Flow Control in Stochastic Networks

François Baccelli

INRIA & ENS

E-mail: Francois.Baccelli@ens.fr

Madison Workshop 2000

Structure of the Course

1. Lecture 1 TCP and its evolution equations;
   joint work with Dohy Hong, Sigcomm '00
2. Lecture 2 The (max,plus) algebra: Spectral properties of matrices and fixed point equations
3. Lecture 3 The (max,plus) algebra: Random media
4. Lecture 4 TCP with cross traffic;
   joint work with Thomas Bonald, Questa (1999)
5. Lecture 5 Beyond (max,plus).

Summary of Lecture 2

1. Part I Linear Systems in the (max,plus) Algebra
   - From an implicit system to a product of matrices
   - Open systems
   - Petri nets and Event Graphs
2. Part II Linear Algebra
   - Spectral theory of matrices
   - Affine equations
   - Cyclicity theorem
3. Part III Periodic Regimes of TCP and Spectral Properties
   - Deterministic case
   - Random losses

PART I: THE (max,plus) ALGEBRA

Maslov 73, Gondran-Minoux 77, Cuninghame Green 79, Cohen-Quadrat-Viot 83, FB-Cohen-Olsder-Quadrat 92

1. Scalar Semi-Field $(\mathbb{R}_{max}, \oplus, \otimes)$

– $\mathbb{R}_{max} = \mathbb{R} \cup \{\varepsilon = -\infty\}$, $\oplus = \max$, $\otimes = +$

– $(\mathbb{R}_{max}, \oplus)$ commutative idempotent semi-group

$$a \oplus b = b \oplus a$$

$$a \oplus \varepsilon = \varepsilon \oplus a = a$$

$$a \oplus a = a$$

– $(\mathbb{R}_{max}, \otimes)$ commutative group neutral element $e = 0$

– $\otimes$ distributes over $\oplus$

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

2. Matrix Semi-Ring $(\mathbb{R}_{max}^{K,K}, \oplus, \otimes)$

– Matrix addition Neutral $\mathcal{E}$, $\mathcal{E}_{ij} = \varepsilon$

$$(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$$

– Matrix multiplication Neutral E, $E_{ij} = \varepsilon,\ i \neq j$, $E_{ii} = e$

$$(A \otimes B)_{ij} = \bigoplus_k A_{ik} \otimes B_{kj}$$

Example

$$\begin{pmatrix} 1 & 4 \\ -3 & 0 \end{pmatrix} \otimes \begin{pmatrix} -7 & 2 \\ \varepsilon & 3 \end{pmatrix} = \begin{pmatrix} -6 & 7 \\ -10 & 3 \end{pmatrix}$$

TRANSFORMATION OF AN IMPLICIT (max,plus) LINEAR RECURRENCE

– $M \in I\!N$: order of the recurrence

– $X_n^k \in I\!R$: state variables

– $A(n,p) \in I\!R_{max}^{K \times K}$: matrices

Recurrence Equation $X_n = (X_n^k) \in I\!R_{max}^{K \times 1}$

$$X_n = A(n,0) \otimes X_n \oplus A(n,1) \otimes X_{n-1} \ldots \oplus A(n,M) \otimes X_{n-M}$$

TRANSFORMATION 1: CONSTRUCTIVE RECURRENCES

- If the matrix $A(n,0)$ is strictly lower triangular up to a change of basis then following series converges

$$A(n,0)^\star = E \oplus A(n,0) \oplus A(n,0)^2 \oplus \ldots$$

- Transformation $X_n = A(n,0)X_n \oplus A(n,1) \otimes X_{n-1} .. A(n,M)X_{n-M} \to$

$$X_n = \overline{A}(n,1)X_{n-1} \oplus \ldots \oplus \overline{A}(n,M)X_{n-M}$$

with $\overline{A}(n,p) = A(n,0)^\star A(n,p)$

TRANSFORMATION 2: RECURRENCE OF ORDER 1

State extension $\rightarrow$ Standard form

$$\begin{pmatrix} X_n \\ X_{n-1} \\ \vdots \\ X_{n-M+1} \end{pmatrix} = \begin{pmatrix} \overline{A}(n,1) & \overline{A}(n,2) & \dots & \dots & \overline{A}(n,M) \\ E & \mathcal{E} & \dots & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & E & \ddots & \vdots & \mathcal{E} \\ \vdots & \ddots & E & \mathcal{E} & \mathcal{E} \\ \mathcal{E} & \dots & \mathcal{E} & E & \mathcal{E} \end{pmatrix} \begin{pmatrix} X_{n-1} \\ X_{n-2} \\ \vdots \\ X_{n-M} \end{pmatrix}$$

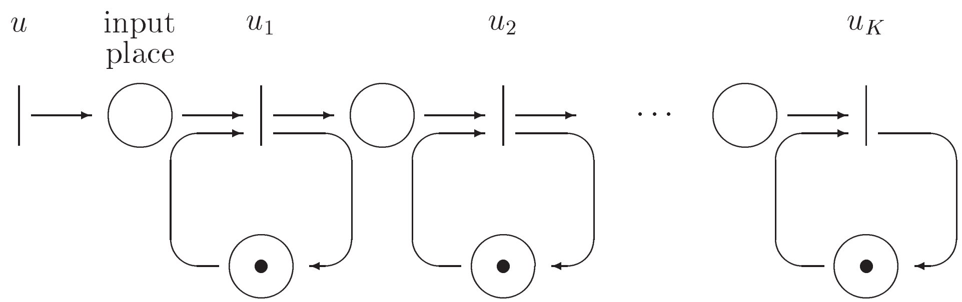
CONCLUSION FROM TRANSFORMATIONS 1 & 2

A recurrence of order M admits a representation as a $(max, plus)$-linear recurrence of order 1

$$\widetilde{X}_{n+1} = I\!P_n \otimes \widetilde{X}_n$$

Builds on result by Cohen-Dubois-Quadrat-Viot 83, CDC on the deterministic case $(A(n,k) = A_k, \forall n)$.

