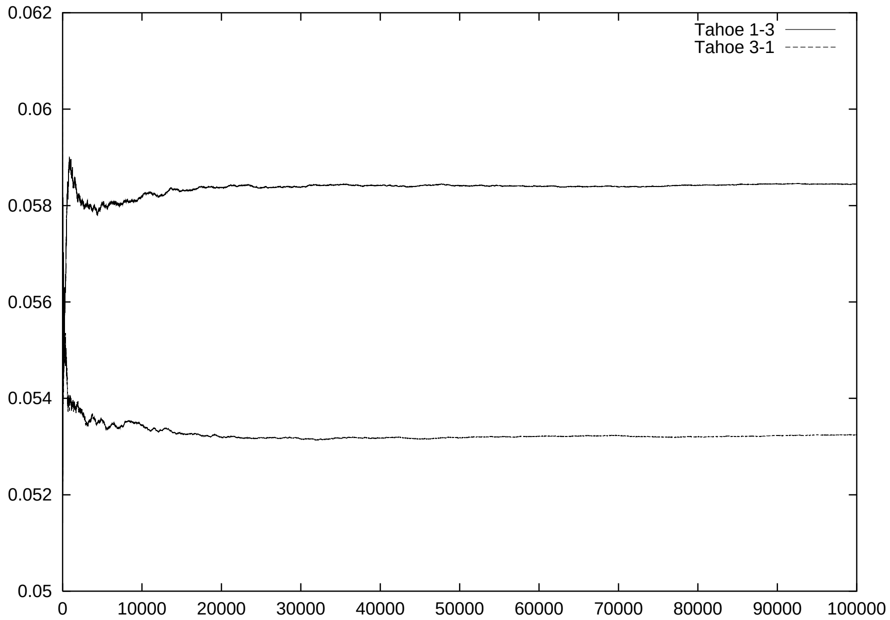
Numerical Example (*continued*)

Evolution of $W(n)$ for TCP Tahoe model with slow start



Numerical Example (*continued*)

Throughputs when exchanging the statistics of $\sigma_1(n)$ and $\sigma_3(n)$:
$\lambda_1 = 0.058,\ \lambda_2 = 0.053$



Numerical Example (*continued*)

- The permutation of the characteristics of two routers may influence the value of the throughput:
- The variance has an impact too: when moving to deterministic service times equal to the mean values of the corresponding multinomial distributions, we find a throughput of 0.062 for model 1.

STRUCTURE OF LYAPUNOV EXPONENTS – FIXED SUPPORT CASE

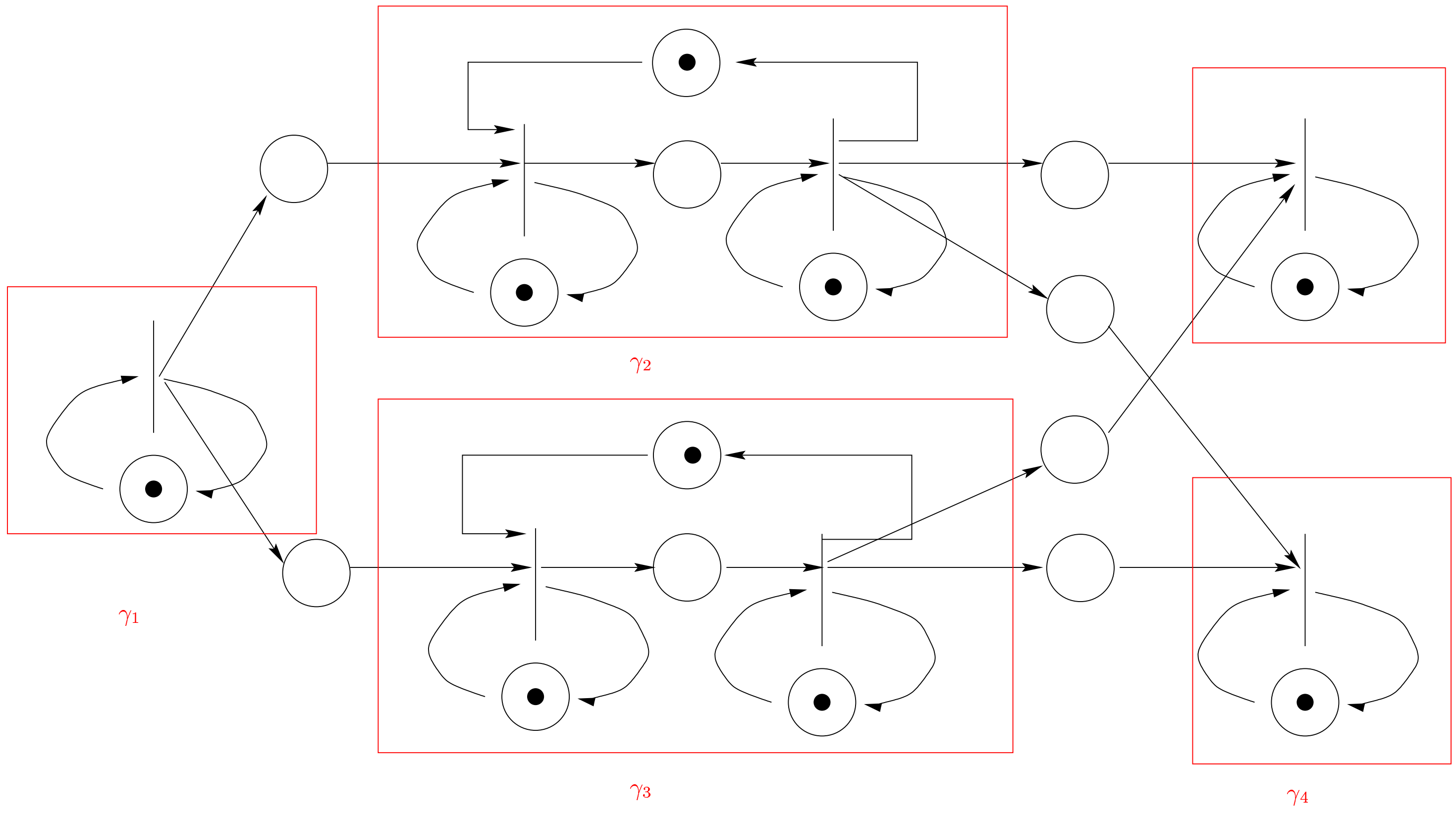
Fixed support: For all j, k, either $(A_0)_{kj} > \varepsilon$ a.s or $(A_0)_{kj} = \varepsilon$ a.s.
FIFO: $\forall j,\ (A_0)_{jj} \geq 0$ a.s.
Graph of A_0: Directed graph $j \to k$ if $(A_0)_{kj} > \varepsilon$ a.s
CC: Equivalence classes of $i\mathcal{R}j$ if $i \to^\star j$ and $j \to^\star i$

$$\forall i \in cc\ \tilde{u}, \lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \oplus_{\tilde{v} \in {}^\star\tilde{u}}\, \gamma_{\tilde{v}}$$

– ${}^\star\tilde{u}$: set of all ancestors of $\tilde{u}$ in reduced graph

– $\gamma_{\tilde{v}}$: Lyapunov exponent of the restriction of A_n to the coordinates of $\tilde{v}$

FB & Z. Liu 92, Annals Prob.
Various extensions to the non-fixed support case.



$$\lim_n \frac{X_n^i(X_0)}{n} = \lim_n E\left(\frac{X_n^i(X_0)}{n}\right) = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 \oplus \gamma_4, \;\; i \in cc\ 4$$