OPEN EXAMPLE: TANDEM QUEUES



$$X_n = A(n,0)X_n \oplus A(n,1)X_{n-1} \oplus B(n,0)U_n$$

$$A(n,0) = \begin{pmatrix} \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma_n^2 & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \cdots & \sigma_n^{K-2} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_n^{K-1} & \varepsilon \end{pmatrix} A(n,1) = \begin{pmatrix} \sigma_{n-1}^1 & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \sigma_{n-1}^2 & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma_{n-1}^3 & \cdots & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \sigma_{n-1}^{K-1} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_{n-1}^{K} \end{pmatrix} B(n,0) = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}$$

Transformation 1

$$X_n = A(n,0)X_n \oplus A(n,1)X_{n-1} \oplus B(n,0)U_n$$

$$X_n = A(n,0)^{\star}A(n,1) \otimes X_{n-1} \oplus A(n,0)^{\star} \otimes B(n,0)U_n$$

Standard Representation for open systems

$$\widetilde{X}_{n+1} = I\!P_n \otimes \widetilde{X}_n \oplus \widetilde{B}_{n+1}U_{n+1}$$

Here

$$(I\!P_n)_{ij} = \begin{cases} \varepsilon & \text{if} \quad i < j \\ \sum_{k=j}^{i-1} \sigma_{n+1}^k + \sigma_n^j & \text{if} \quad i \geq j \end{cases}, \qquad \widetilde{B}_n = \begin{pmatrix} 0 \\ \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 \\ \sigma_n^1 + \sigma_n^2 + \sigma_n^3 \\ \vdots \\ \sum_{k=1}^{K-1} \sigma_n^k \end{pmatrix}.$$

REPRESENTATION FORMULAS: OPEN CASE

Reference equation – 1 dim input.

$$X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$$

Waiting Times

$$W_n^i = X_n^i - T_n$$

Equation for Waiting Times by subtracting T_{n+1} on both sides of (1)

$$W_{n+1} = A_n \otimes C(-\tau_n) \otimes W_n \oplus B_{n+1}$$

$C(x) = x \otimes E$, $\tau_n = T_{n+1} - T_n$

Particular cases:

– 1-dim, with $B_n = 0$: $W_{n+1} = W_n \otimes (\sigma_n) \otimes (-\tau_n) \oplus 0$

– Deterministic case: $W_{n+1} = A \otimes C(-\tau) \otimes W_n \oplus B$

EXTENSIONS: HIGHER DIMENSION INPUT

Division $u \in \mathbb{R}^m_{max}$, $v \in \mathbb{R}^n_{max}$

$$\frac{u}{v} \in \mathbb{R}^{n\times m}_{max}, \quad (\frac{u}{v})_{ij} = u_i - v_j$$

T dim J $X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$

Extended Waiting Time Variables

$$W_n = \frac{X_n}{T_n} \in \mathbb{R}^{K\times J}_{max}$$

Waiting Time Equation $U_n^+ = \frac{T_{n+1}}{T_n}$, $U_n^- = \frac{T_n}{T_{n+1}}$

$$W_{n+1} = A_n \otimes W_n \otimes U_n^- \oplus B_{n+1} \otimes U_n^+ \otimes U_n^-$$

PETRI NETS

1. Bipartite Directed Graph with two types of nodes
   - $\mathcal{P}$ set of places reservoirs
   - $\mathcal{T}$ set of transitions synchronizers
2. Arcs
   - $p^\bullet$ $({}^\bullet p)$ set of transitions which follow (precede) p
   - $t^\bullet$ $({}^\bullet t)$ set of places which follow (precede) t
3. Initial Marking $\mu : \mathcal{P} \to I\!N$

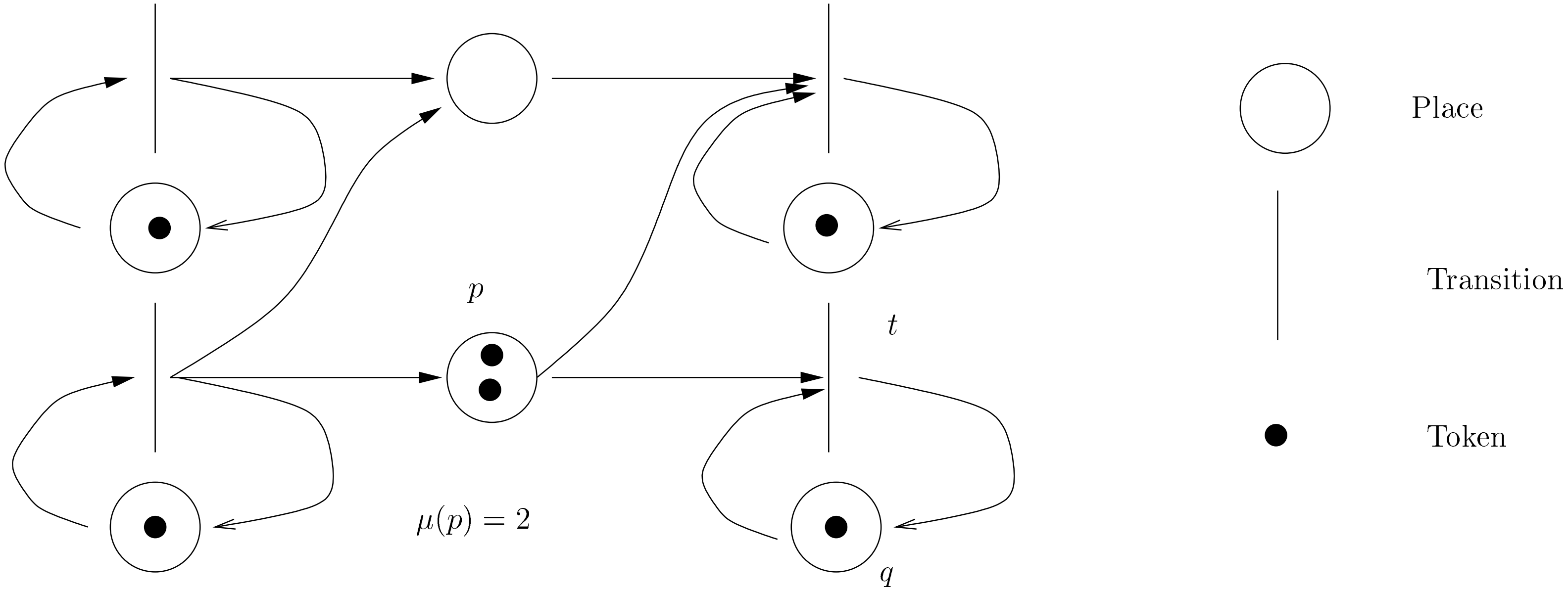


Figure 1: ${}^{\bullet}t = \{p, q\}$

1: LOCAL DYNAMICS OF SYNCHRONIZATION

1. Enabling of a transition $\forall p \in {}^{\bullet}t, \quad \mu(p) \geq 1$

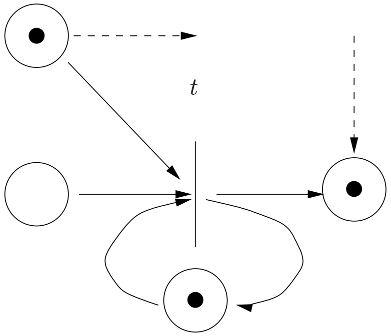


Figure 2: Transition t is not enabled

2. Firing of an enabled transition
   Tokens consumed by the firing of t
   $\forall p \in^{\bullet} t, \quad \mu(p) := \mu(p) - 1$
   Tokens produced by the firing of t
   $\forall p \in \ t^{\bullet}, \quad \mu(p) := \mu(p) + 1$