PART II: OPEN CASE – REPRESENTATION FORMULAS

Reference equation – 1 dim input. Assumptions: FIFO + Fixed Support

$$X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$$

Waiting Times

$$W_n^i = X_n^i - T_n$$

Equation for Waiting Times by subtracting T_{n+1} on both sides of (1)

$$W_{n+1} = A_n \otimes C(-\tau_n) \otimes W_n \oplus B_{n+1}$$

$C(x) = x \otimes E$, $\tau_n = T_{n+1} - T_n$

Particular cases:

- 1-dim, with $B_n = 0$: $W_{n+1} = W_n \otimes (\sigma_n) \otimes (-\tau_n) \oplus 0$
- Deterministic case: $W_{n+1} = A \otimes C(-\tau) \otimes W_n \oplus B$

REPRESENTATION FORMULAS: OPEN CASE

Backward Coupling

– $\{A_n, B_n, \tau_n\}$ is stationary and ergodic, with all entries non ε integrable

– γ top Lyapunov exponent of $\{A_n\}$

If $\rho = \lambda\gamma < 1$, with $\lambda^{-1} = E(\tau_0)$, $\{W_n\}$ converges in total variation to a unique stationary sequence. The stationary law of W_n is that of

(REP1) $$W = B_0 \oplus \bigoplus_{n\geq 1} C(T_{-n}) \otimes D_n, \qquad D_n = A_{-1} \otimes \ldots \otimes A_{-n} \otimes B_{-n}$$

If $\rho > 1$, $|W_n|_{\oplus}$ tends to ∞ a.s. FB 92, Annals Prob.

Particular cases:

– 1-dim, with $B_n = 0$: W: Loynes's variable for G/G/1/∞ queue

– Deterministic case: $W = (A \otimes C)^* \otimes B$, unique solution of $W = A \otimes C \otimes W \oplus B$

EXTENSIONS: HIGHER DIMENSION INPUT

Division $u \in I\!R^m_{max}$, $v \in I\!R^n_{max}$

$$\frac{u}{v} \in I\!R^{n \times m}_{max}, \quad (\frac{u}{v})_{ij} = u_i - v_j$$

T dim J $X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$

Extended Waiting Time Variables

$$W_n = \frac{X_n}{T_n} \in I\!R^{K \times J}_{max}$$

Waiting Time Equation $U^+_n = \frac{T_{n+1}}{T_n}$, $U^-_n = \frac{T_n}{T_{n+1}}$

$$W_{n+1} = A_n \otimes W_n \otimes U^-_n \oplus B_{n+1} \otimes U^+_n \otimes U^-_n$$

For extending the ergodic theorem, $\{U^+_n\}$ should be stationary ergodic with $EU^+_{ii} = \lambda^{-1}$

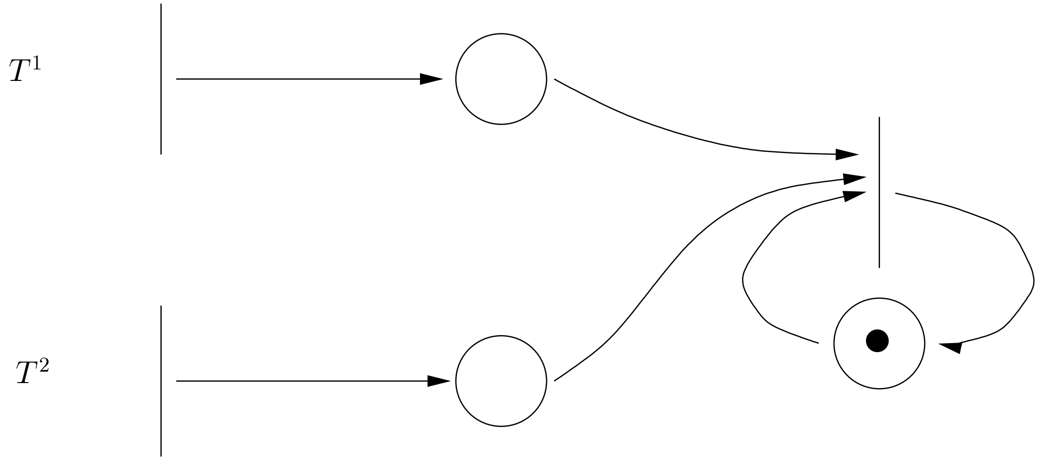


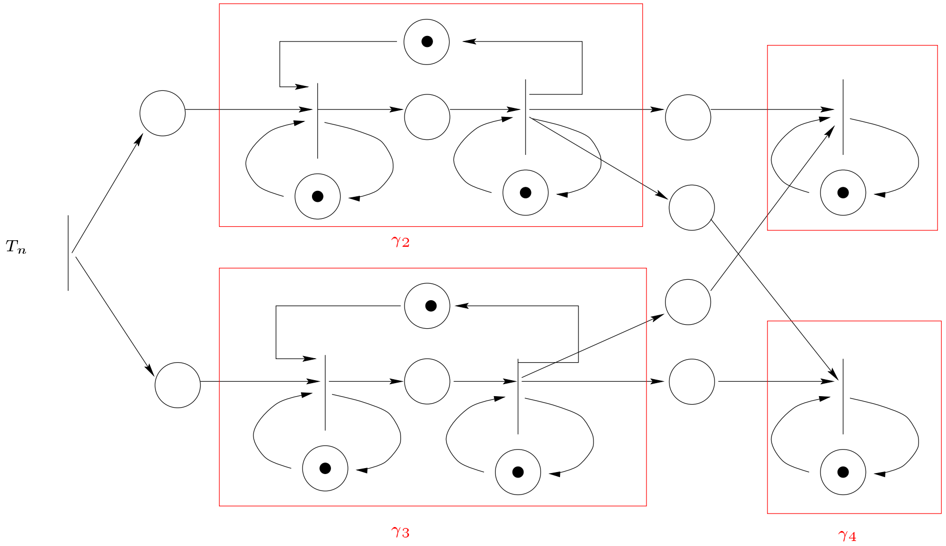
Figure 2: Solution to Harrison's stability problem in assembly lines

EXTENSIONS

A_n non-irreducible

– If $\oplus_{\tilde{v}} \gamma_{\tilde{v}} < \lambda^{-1}$, W_n converges in total variation to a ! finite limit

– Otherwise, for $\tilde{u}_0$ s.t. $\oplus_{\tilde{v} \in \bullet_{\tilde{u}_0}} \gamma_{\tilde{v}} > \lambda^{-1}$, $(W_n)_{i.}$ diverges a.s. for all i in cc $\tilde{u}_0$


T_n
γ_2
γ_3
γ_4

Open Systems: Computational Aspects

Two examples (with no direct motivations nor implications on TCP modelling)

– Light traffic expansions for the case with Poisson input

– Asymptotics for the case with heavy tailed service times

LIGHT TRAFFIC EXPANSIONS: MEAN RESPONSE TIMES
Open event graph with 1-dim Poisson input $W_n^i = X_n^i - T_n$

- $\{T_n\}$ Poisson process of intensity λ
- $\{\sigma_n^k,\ k = 1, \ldots, K\}$ i.i.d. sequence, for all k, $E[(\sigma_n^k)^{m+3}] < \infty$

From (REP1)

$$W = B_0 \oplus \bigoplus_{n \geq 1} C(T_{-n}) \otimes D_n,\ \ D_n = A_{-1} \otimes \ldots \otimes A_{-n} \otimes B_{-n}$$

$$\Downarrow$$

$$EW^i = \sum_{k=0}^{m} \lambda^k E p_{k+1}(D_0^i, D_1^i, \ldots, D_k^i)\ +\ \mathcal{O}(\lambda^{m+1})$$