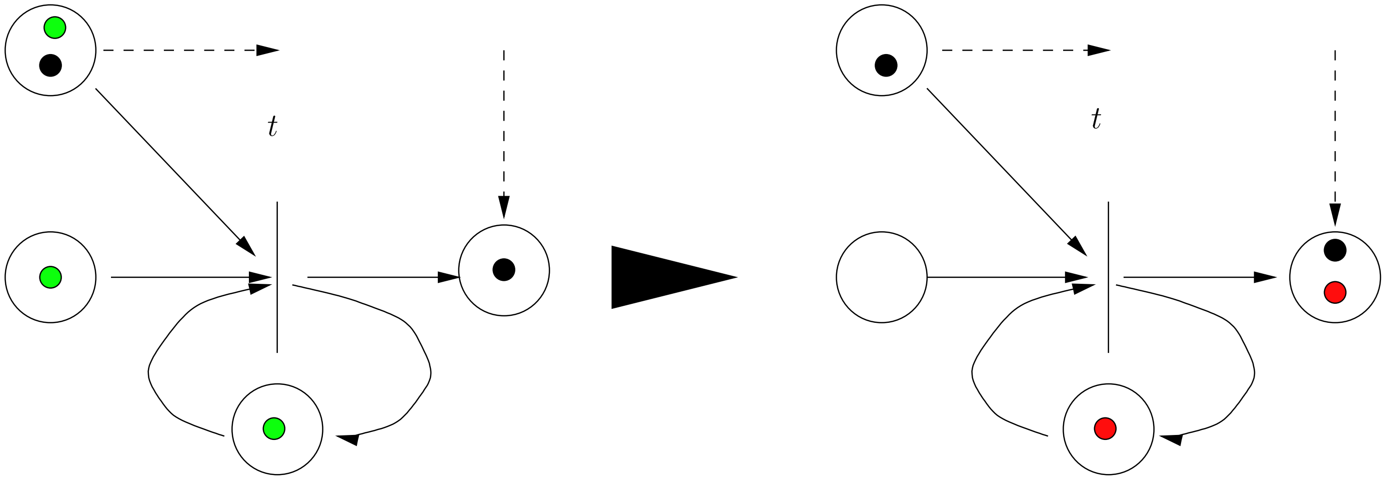
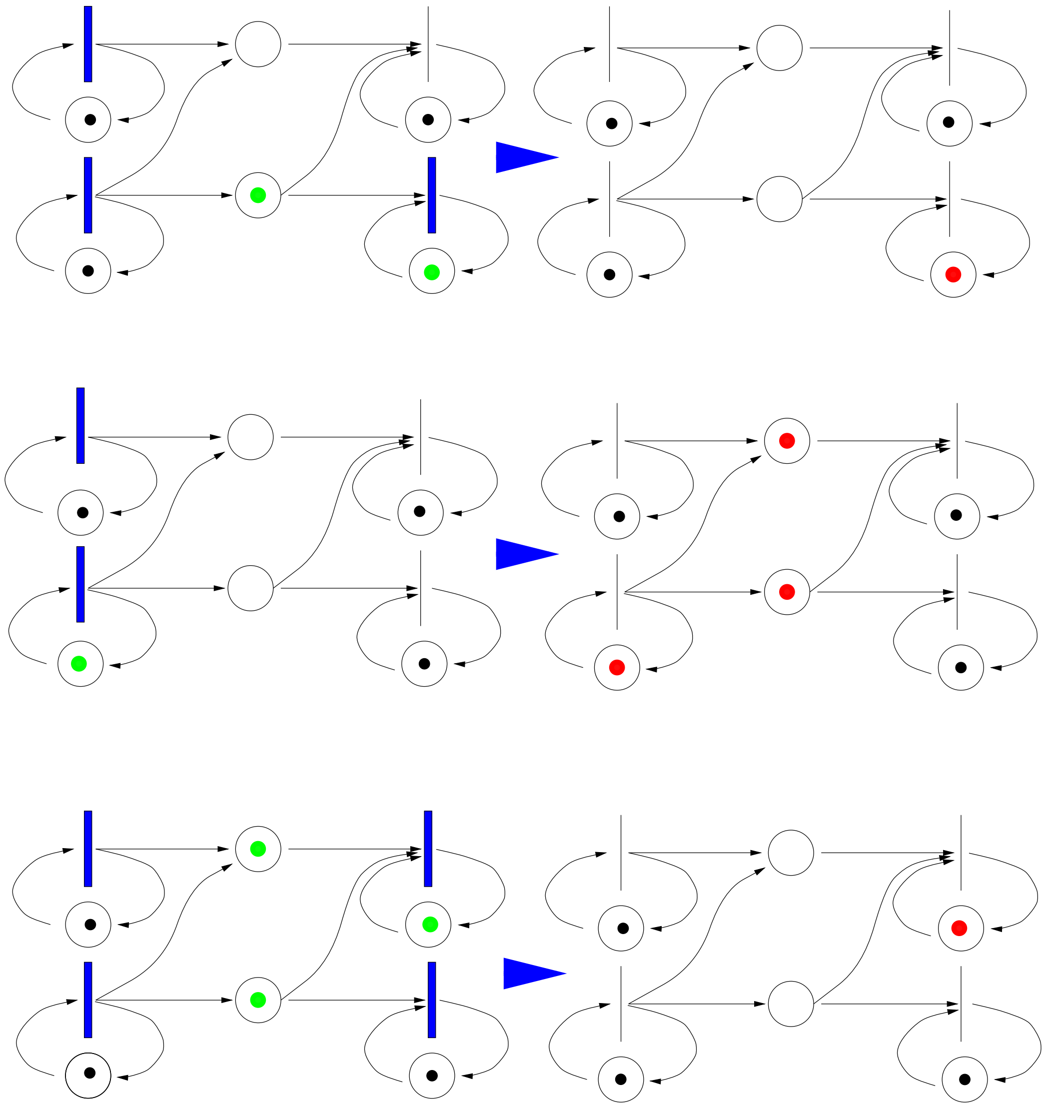