If λ small enough, with $p_k(x_0, \ldots, x_{k-1})$ class of combinatorial polynomials
FB & V. Schmidt 96, Annals Appl. Prob.

$$p_k(x_0,\ldots,x_{k-1}) = \sum_{i=(i_0,\ldots,i_{k-1})\in S_k} (-1)^{q_k(i)}\, \frac{x_0^{i_0}}{i_0!}\frac{x_1^{i_1}}{i_1!}\cdots\frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!}$$

$$S_k = \{i \in I\!N^k : i_0 + .. + i_{k-1} = k;\ \text{if } i_s = l > 1,\ i_{[s-1]_k} = .. = i_{[s-l+1]_k} = 0\}$$

$$q_k(i_0,i_1,\ldots,i_{k-1}) = 1 + \sum_{s=0}^{k-1} \mathbf{1}(i_s > 0)$$

$$\begin{aligned}
p_1(x_0) &= x_0 \\
p_2(x_0,x_1) &= \frac{1}{2}\left[x_0^2 + x_1^2 - 2x_0x_1\right] \\
p_3(x_0,x_1,x_2) &= \frac{1}{6}\left[x_0^3 + x_1^3 + x_2^3 - 3(x_0^2x_1 + x_1^2x_2 + x_2^2x_0) + 6x_0x_1x_2\right] \\
p_4(x_0..x_3) &= \frac{1}{24}\left[x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4(x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0)\right. \\
&\quad -6(x_0^2x_2^2 + x_1^2x_3^2) + 12(x_0^2x_1x_2 + x_1^2x_2x_3 + x_2^2x_3x_0 + x_3^2x_0x_1) \\
&\quad \left.-24x_0x_1x_2x_3\right]
\end{aligned}$$

POLYNOMIALS – FIRST PROPERTIES

1. special values For all $k \geq 2$

$$p_k(0, \ldots, 0) = 0, \qquad p_k(1, \ldots, k) = \frac{1}{2}.$$

2. invariance by circular permutation: $k \geq 1$

$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_1, x_2 \ldots, x_{k-1}, x_0)$$

3. translation-invariance for all $t \in I\!R, k \geq 2$,

$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_0 + t, x_1 + t, \ldots, x_{k-1} + t)$$

4. multiplicative-invariance for all $t \in I\!R$,

$$p_k(tx_0, tx_1, \ldots, tx_{k-1}) = t^k p_k(x_0, x_1 \ldots, x_{k-1}).$$

5. differences

$$p_k(x_0, x_1, \ldots, x_{k-1}) - p_k(0, x_1, \ldots, x_{k-1})$$

$$= \frac{x_0^k}{k!} - \sum_{j=1}^{k-1} \frac{x_0^{k-j}}{(k-j)!}$$

$$\{ \; p_j(x_1, x_2, \ldots, x_j) - p_j(0, x_1, \ldots, x_{j-1}) \; \}$$

$$p_{k+1}(x_0, \ldots, x_{k-1}, x_k) - p_k(x_0, \ldots, x_{k-1}, 0)$$

$$= \frac{x_k^{k+1}}{(k+1)!} - \sum_{j=1}^{k} \frac{(x_{k-j})^{k+1-j}}{(k+1-j)!}$$

$$\{p_j(x_{k+1-j}..x_k) - p_j(x_{k+1-j}..x_{k-1}, 0)\}$$

POLYNOMIALS – COMBINATORIAL PROPERTIES

FB, S. Hasenfuss & V. Schmidt 98, Stoch. Models

$$p_1(x_0) = x_0$$

and Integral recurrence

$$p_{k+1}(x_0, \ldots, x_k) = \sum_{n=0}^{k-1} \int_{x_n - x_0}^{x_{n+1}-x_0}$$

$$\Big[p_k(\underbrace{x_0, \ldots, x_0}_{n}, x_{n+1} - u, \ldots, x_k - u)$$

$$-p_k(\underbrace{x_0, \ldots, x_0}_{n+1}, x_{n+1} - u, \ldots, x_{k-1} - u) \Big] \, du.$$

Direct recurrence

$$p_{k+1}(x_0, \ldots, x_k) = \frac{1}{k+1} \sum_{m=0}^{k} \{x_m - x_{(m-1 \bmod k+1)}\} p_k(x_{([m]_{k+1})}, \ldots, x_{([m+k-1]_{k+1})})$$

EXAMPLE: SINGLE SERVER QUEUE

Lindley's equation $W_{n+1} = (\sigma_n \otimes (-\tau_n) \otimes W_n) \oplus e$

$D_0 = 0$ and $D_k = \Sigma_{n=1}^{k} \sigma_{-n}$

$$p_2(D_0, D_1) = \frac{\sigma_{-1}^2}{2}, p_3(D_0..D_2) = \frac{\sigma_{-2}^3 - \sigma_{-1}^3 + 3\sigma_{-1}\sigma_{-2}^2}{6}$$

$$\begin{aligned} p_4(D_0, \ldots, D_3) = \frac{1}{24}(\sigma_{-1}^4 - 2\sigma_{-2}^4 + \sigma_{-3}^4 \\ +4(\sigma_{-1}\sigma_{-3}^3 - 2\sigma_{-1}\sigma_{-2}^3 + \sigma_{-2}\sigma_{-3}^3)) \\ -6(\sigma_{-1}^2\sigma_{-2}^2 - \sigma_{-2}^2\sigma_{-3}^2) \\ +12\left(\sigma_{-1}\sigma_{-2}\sigma_{-3}^2\right) \end{aligned}$$

Assume $\{\sigma_n\}$ i.i.d.

$$Ep_2(D_0, D_1) = \frac{E[\sigma_n^2]}{2}, \quad Ep_3(D_0, D_1, D_2) = \frac{E[\sigma_n^2]E[\sigma_n]}{2}$$

$$Ep_4(D_0, D_1, D_2, D_3) = \frac{E[\sigma_n^2]E[\sigma_n]^2}{2}$$

COROLLARIES

Assume moment and stability conditions

1. Pollaczek-Khinchine Formula

$$EW = \lambda\frac{E[\sigma_n^2]}{2} + \lambda^2\frac{E[\sigma_n^2]E[\sigma_n]}{2} + \lambda^3\frac{E[\sigma_n^2]E[\sigma_n]^2}{2} \quad + \mathcal{O}(\lambda^4).$$

$$EW = \frac{\lambda E[\sigma_n^2]}{2(1-\lambda E[\sigma_n])} = \frac{E[\sigma_n^2]}{2}\sum_{k=1}^{\infty}\lambda^k E[\sigma_n]^{k-1}$$

2. Conditional expectation formula $\mathcal{F}_\sigma$: σ–algebra generated by $\{\sigma_n\}$

$$\begin{aligned} E[W|\mathcal{F}_\sigma] \; = \; & \frac{\lambda}{2}\sigma_{-1}^2 + \frac{\lambda^2}{6}\left(\sigma_{-2}^3 - \sigma_{-1}^3 + 3\sigma_{-1}\sigma_{-2}^2\right) \\ & + \frac{\lambda^3}{24}(\sigma_{-1}^4 - 2\sigma_{-2}^4 + \sigma_{-3}^4 + 4(\sigma_{-1}\sigma_{-3}^3 - 2\sigma_{-1}\sigma_{-2}^3 + \sigma_{-2}\sigma_{-3}^3) \\ & - 6(\sigma_{-1}^2\sigma_{-2}^2 - \sigma_{-2}^2\sigma_{-3}^2) + 12(\sigma_{-1}\sigma_{-2}\sigma_{-3}^2)) + \ldots \end{aligned}$$

M/D TYPE : FORMULA FOR STATIONARY WAITING TIMES

Case D_n ultimately 1-pseudo-periodic (A CYC 1):

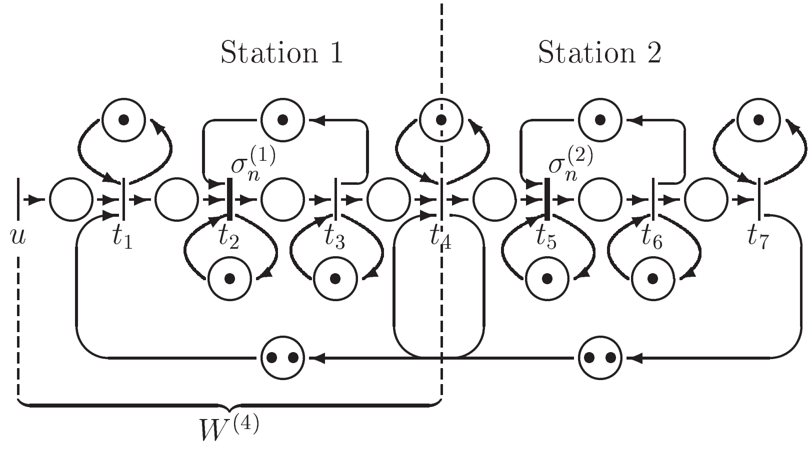
$$D_k = \begin{cases} \eta_k, & \text{for} \quad k = 0, \ldots, \xi - 1, \\ \eta_\xi + (k - \xi)\gamma, & \text{for} \quad k \geq \xi, \end{cases} .$$

$$\begin{aligned} I\!E\left[W^{\{\xi+1\}}\right] &= \frac{\gamma\rho}{2(1-\rho)} + \eta_\xi - \frac{\xi}{\lambda} \\ &+ \frac{(1-\rho)}{\lambda}\left\{ \sum_{m=0}^{\xi-1} e^{-\lambda(\eta_\xi + (m-\xi)\gamma - \eta_m)} + \sum_{m=0}^{\xi-2} e^{-\lambda(\eta_\xi + (m-\xi)\gamma - \eta_m)} \right. \\ &+ \sum_{l=1}^{\xi-1-m} e^{-\rho l}\{p_l(\lambda(\eta_\xi + (l+m-\xi)\gamma), \lambda\eta_{m+1}, \ldots, \lambda\eta_{m+l-1}) \\ &\left. - p_l(\lambda\eta_{m+1}, \ldots, \lambda\eta_{m+l})\} \right\} . \end{aligned}$$

M/D TYPE : FORMULA FOR STATIONARY WAITING TIMES *(continued)*

$$
\begin{aligned}
I\!E\left[W^{\{1\}}\right] &= \frac{\gamma\rho}{2(1-\rho)} + \eta_0 \\
I\!E\left[W^{\{2\}}\right] &= \frac{\gamma\rho}{2(1-\rho)} + \eta_1 - \frac{1}{\lambda}\left\{1-(1-\rho)e^{-\lambda(\eta_1-\gamma-\eta_0)}\right\} \\
I\!E\left[W^{\{3\}}\right] &= \frac{\gamma\rho}{2(1-\rho)} + \eta_2 - \frac{2}{\lambda} + \frac{(1-\rho)}{\lambda} \\
&\quad \left\{ e^{-\lambda(\eta_2-2\gamma-\eta_0)} + e^{-\lambda(\eta_2-\gamma-\eta_1)} + \lambda(\eta_2-\gamma-\eta_1)e^{-\lambda(\eta_2-\gamma-\eta_0)} \right\}
\end{aligned}
$$

EXAMPLE: TWO WINDOW FLOWS IN TANDEM



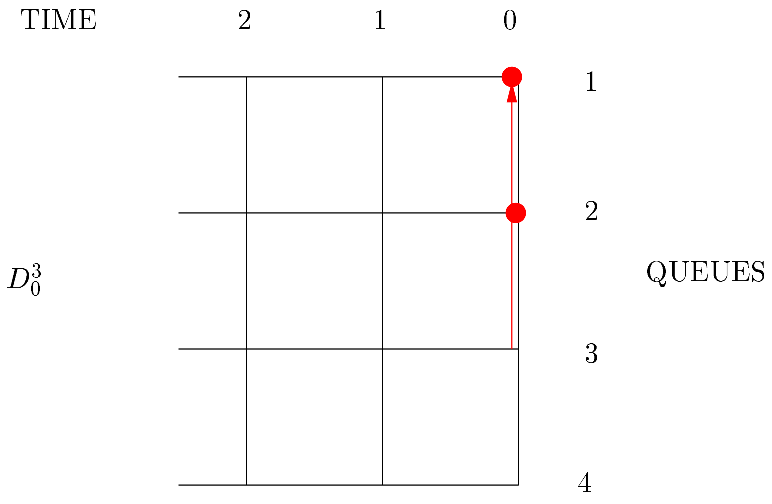
If $\sigma_n^{(1)} = 3$, $\sigma_n^{(2)} = 5$: $D_0^4 = 3$, $D_1^4 = 6$, $D_2^4 = 9$, $D_n^4 = 13 + 5\,(n-3)$, $n \geq 3$

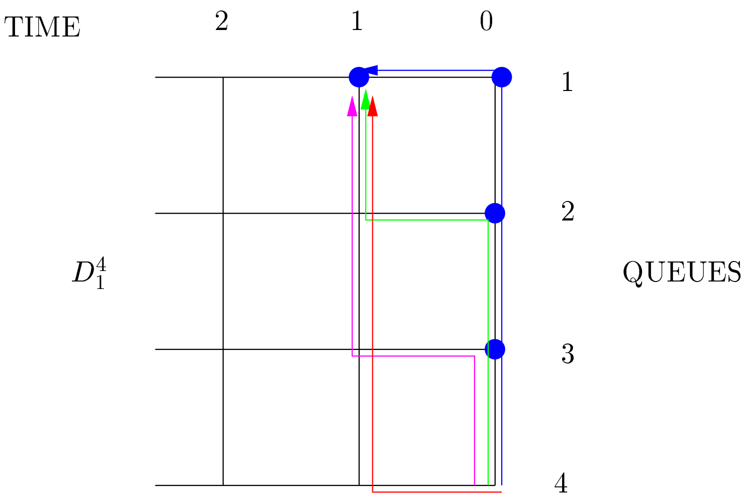
For $\lambda \in [0, 0.2)$

$$I\!E\,[W^4] = \frac{25\,\lambda}{2 - 10\,\lambda} + 13 - \frac{3}{\lambda} + \frac{(1-5\,\lambda)\left(e^{5\,\lambda} + e^{3\,\lambda} + e^{\lambda} - 3\,\lambda - \lambda\,e^{-2\,\lambda} - \frac{5}{2}\,\lambda^2\,e^{-5\,\lambda}\right)}{\lambda}$$

EXAMPLE M/GI-TYPE: QUEUES IN TANDEM: GENERAL CASE

$$D_n^i = \max_{1 \le l_n \le \ldots \le l_1 \le i} \left\{ \sum_{k=l_1}^{i-1} \sigma_0^k + \sum_{k=l_2}^{l_1} \sigma_{-1}^k + \ldots + \sum_{k=1}^{l_n} \sigma_{-n}^k \right\}$$