Figure 3: Local marking before and after the firing of t



BASIC ELEMENTS FOR SYNCHRONIZATION

On Transitions

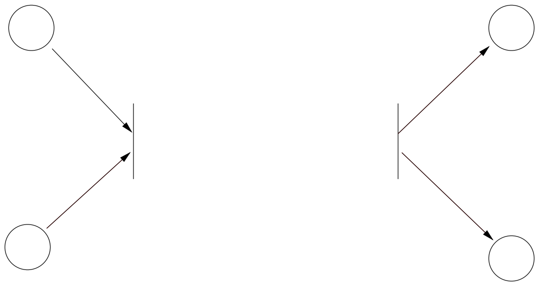


Figure 4: Convergence to a transition: In Synchronization – Divergence from a transition: Out Synchronization

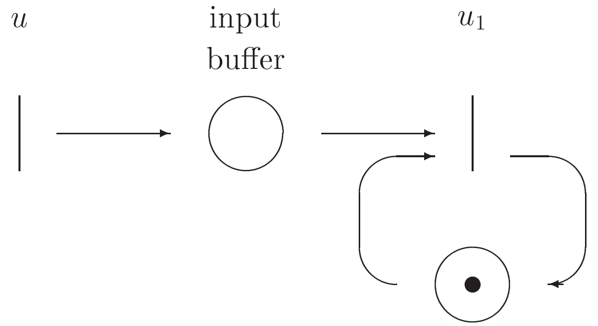
CLASSES OF NETS: EVENT GRAPHS

Pure Synchronization Each place has

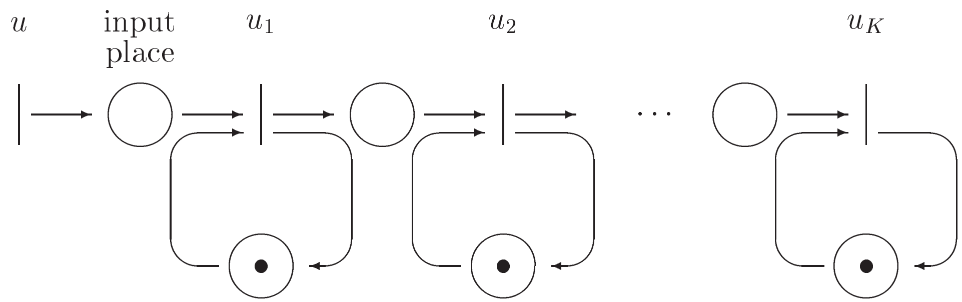
– at most one input transition

– at most one output transition

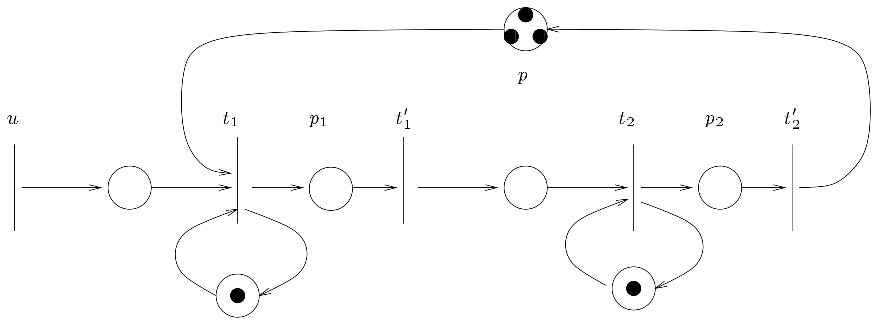
Example: G/G/1/∞ queue



EXAMPLES: TANDEM QUEUES

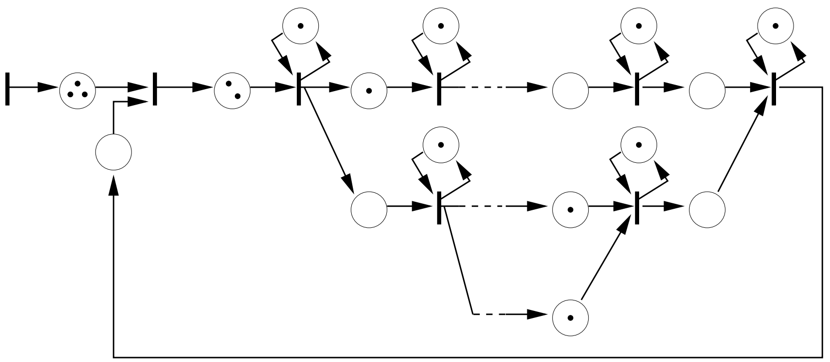


WINDOW FLOW CONTROL WITH FIXED WINDOW SIZE



u: Input Flow, t_i: Routers, t_i': Transmission Delays, $\mu(p)$: Window Size.

FLOW CONTROL OVER MULTICAST



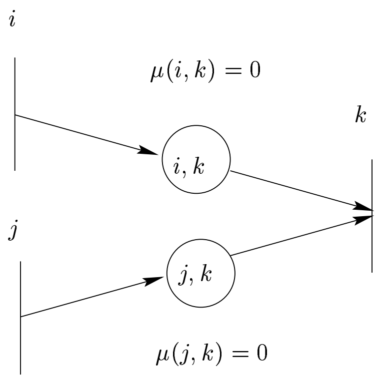
EVENT GRAPHS AS (max,plus) LINEAR SYSTEMS

- $M \in \mathbb{N}$: Maximal initial marking
- $\sigma_n^k \in \mathbb{R}$: Duration of n-th firing of transition k (data)
- $X_n^k \in \mathbb{R}$: Date of n-th enabling of transition k

σ_n^k: Random variables on $(\Omega, \mathcal{F}, P)$

- G: the sequences $\{\sigma_n^k\}$ are jointly stationary and ergodic
- GI: the sequences $\{\sigma_n^k\}$ are independent and i.i.d.
- M: the sequences $\{\sigma_n^k\}$ are independent, i.i.d. and exponentially distributed.