EXAMPLE M/GI-TYPE: QUEUES IN TANDEM: I.I.D. Case

$$D_1^i - D_0^i = \max_{1 \le l \le i} \left\{ \sigma_{-1}^{l} + \sum_{k=1}^{l-1} (\sigma_{-1}^{k} - \sigma_0^k) \right\}$$

$$D_2^i - D_0^i = \max_{1 \le l_2 \le l_1 \le i} \left\{ \sigma_{-2}^{l_2} + \sum_{k=1}^{l_2-1} (\sigma_{-2}^{k} - \sigma_0^k) + \sigma_{-1}^{l_1} + \sum_{k=l_2}^{l_1-1} (\sigma_{-1}^{k} - \sigma_0^k) \right\}$$

$$\mathbb{E} p_1(D_0^i) = \sum_{k=1}^{i-1} \mathbb{E} \sigma_0^k, \quad \mathbb{E} p_2(D_0^i, D_1^i) = \frac{1}{2} \mathbb{E}(D_1^i - D_0^i)^2$$

$$\begin{aligned} \mathbb{E} p_3(D_0^i, D_1^i, D_2^i) \ = \ & \frac{1}{6} \{ \mathbb{E}(D_1^i - D_0^i)^3 + \mathbb{E}(D_2^i - D_0^i)^3 \\ & -3 \mathbb{E}(D_1^i - D_0^i)^2 (D_2^i - D_0^i) \} \end{aligned}$$

EXAMPLE M/GI-TYPE
Case with identical random service times on all nodes

$$\begin{aligned} D_0^3 &= 2\sigma(0), \quad D_1^3 = \sigma(1) + 2\max\{\sigma(0), \sigma(1)\} \\ D_2^3 &= \sigma(1) + \sigma(2) + 2\max\{\sigma(0), \sigma(1), \sigma(2)\}, \end{aligned}$$

$$\begin{aligned} \mathbb{E}[W^3] \;=\; & 2\mathbb{E}[\sigma(0)] + \frac{\lambda}{2}\mathbb{E}[(2(\sigma(1)-\sigma(0))^+ + \sigma(1))^2] \\ + \; & \frac{\lambda^2}{6}\Big\{ \mathbb{E}\Big[\big(2(\sigma(1)-\sigma(0))^+ + \sigma(1)\big)^3 \\ & \quad + \big(2\max(\sigma(1)-\sigma(0), \sigma(2)-\sigma(0), 0) + \sigma(1) + \sigma(2)\big)^3 \\ & \quad - 3\big(2(\sigma(1)-\sigma(0))^+ + \sigma(1)\big)^2 \\ & \qquad \times \big(2\max(\sigma(1)-\sigma(0), \sigma(2)-\sigma(0), 0) + \sigma(1) + \sigma(2)\big)\Big] \Big\} \\ + \; & \mathcal{O}(\lambda^3) \end{aligned}$$

GENERIC EXPANSION THEOREM
Open event graph with 1-dim Poisson input

– $f : I\!R^+ \to I\!R^+$ be a Riemann-integrable, non-negative function, $f(x) \leq c\, x^\nu$ for all $x \geq 0$. $f(x) = x^p \to$ moments, $f(x) = e^{-sx} \to$ Laplace Transform ...

– $F^{[0]}(x) = f(x)$,

$$F^{[n]}(x) = \int F^{[n-1]}(x)\, dx.$$

– $q_1(x_0) = f(x_0)$, and

$$q_{k+1}(x_0, x_1, \ldots, x_k) = \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} F^{[k]}(x_n) - \sum_{n=0}^{k-1} \sum_{j=n}^{k-1} \binom{j}{n} (-1)^{j-n} F^{[j]}(x_n) \{p_{k-j}(x_{n+1}, \ldots, x_{k-j+n}) - p_{k-j}(x_n, \ldots, x_{k-j+n-1})\}$$

EXPANSION THEOREM

$$G_n = \bigoplus_i \left\{ \left(A_{-1} \otimes A_{-2} \otimes \ldots \otimes A_{-n} \otimes (B_{-n} \oplus O) \right)^i \right\}$$

$$H_{n,\ell} = \bigoplus_i \left\{ \left(A_{-(n\ell+1)} \otimes A_{-(n\ell+2)} \otimes \ldots \otimes A_{-(n+1)\ell} \otimes (B_{-(n+1)\ell} \oplus O) \right)^i \right\}$$

1. If the $(\nu + m + 2)$-th moment of H_n is finite,

$$\mathbb{E}[f(W^i)] = \sum_{k=0}^{m} \lambda^k \, \mathbb{E}[\, q_{k+1}(D_0^i, D_1^i, \ldots, D_k^i) \,] + \mathcal{O}(\lambda^{m+1}).$$

2. If $\exists\ \theta \in [0, \theta^*)$, $\mathbb{E}[\, e^{\theta G_n}] \leq L_\theta \left(\phi(\theta)\right)^n$, for finite functions $\phi(\theta) > 1$ and L_θ, then, $\mathbb{E}[f(W^i)]$ is infinitely differentiable in 0^+, the expansion holds for all m, and

$$\lim_{\lambda \downarrow 0} \frac{d^n}{d\lambda^n} \mathbb{E}[f(W^i)](\lambda) = \mathbb{E}[\, q_{n+1}(D_0^i, D_1^i, \ldots, D_n^i) \,]$$

FB, S. Hasenfuss & V. Schmidt 97, Questa

APPLICATION TO LAPLACE TRANSFORMS

$$\mathbb{E}[e^{-sW^i}] = \sum_{k=0}^{m} \lambda^k \, \mathbb{E}[\, q_{k+1}(s; D_0^i, D_1^i, \ldots, D_k^i) \,] + \mathcal{O}(\lambda^{m+1}),$$

where $q_1(s; x_0) = e^{-sx_0}$,

$$q_{k+1}(s; x_0, x_1, \ldots, x_k)$$

$$= \frac{1}{s^k} \sum_{n=0}^{k} \binom{k}{n} (-1)^n e^{-sx_n} - \sum_{j=0}^{k-1} \frac{1}{s^j} \sum_{n=0}^{j} \binom{j}{n} (-1)^n e^{-sx_n}$$
$$\{p_{k-j}(x_{n+1}, \ldots, x_{k-j+n}) - p_{k-j}(x_n, \ldots, x_{k-j+n-1})\}$$

$$= \frac{q_k(s; x_0, x_1, \ldots, x_{k-1}) - q_k(s; x_1, x_2, \ldots, x_k)}{s}$$
$$- e^{-sx_0}[p_k(x_1, x_2, \ldots, x_k) - p_k(x_0, x_1, \ldots, x_{k-1})],$$

p_k polynomials defined before

APPLICATION: SECOND MOMENT, VARIANCE

$$E(W^i)^2 = \sum_{k=0}^{m} \lambda^k \, E q_{k+1}(D_0^i, D_1^i, \ldots, D_k^i) + \mathcal{O}(\lambda^{m+1}),$$

with

$$\begin{aligned} q_1(x_0) &= x_0^2 \\ q_2(x_0, x_1) &= \frac{1}{3}\left(2x_0^3 + x_1^3 - 3x_0^2 x_1\right) \\ q_3(x_0, x_1, x2) &= \frac{1}{12}\left(3x_0^4 + 2x_1^4 + x_2^4 - 8x_0^3 x_1\right. \\ &\quad \left. -4x_1^3 x_2 - 6x_0^2 x_2^2 + 12 x_0^2 x_1 x_2\right). \end{aligned}$$

$$\mathrm{Var}\left(W^i\right) = \sum_{k=0}^{m-1} \lambda^k \, q^\star_{k+1}(D_0^i, D_1^i, \ldots, D_k^i) + \mathcal{O}(\lambda^m),$$

with

$$q^\star_{k+1}(\ldots) = E\, q_{k+1}(\ldots) - \sum_{j=0}^{k} E\, p_{j+1}(\ldots)\, I\!E\, p_{k+1-j}(\ldots)$$