X_n^k: State Variables



Evolution equation in FIFO case

$$X_n^k = \max_{l \in \bullet\bullet k} X^l_{n-\mu(l,k)} + \sigma^l_{n-\mu(l,k)} = \bigoplus_p \bigoplus_{l \in S_{k,p}} X^l_{n-p} \otimes \sigma^l_{n-p}$$

Evolution equation in $\mathbb{R}^K_{max}$ $X_n = (X_n^k) \in \mathbb{R}^{K\times 1}_{max}$

$$X_n = A(n,0) \otimes X_n \oplus A(n,1) \otimes X_{n-1} \ldots \oplus A(n,M) \otimes X_{n-M}$$

$A(n,p) \in \mathbb{R}^{K\times K}_{max}$

$$A(n,p)_{kl} = \begin{cases} \sigma^l_{n-p} & \text{if } l \in {}^{\bullet\bullet}k,\ \mu(l,k) = p \\ \varepsilon & \text{otherwise} \end{cases}$$

VARIATIONS

- Alternate state variables: Y_n^k completion of n-th firing of transition k

$$Y_n = A(n,0) \otimes Y_n \oplus A(n,1) \otimes Y_{n-1} \ldots \oplus A(n,M) \otimes Y_{n-M}$$

$A(n,p) \in I\!R_{max}^{K \times K}$

$$A(n,p)_{kl} = \begin{cases} \sigma_n^k & \text{if } l \in {}^{\bullet\bullet}k,\ \mu(l,k) = p \\ \varepsilon & \text{otherwise} \end{cases}$$

Much nicer: i.i.d matrices if the random variables $\{\sigma_n^k\}$ are i.i.d in n.

- Alternate description of delays: delays on places (mainly for deterministic case).

BASIC ELEMENTS FOR ROUTING

On Places

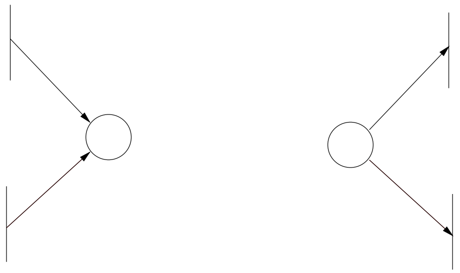


Figure 5: Convergence to a place: Superposition – Divergence from a place: Choice)

CLASSES OF NETS: FREE CHOICE NETS

In-Synchronization and Choice are separated

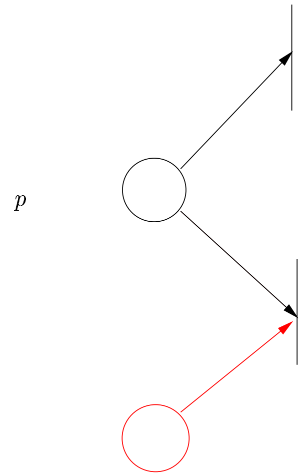
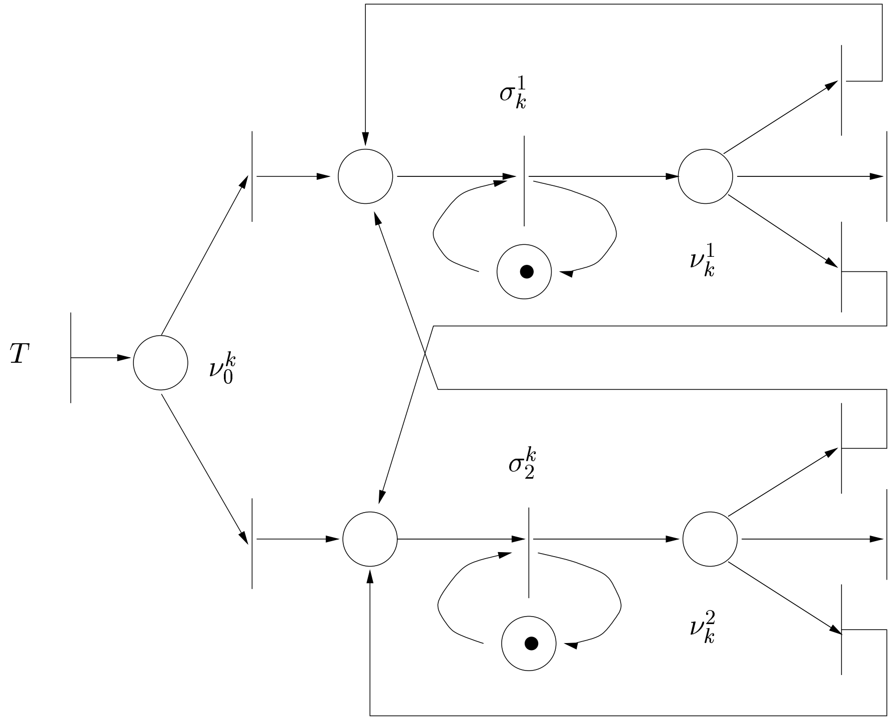


Figure 6: $Card(p^\bullet) > 1$ implies $Card(^\bullet t) = 1$ for all $t \in p^\bullet$

An Event Graph is a Free Choice net

GENERALIZED JACKSON NETWORK AS A FREE CHOICE NET



Data: For all $q \in \mathcal{T}$, $\sigma_n^q \in I\!R$ For all routing place p, $\nu_n^p \in p^\bullet$

WINDOW FLOW CONTROL WITH FIXED WINDOW SIZE & CROSS TRAFFIC

W: window size (fixed), K: number of routers Not a Free Choice Net

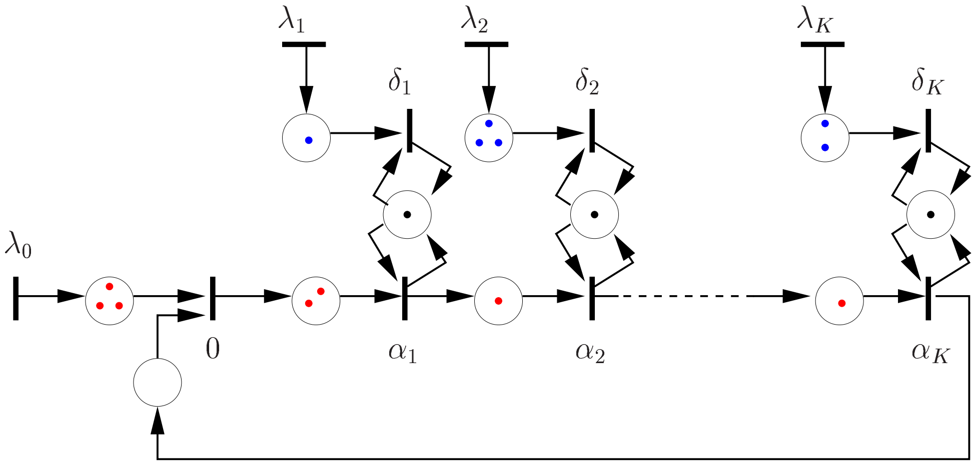


Figure 7: Controlled flow Cross flows

PART II: EIGENVALUE, EIGENVECTORS
Cohen Dubois Quadrat Viot 83

- A irreducible matrix of $\mathbb{R}_{max}^{K,K}$

- $\rho \in \mathbb{R}$, $X \in \mathbb{R}_{max}^{K}$ eigenpair of A if $\quad A \otimes X = \rho \otimes X.$

- Interpretation for an event graph of equation $X_{n+1} = AX_n$:
if $X_n = X \otimes c$, c constant, then for all $k \geq 0$,
$$X_{n+k} = X_n A^k = \rho^k \otimes c \otimes X \quad \text{periodic regime}$$

- ρ inverse of the asymptotic throughput

EIGENVALUE THEOREM

Theorem An irreducible matrix $I\!P$ of $I\!R_{max}^{K,K}$ admits a *unique* eigenvalue

$$\rho(I\!P) = \max_{k=1}^{K} \frac{\max_{i=1}^{K} I\!P_{ii}^{k}}{k}.$$

- Mean weight of circuit $s = (i_1, \ldots, i_l)$

$$\frac{p(s)}{|s|} = \frac{I\!P_{ii_1} + I\!P_{i_1 i_2} + \cdots + I\!P_{i_{l-1} i}}{l}.$$

$$\rho(I\!P) = \max_{l=1}^{K} \max_{s,\ |s|=l} \frac{p(s)}{l} = \max_{l=1}^{\infty} \max_{s,\ |s|=l} \frac{p(s)}{l}.$$

PROOF IN 3 LEMMAS: LEMMA 1

Lemma 1

If $\rho(I\!\!P) \leq 0$, the matrices $I\!\!P^* = \oplus_{l \geq 0} I\!\!P^l$ and $I\!\!P^+ = \oplus_{l \geq 1} I\!\!P^l$ are finite.

Proof

- For all $i_0, i_1, \ldots, i_k$ with $1 \leq i_n \leq K$, let

$$L = I\!\!P_{i_0 i_1} + I\!\!P_{i_1 i_2} + \cdots + I\!\!P_{i_{k-1} i_k}$$

$$j^* = \inf\{j \geq 1 \text{ s. t. } \exists l, 0 \leq l < j \text{ with } i_l = i_j\}.$$

  There exists such a j if $k \geq K$.

- Let l^*: $i_{l^*} = i_{j^*}$

$$L \leq I\!\!P_{i_0 i_1} + I\!\!P_{i_1 i_2} + \cdots + I\!\!P_{i_{l^*-1}, i_{l^*}} + I\!\!P_{i_{j^*}, i_{j^*+1}} + \cdots + I\!\!P_{i_{k-1} i_k}$$

$$L \leq \max_{k=0}^{K} \max_{i,j=1}^{K} (I\!\!P^k)_{ij} < \infty.$$

LEMMA 2

- $\varepsilon \neq u \in I\!R^K_{max}$ sub eigenvector of $I\!P$ for λ if $I\!P u \leq \lambda u$
- $\varepsilon \neq u \in I\!R^K_{max}$ super eigenvector of $I\!P$ for λ if $I\!P u \geq \lambda u$

Lemma 2

If $I\!P$ irreductible, $\lambda \geq \rho(I\!P)$ iff $I\!P$ has a sub eigenvector for λ.

Proof

For λ scalar, let $I\!P_\lambda = \lambda^{-1} I\!P$

If $\lambda \geq \rho(I\!P)$, $\rho(I\!P_\lambda) \leq 0$, so that $I\!P^*_\lambda$ et $I\!P^+_\lambda$ finite. As

$$I\!P_\lambda I\!P^*_\lambda = I\!P^+_\lambda = \bigoplus_{k \geq 1} I\!P^k_\lambda \leq \bigoplus_{k \geq 0} I\!P^k_\lambda = I\!P^*_\lambda,$$

each column of $I\!P^*_\lambda$ is a sub-eigenvector of $I\!P$ for λ.

Conversely

Suppose there exists $u \in I\!R^K_{max}$, with $I\!Pu \leq \lambda u$. Then

$$I\!P^k u \leq \lambda^k u,$$

for all $k \geq 1$. From the irreductibility $u(i) > \varepsilon$ for all i. So

$$u(i)(I\!P^k)_{ii} \leq (I\!P^k u)_i \leq \lambda^k u(i)$$

and

$$\lambda \geq \max_k \frac{\max_i I\!P^k_{ii}}{k} = \rho(I\!P).$$

LEMMA 3

Lemma 3

$\lambda \le \rho(I\!\!P)$ iff $I\!\!P$ has a super eigenvector for λ.

Proof

Let $u \in I\!\!R^K_{max}$, with $u(i_0) > \varepsilon$ and $I\!\!Pu \ge \lambda u$.

$$\begin{aligned}
\lambda u(i_0) &\le (I\!\!Pu)_{i_0} &= u(i_1) I\!\!P_{i_0 i_1} \\
\lambda u(i_1) &\le (I\!\!Pu)_{i_1} &= u(i_2) I\!\!P_{i_1 i_2} \\
&\cdots \\
\lambda u(i_{K-1}) &\le (I\!\!Pu)_{i_{K-1}} &= u(i_K) I\!\!P_{i_{K-1} i_K},
\end{aligned}$$

for indices i_n s.t. $u(i_n) > \varepsilon$.

This sequence contains a circuit. So $\exists i$ s.t. $u(i) > \varepsilon$ and $l \le K$ s.t.

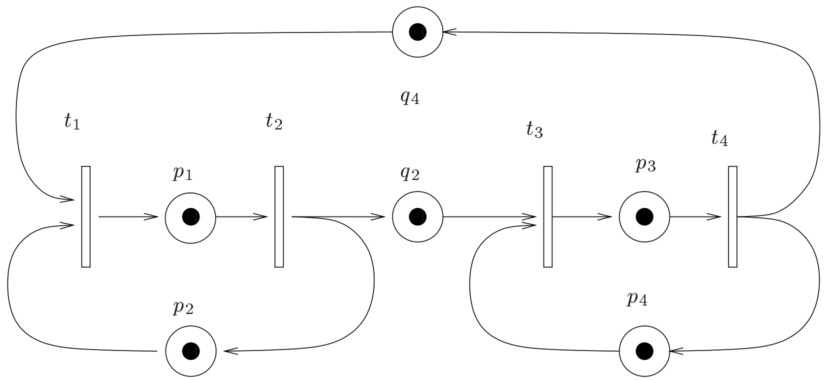
$$\lambda^l u(i) \le u(i) I\!\!P_{i i_m} I\!\!P_{i_m i_{m+1}} \cdots I\!\!P_{i_{m+l-2} i} \quad \Rightarrow \quad \lambda \le \rho(I\!\!P).$$

EIGENVECTOR THEOREM

- Critical Circuit: $i_1, \ldots, i_k$ s.t. $I\!\!P_{i_1 i_2} + \cdots + I\!\!P_{i_k i_1} = k\rho$.