TAYLOR EXPANSIONS FOR TRANSIENT WAITING TIMES
Non-autonomous event graph with 1-dim Poisson input

If

$$\mathbb{E}\left[e^{aD_n^i}\right] < \infty$$

for some positive a, then $\mathbb{E}W_n^i$ is an analytical function of λ in 0, and for each $\lambda \geq 0$,

$$\begin{aligned}\mathbb{E}[W_{n+1}^i] &= \sum_{k=0}^{n} \lambda^k \,\mathbb{E}\left[p_{k+1}(D_0^i, \ldots, D_k^i)\right] \\ &+ \sum_{k=n+1}^{\infty} \lambda^k \,\mathbb{E}[p_{k+1}(D_0^i, \ldots, D_{n-1}^i, \underbrace{D_n^i, \ldots, D_n^i}_{k+1-n})]\end{aligned}$$

REPRESENTATION OF TRANSIENT WAITING TIMES

For each arrival intensity $\lambda > 0$ and for all $n \geq 0$,

$$\begin{aligned} I\!E[W_{n+1}] &= \frac{1}{\lambda}\sum_{m=0}^{n-1} I\!E\left[e^{-\lambda(D_n-D_m)} - 1\right] + I\!E[D_n] \\ &+ \sum_{j=0}^{n-2} \lambda^j \sum_{m=0}^{n-j-2} I\!E\left[e^{-\lambda(D_n-D_m)} \times \right. \end{aligned}$$

$$\left. \{p_{j+1}(D_n, D_{m+1}..D_{m+j}) - p_{j+1}(D_{m+1}..D_{m+1+j})\}\right]$$

First terms:

$$\begin{aligned} I\!E[W_1] &= I\!E[D_0], \\ I\!E[W_2] &= \frac{1}{\lambda} I\!E[e^{-\lambda(D_1-D_0)} - 1] + I\!E[D_1], \\ I\!E[W_3] &= \frac{1}{\lambda}\{I\!E[e^{-\lambda(D_2-D_0)} - 1] + I\!E[e^{-\lambda(D_2-D_1)} - 1]\} \\ &\quad + \; + I\!E[D_2] + I\!E\left[(D_2 - D_1)\, e^{-\lambda(D_2-D_0)}\right] \end{aligned}$$

B H S, 1996

IDEA OF PROOF FOR EXPANSIONS

Reiman & Simon light traffic formulas for functionals of Poisson PP

– $D = \{D_n\}$, $I\!R^K$-valued random variables

– $T = \{T_n\}$, Poisson point process of intensity λ

Under some regularity conditions, $f(\lambda) = I\!E\,\psi(T, D)$ is infinitely differentiable in λ in 0^+ and

$$\lim_{\lambda \downarrow 0} \frac{d^k}{d\lambda^k} I\!E\,\psi(T,D) = I\!E \int_{-\infty}^{0} \ldots \int_{-\infty}^{0} \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell}\, \psi\Big(\sum_{j=1}^{\ell} \delta_{s_j}, D\Big)\, ds_1 \ldots ds_k$$

Take

$$\psi(T,D) = W(T,D) \quad = \quad \Big(D_0 \vee \bigvee_{n \geq 1} (D_n + T_{-n})\Big).$$

and you find the polynomials (thanks to Maple).

IDEA OF PROOF FOR EXPANSIONS: REGULARITY

Finite Horizon: T^t, restriction of T on $[t, 0)$

$$\mathbb{E}\,\psi(T^t, D) \;=\; e^{\lambda t} \sum_{k=0}^{\infty} \frac{(-\lambda t)^k}{k!}\, \beta_k(t),$$

where

$$\beta_k(t) \;=\; \frac{1}{(-t)^k} \int_t^0 \ldots \int_t^0 \;\mathbb{E}\,\psi\Big(\sum_{j=1}^{k} \delta_{u_j}, D\Big)\, du_1 \ldots du_k.$$

M/D TYPE : FORMULA FOR STATIONARY WAITING TIMES *(continued)*

Interchange of limits in this series Assume that $I\!E|\psi(T^t, D)| < \infty$ for $\lambda = \lambda_0$. Then

(a) the function $f_t(\lambda) = I\!E\psi(T^t, D)$ is finite and infinitely differentiable on the interval $[0, \lambda_0)$;

(b) for all $\ell = 1, 2, \ldots$ and for all $\lambda \in [0, \lambda_0)$,

$$\int_t^0 \ldots \int_t^0 \; I\!E \mid \psi(T^t + \sum_{j=1}^{\ell} \delta_{s_j}, D) \mid ds_1 \ldots ds_\ell \; < \; \infty;$$

(c) for all $n = 1, 2, \ldots$ and $\lambda \in [0, \lambda_0)$,

$$\frac{d^n}{d\lambda^n} f_t(\lambda) = \int_t^0 \ldots \int_t^0$$

$$\sum_{\ell=0}^{n} \binom{n}{\ell} (-1)^{n-\ell} \; I\!E\, \psi(T^t + \sum_{j=1}^{\ell} \delta_{s_j}, D)\, ds_1 \ldots ds_n$$

REGULARITY – INFINITE HORIZON

If $I\!\!E\psi(T^t, D)$ converges uniformly in λ to $I\!\!E\psi(T, D)$ when $t \to \infty$, proof concluded.

Sufficient conditions for uniform convergence:

– Uniform Coupling: for all k, there exists a random variable τ_k such that

$$\psi(T + \sum_{j=1}^{\ell} \delta_{s_j}, Z) \;=\; \psi(T^t + \sum_{j=1}^{\ell} \delta_{s_j}, Z),$$

for all $t \le s_j \le 0$ and $\ell \le k$ when $t < \tau_k$

– Uniform Boundedness: for $k > 0$, there is a random variable η_k such that $I\!\!E\,\eta_k < \infty$ and

$$\int_t^0 \ldots \int_t^0 \left| \sum_{\ell=0}^{k} \binom{k}{\ell} (-1)^{k-\ell}\, \psi(T^t + \sum_{j=1}^{\ell} \delta_{s_j}, Z) \right| ds_1 \ldots ds_k \le \eta_k,$$

for all $t \le 0$ and for all $\lambda \in [0, \lambda_0]$.