- Critical Matrix: $C(I\!\!P)_{ij} = I\!\!P_{ij}$ if (i,j) belongs to a critical circuit, ε otherwise

EXAMPLE



$X_n = (X_n^1, X_n^2, X_n^3, X_n^4)^t$, $X_{n+1} = I\!P \otimes X_n$ with

$$I\!P = \begin{pmatrix} \varepsilon & \sigma_2 & \varepsilon & \tau_4 \\ \sigma_1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_2 & \varepsilon & \sigma_4 \\ \varepsilon & \varepsilon & \sigma_3 & \varepsilon \end{pmatrix}.$$

$$\mathbb{P} = \begin{pmatrix} \varepsilon & \sigma_2 & \varepsilon & \tau_4 \\ \sigma_1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \tau_2 & \varepsilon & \sigma_4 \\ \varepsilon & \varepsilon & \sigma_3 & \varepsilon \end{pmatrix}.$$

Assume $\sigma_2 = \sigma_4 = \tau_2 = 0$.

– If $\sigma_1 = \sigma_3 = 2$ and $\tau_4 = 0$, $\mathbb{P} = C(\mathbb{P})$

– If $\sigma_1 = \sigma_3 = 2$ and $\tau_4 = 1$,

$$C(\mathbb{P}) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 1 \\ 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & \varepsilon \end{pmatrix}.$$

EIGENVECTOR THEOREM

Theorem For all k in the critical graph of $I\!\!P$, the k-th column of $\widetilde{I\!\!P}^*$ with $\widetilde{I\!\!P} = I\!\!P_{\rho(I\!\!P)}$ is an eigenvector of $I\!\!P$ for $\rho(I\!\!P)$.

Proof $\widetilde{I\!\!P}^*$ is finite. A state k in the critical graph belongs to a circuit of weight 0 in $\widetilde{I\!\!P}$:

$$\widetilde{I\!\!P}^+_{kk} = 0.$$

So for all j,

$$\widetilde{I\!\!P}^*_{jk} = \bigoplus_{i \geq 0} \widetilde{I\!\!P}^i_{jk} = \bigoplus_{i \geq 1} \widetilde{I\!\!P}^i_{jk} = \widetilde{I\!\!P}^+_{jk}.$$

As $\widetilde{I\!\!P}^+ = \widetilde{I\!\!P}\,\widetilde{I\!\!P}^*$,

$$(\widetilde{I\!\!P}\,\widetilde{I\!\!P}^*)_{jk} = \widetilde{I\!\!P}^*_{jk}$$

MORE ON EIGENVECTORS

- Colinearity: $u \equiv v \in I\!R^K_{\max}$ if there exists c such that $u(i) = v(i) + c$ for all i.

  Lemma Let k et l two states of the critical graph in the same communication class of the critical graph. The eigenvectors u and v: $u(j) = \widetilde{I\!P}^*_{jk} = \widetilde{I\!P}^+_{jk} =$ and $v(j) = \widetilde{I\!P}^*_{jl} = \widetilde{I\!P}^+_{jl}$ are colinear.

  Proof

$$\widetilde{I\!P}^+_{jk} \geq \widetilde{I\!P}^+_{jl}\widetilde{I\!P}^+_{lk} \geq \widetilde{I\!P}^+_{jk}\widetilde{I\!P}^+_{kl}\widetilde{I\!P}^+_{lk} = \widetilde{I\!P}^+_{jk}$$

  because $\widetilde{I\!P}^+_{kl}\widetilde{I\!P}^+_{lk} = 0$ if k and l belong to the same communication class. So

$$(\widetilde{I\!P}^+)_{jk} = \widetilde{I\!P}^+_{jl}\widetilde{I\!P}^+_{lk}, \quad \forall j.$$

MORE ON EIGENVECTORS

- Let $\mathcal{K}_1, \ldots, \mathcal{K}_q$ be the communication classes of $C(I\!\!P)$ containing at least a critical state.

Lemma All $I\!\!R_{\max}$-linear combination of eigenvectors of $I\!\!P$ is an eigenvector of $I\!\!P$. If $I\!\!P$ is irreductible, let $k_m \in \mathcal{K}_m$ be a critical state in $\mathcal{K}_m$, $m = 1, \ldots, q$. Each eigenvector of $I\!\!P$ for $\rho(I\!\!P)$ is a $I\!\!R_{\max}$-linear combination of $u_1, \ldots, u_q$, with $u_m(j) = \widetilde{I\!\!P}^*_{j,k_m}$.

CYCLICITY: EXAMPLE

$$\mathbb{P} = \begin{pmatrix} \varepsilon & 3 & \varepsilon & \varepsilon \\ 1 & \varepsilon & 1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ 0 & \varepsilon & 1 & \varepsilon \end{pmatrix}, \quad C(\mathbb{P}) = \begin{pmatrix} \varepsilon & 3 & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 3 \\ \varepsilon & \varepsilon & 1 & \varepsilon \end{pmatrix}$$

and

$$\widetilde{\mathbb{P}} = \begin{pmatrix} \varepsilon & 1 & \varepsilon & \varepsilon \\ -1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 1 \\ -2 & \varepsilon & -1 & \varepsilon \end{pmatrix}.$$

$\rho(\mathbb{P}) = 2$ and $C(\mathbb{P})$ has two communication classes.

For all $n \geq 1$

$$\widetilde{I\!P}^{2n} = \begin{pmatrix} 0 & \varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 0 \\ -1 & \varepsilon & 0 & \varepsilon \\ \varepsilon & -1 & \varepsilon & 0 \end{pmatrix}, \quad \widetilde{I\!P}^{2n+1} = \begin{pmatrix} \varepsilon & 1 & \varepsilon & 1 \\ -1 & \varepsilon & -1 & \varepsilon \\ \varepsilon & 0 & \varepsilon & 1 \\ -2 & \varepsilon & -1 & \varepsilon \end{pmatrix}, \quad \widetilde{I\!P}^{*} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix}.$$

$u = (0,1,0,1)$ and $v = (0,1,1,2)$ are two different eigenvectors.

- The powers of $\widetilde{I\!P}^{n}$ are periodic of period 2: for the initial condition $X_0 = (0,0,0,0)$, the equivalence class of X_n (for colinearity) is also of period 2; the maximal marking of q_4 is 2.
- For the 1-periodic regime associated with v, the maximal marking of q_4 is 1.