CURRENT DIRECTIONS OF RESEARCH ON LIGHT TRAFFIC EXPANSIONS

1. Region of analyticity of expansions
   - 1.1. less than stability region A. Jean-Marie 97, INRIA Report
   - 1.2. via contraction FB & D. Hong 99, $\rightarrow$ JAP
   - 1.3. via weak derivatives B. Heidergott 99, INFORMS
   - 1.4. via rational languages S. Gaubert & D. Hong 99, INFORMS
2. Further functionals or variables
   - 2.1. Transient FB, S. Hasenfuss & V. Schmidt 97, Questa and B. Heidergott 99
   - 2.2. Joint Laplace transforms of two waiting times H. Ahyan & FB 99, INFORMS
   - 2.3. Complexity V. Blondel, S. Gaubert, J. Tsitisklis 99

SUBEXPONENTIAL ASYMPTOTIC TAIL BEHAVIOR
1-D Problem: $GI/GI/1/\infty$ Queue
J.W. Cohen 73, A.G. Pakes 75, N. Veraverbeke 77

– $\{\tau_n\}$ i.i.d., integrable

– $\{\sigma_n\}$ i.i.d., integrable, such that $F_\sigma(x) = P(\sigma < x)$ and

$$F^s_\sigma(x) = \frac{\int_0^x 1 - F_\sigma(u)du}{\int_0^\infty 1 - F_\sigma(u)du}$$

both subexponential: $\mathcal{S}^*$ (e.g. Pareto, Log Normal, Weibull)

$$W_{n+1} = (W_n + \sigma_n - \tau_n)^+ \to_{n\to\infty} W$$

If $\rho = \frac{E(\sigma)}{E(\tau)} < 1$

$$P(W > x) \sim \frac{\rho}{1-\rho} F^s_\sigma(x)$$

SUBEXPONENTIAL ASYMPTOTIC TAIL BEHAVIOR
General Dimension

Technical Assumptions $W_{n+1} = A_n \otimes C(-\tau_n) \otimes W_n \oplus B_{n+1}$

– A_n, B_n i.i.d., fixed Support, D_n non decreasing, stability;

– $\max_i (A_{-1} \otimes \cdots \otimes A_{-n} \otimes B_{-n})_i \leq \max_{ij} (A_{-1} \otimes \cdots \otimes A_{-n} \otimes A_{-n-1})_{ij}$

– There exists a coordinate i_0 such that

* $F_{i_o}(x) = P[(A_0)_{i_0 i_0} \leq x] \in \mathcal{S}^*$.
* $\max_{ij} (A_{-1} \otimes \cdots \otimes A_{-n})_{ij} \leq \Sigma_{l=1}^n (A_{-l})_{i_0 i_0} + H_n$
  with
  · H_n independent of $\Sigma_{l=1}^n (A_{-l})_{i_0 i_0}$
  · $P[H_n > x] = o(1 - F_{i_0}(x))$

MAIN RESULT – IRREDUCIBLE CASE

Under the technical assumptions, $W_n \to W$ in total variation and

$$P(W > x) \sim \frac{\lambda \mu_{i_0}}{1 - \lambda\gamma} F^s_{i_0}(x)$$

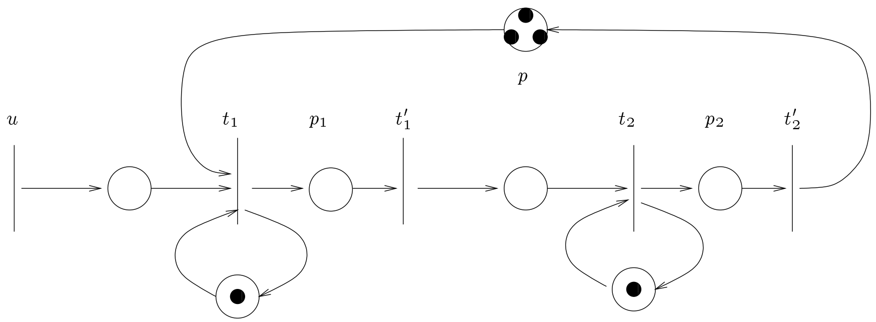
with

- $\lambda = E(\tau)^{-1}$
- γ Lyapunov exponent of $\{A_n\}$
- μ_{i_0} mean value of F_{i_0}.
- $F^s_{i_0}$ excess distribution of F_{i_0}.

FB, S. Schlegel & V. Schmidt 99, Questa

EXAMPLE: WINDOW FLOW CONTROL

Two Routers, Window equal to K, σ_n^2 subexponential



For appropriate state variables (after state extension and reduction)

$$A_{n-1} \in I\!R^{K+1,K+1} = \begin{pmatrix} \sigma_n^1 & \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 & \sigma_n^2 & \varepsilon & \cdots & \varepsilon & \sigma_n^1 + \sigma_n^2 \\ \varepsilon & 0 & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & & \vdots & \varepsilon \\ \vdots & \vdots & & \ddots & \vdots & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & 0 & \varepsilon \end{pmatrix}, \quad B_n = \begin{pmatrix} \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix}$$

IDEA OF PROOF

Two key sequences Fix i, with $B_n^i \geq 0$, fix $p \geq q$

- $g_{p,q} = \max_i (A_{p-1} \otimes \cdots \otimes A_{q-1})_{ii}$: super additive;
- $G_{p,q} = \max_{jj'} (A_{p-1} \otimes \cdots \otimes A_{q-1})_{jj'}$: sub additive;

From ergodic theorems, both satisfy a SLLN with slope γ.

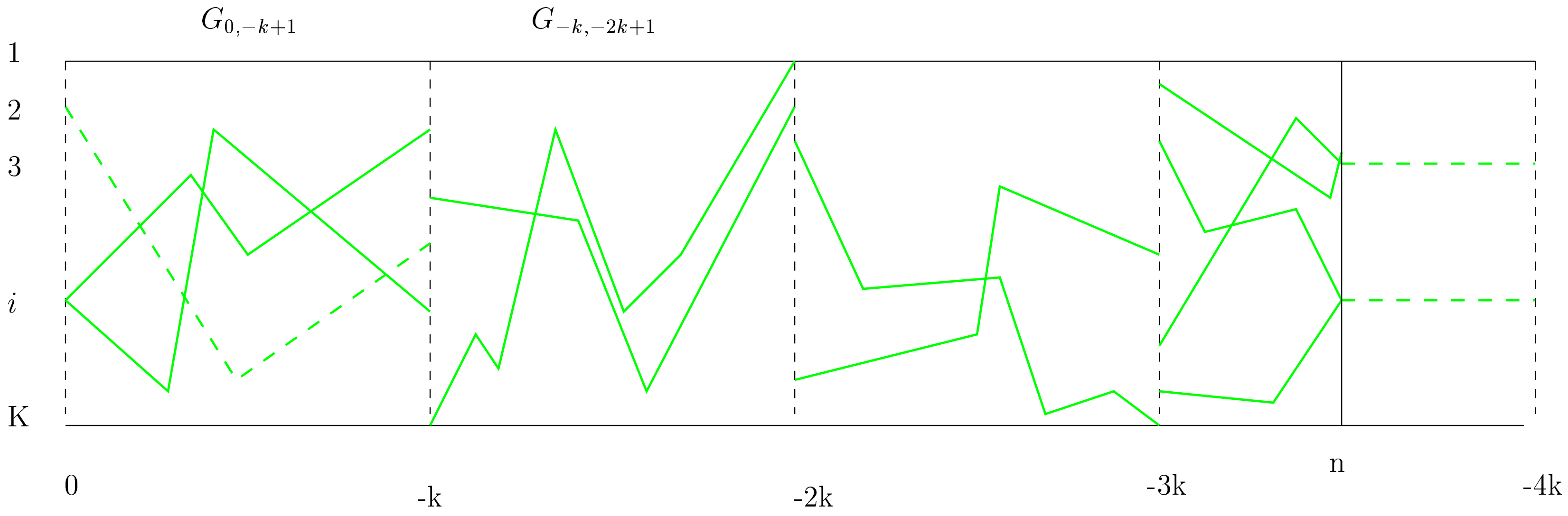
Upper Bound:

From Assumptions $\quad D_n^i \leq \Sigma_{l=0}^{\lfloor \frac{n+1}{k} \rfloor} G_{-lk,-(l+1)k+1}$

From (REP1) $\quad W^i \leq \sup_{p \geq 0} \Sigma_{l=0}^{p} G_{-lk,-(l+1)k+1} + T_{-pk}$

Max of a random walk again!

$$\limsup \frac{P[W^i > x]}{\overline{G}_k^s(x)} \leq \frac{\frac{\lambda}{k} E G_{1,k}}{1 - \frac{\lambda}{k} E G_{1,k}}, \qquad \overline{G}_k^s(x) \sim \frac{k \mu_{i_0}}{E G_{1,k}} \overline{F}_{i_0}^s(x)$$


$G_{0,-k+1}$
$G_{-k,-2k+1}$
1
2
3
i
K
0
-k
-2k
-3k
n
-4k

MORE ON SUBEXPONENTIAL TAILS

1. non irreducible case and several subexponential distributions FB, S. Schlegel & V. Schmidt 99, Questa
2. via Harris recurrence T. Huang & K. Sigman 99, Questa for tandem queues
3. Exact asymtotics for the reducible case FB, S. Foss 00

SIMULATION

- Principles: do matrix products (or even better, matrix-vector products) rather than discrete event simulation;
- Complexity: cost for simulating the transmission of n packets over K routers linear in n and a most quadratic in Kw^* (in fact $\sim 2K^2$).

PARALLEL SIMULATION

Time parallel algorithm – Basic Model

Estimation of throughput via the ergodic theorem

$$\gamma^i = \lim_n \frac{X_n^i}{n} \quad P\ a.s.$$

Non-autonomous equation

$$X_{n+1} = A_n \otimes X_n \oplus H \otimes T_{n+1}$$

Autonomous equivalent form: $Y_{n+1} = B_n \otimes Y_n$

$$Y_{n+1} = \begin{pmatrix} T_{n+2} \\ X_{n+1} \end{pmatrix} = \begin{pmatrix} \tau_{n+1} & \varepsilon \\ H & A_n \end{pmatrix} \otimes \begin{pmatrix} T_{n+1} \\ X_n \end{pmatrix}$$

For instance, the matrices $\{B_n\}$ are i.i.d. if

– input process is renewal

– service times are i.i.d.

Possibility of sampling the B_n matrices independently

ALGORITHM AND COMPLEXITY

Simulation problem Compute

$$Y_{n+1} = B_n \otimes B_{n-1} \otimes \ldots \otimes B_0 \otimes Y_0$$

for n very large

Parallel algorithm – due to

1. associativity of $\otimes$ in $I\!R_{max}^{K,K}$
2. possibility of sampling the matrices B_n independently

Time complexity of the computation of X_n with P processors

$$\mathcal{O}\left(\frac{nK^3}{P} + K^3 \log P\right)$$

FB & Canales, 93 Extension of Greenberg, Mitrani, Lubatchevsky, 91 for $G/G/1$

Extensions to Markov dependent matrices

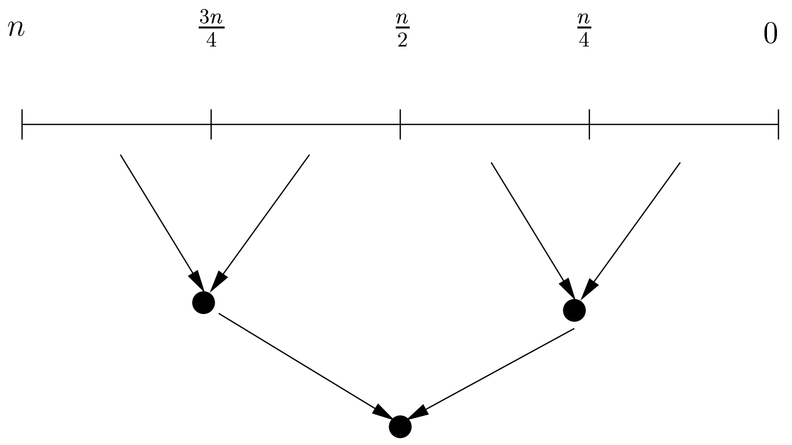


Figure 3: Parallel prefix algorithm – 4 processors

GENERALIZATIONS

1. Beyond Fixed Support

   1.1. Tetris models Gaubert & Mairesse 98, Idempotency, TCP!

2. Inf Convolutions

   2.1. Deterministic service guarantees C.S. Chang 97, Infocom; R. Agrawal & R. Rajan 96, Allerton; C.S. Chang, SPRINGER 00

   2.2. Stochastic case R. Agrawal & FB 98, INRIA report

3. Infinite Dimensional Systems

   3.1. Relations with literature on percolation

   3.2. Tandem case Glynn and Whitt 91 Annals Appl. Prob.; FB, A. Borovkov & J. Mairesse 99, J. Martin 00

2.4: DETERMINISTIC SETTING: TRAFFIC REGULATION

Deterministic Bounds in Stochastic Networks

Yet another semi-ring: Inf Convolutions

- F set of non-decreasing functions $I\!R^+ \to I\!R^+ \cup \{\infty\}$
- $a \oplus b(t) = \min(a(t), b(t))$
- $a \otimes b(t) = \inf_{0 \le u \le t} a(u) + b(t-u)$ with neutral e

Regulation

- a process to be regulated
- b enveloppe process (e.g. $b(t) = \sigma + \rho t$)

What is the largest process c such that

$$c(t) \le a(t), \qquad c(t) - c(u) \le b(t-u), \ \forall u \le t?$$

Answer: $c \le a, \quad b \le c \otimes b \quad \Rightarrow c \le a \oplus c \otimes b$

$$c_0 = a \otimes b^* \qquad \text{with} \quad b^* = e \oplus b \oplus b^2 \oplus \cdots$$

MOTIVATIONS FOR STOCHASTIC FRAMEWORK

– F set of functions $(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^+ \cup \{\infty\}$: $a(s,t), \; s \leq t$

– $a \oplus b(s,t) = \min(a(s,t), b(s,t))$

– $a \otimes b(s,t) = \inf_{s \leq u \leq t} a(s,u) + b(u,t)$

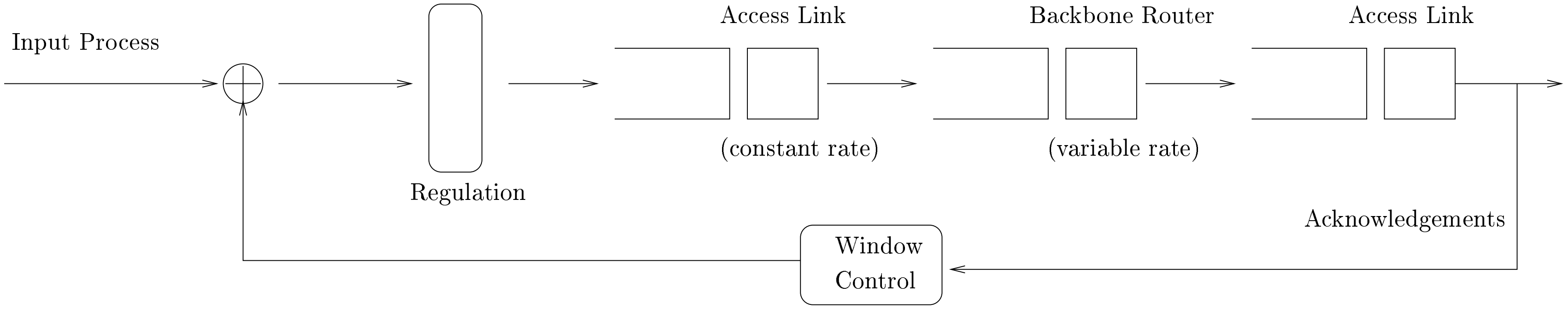


Figure 4: Results in progess: Lyapunov Exponents, Representation Formulas, Tails e.t.c.