CYCLICITY: THEOREM

- Cyclicity c_m of communication class $\mathcal{K}_m$, $m = 1, \ldots, q$:
  GCD of the lengths of circuits in this class
- Cyclicity $\sigma(I\!P)$ of matrix $I\!P$: LCM of cyclicities $c_1, \ldots, c_q$

THEOREM $I\!P$ irreducible. There exists N s.t. for all $n \geq N$,

$$I\!P^{n+\sigma(I\!P)} = (\rho(I\!P))^{\sigma(I\!P)} I\!P^n.$$

PARTIAL PROOF (case $\sigma(I\!P) = 1$, $\rho(I\!P) = 0$)

We have to show that $I\!P^{n+1} = I\!P^{n}$, $n \geq N$.

1. $I\!P^{n}_{kk} = I\!P^{+}_{kk} = 0$ for all k critical and large enough n
   GCD of lengths of the d circuits $s_1, \ldots, s_d$ in the communication class that contains k equal to 1.
   $\exists$ circuit s containing k and each of the s_i's
   For all n large enough, there exists integers $n_1, \ldots, n_d$ s.t.
   $$n = |s| + n_1|s_1| + \cdots + n_d|s_d|.$$
   $\exists$ a circuit of the critical graph, containing k and with length n

2. $I\!P^{n}_{kj} = I\!P^{+}_{kj}$ for all j, for all k critical, for all n large enough
   There exists n_0 s.t. $I\!P^{n_0}_{kj} = I\!P^{+}_{kj}$. Otherwise, there exists $n_l \uparrow$ s.t. $I\!P^{n_l}_{kj} > I\!P^{n_{l-1}}_{kj}$, which implies $I\!P^{p_l}_{kk} > I\!P^{n_{l-1}}_{kk}$, with $p_l = n_l + p$: impossible.
   So for n large enough
   $$I\!P^{n_0+n}_{kj} \geq I\!P^{n}_{kk} I\!P^{n_0}_{kj} = I\!P^{n_0}_{kj} = I\!P^{+}_{kj}.$$

MORE ON CYCLICITY

COROLLARY If $I\!P$ irreducible,

$$\lim_{l\to\infty} \frac{\max_{i,j} I\!P^l_{ij}}{l} = \rho(I\!P).$$

If in addition $I\!P$ is aperiodic, then for all (i,j)

$$\lim_{l\to\infty} \frac{I\!P^l_{ij}}{l} = \rho(I\!P).$$

PROOF For N large enough, for all $k = 0, \ldots, \sigma(I\!P) - 1$, and $p > 0$,

$$\max_{i,j} I\!P_{ij}^{N+k+p\sigma(I\!P)} = p\sigma(I\!P)\rho(I\!P) + \max_{i,j} I\!P_{ij}^{N+k}.$$

In the aperiodic case, $I\!P^n_{ij} > \varepsilon$ for all (i,j) and for n large enough. So

$$I\!P_{ij}^{N+k+p\sigma(I\!P)} = p\sigma(I\!P)\rho(I\!P) + I\!P_{ij}^{N+k}.$$

UNIQUENESS OF EIGENVECTORS – IRREDUCIBLE CASE

- There are as many eigenvectors as there are communication classes in $C(I\!P)$.
- If there is a unique communication class COC 1 in $C(I\!P)$, $I\!P$ has a unique eigenvector;
- If the GCD of the circuits in this class is 1, cyclicity of $I\!P$ is 1 CYC 1. Then

$$\exists N \ s.t. \ \forall n \geq N, \ \ I\!P^{n+1} = \rho \otimes I\!P^{n}$$

AFFINE EQUATIONS

THEOREM
Let $I\!P$ be a non zero matrix of $I\!R_{\max}^{K,K}$ and m be a non zero vector of $I\!R_{\max}^{K}$. The smallest solution of the affine equation with unknown $n \in I\!R_{\max}^{K}$

$$n = nI\!P \oplus m \tag{1}$$

is $n_0 = mI\!P^*$. If this solution is finite and if for all pairs i, j, $(I\!P^l)_{ij} \to -\infty$ when $l \to \infty$, then n_0 is the smallest finite solution of (1).

If $I\!P$ is irreducible, $n_0 < \infty$ iff $\rho(I\!P) \leq 0$, and n_0 is the unique finite solution if $\rho(I\!P) < 0$.

PROOF

n_0 is a solution. For all solutions n,

$$n = n\mathbb{P}^l \oplus m \left(\bigoplus_{p=0}^{l-1} \mathbb{P}^p \right), \quad \Rightarrow \quad n \geq m \left(\bigoplus_{p=0}^{l-1} \mathbb{P}^p \right) \quad \forall l \geq 1,$$

So $n \geq n_0$. If $(\mathbb{P}^l)_{ij} \to -\infty$ when $l \to \infty$, then for all finite n,

$$n\mathbb{P}^l \oplus m \left(\bigoplus_{p=0}^{l-1} \mathbb{P}^p \right) \to m \left(\bigoplus_{p=0}^{\infty} \mathbb{P}^p \right) \quad \Rightarrow \quad n = n_0.$$

If $\mathbb{P}$ irreductible,

$$\lim_{l \to \infty} \frac{\max_{i,j} \mathbb{P}^l_{i,j}}{l} = \rho(\mathbb{P}).$$

So, if $\rho(\mathbb{P}) < 0$, then n_0 is the unique finite solution of (1).

PART III: Periodic Regimes of TCP and Spectral Theory of Matrices

Deterministic TCP Models

- Services are all deterministic, rational and constant in n: the matrices A_i do not depend on n

Theorem 1 *Under any of the above assumptions concerning the protocol (e.g. Reno or Tahoe with or without slow start), and the form of feedback (e.g. G2, or G2 and G3, or G1, etc.), the sequence of reference windows $\{w_n\}$ becomes ultimately periodic, with values in an integer interval of the form $[a^*, b^*]$, such that such that $1 \leq a^* \leq b^* \leq w^*$.*

(G1) Example

- If the period is made of a single increasing phase:
  - t_i the number of occurrences of $a^* \leq i \leq b^*$ during a period
  - T the period: $T = t_{a^*} + t_{a^*+1} + \cdots + t_{b^*}$.
  - One can then simplify the dynamical system by reducing it to the pure max-plus recurrence relation

$$Z(n) = A_{w_{n-1}}(n) \otimes Z(n-1),$$

where $\{w_n\}$ is the periodic window size.

(G1) Example (*continued*)

THEOREM

There exists a square matrix A' of dimension Kw^* describing the transient phase of the window size, and an integer m describing the number of packets sent in this transient phase, such that for all $n \geq 0$,

$$Z(nT + m) = \left(A_{b^*}^{t_{b^*}} \otimes A_{b^*-1}^{t_{b^*-1}} \otimes \cdots \otimes A_{a^*}^{t_{a^*}}\right)^n \otimes A' \otimes Z(0).$$

Link between Throughput and max-plus Eigenvalues

THEOREM
If the square matrix (of dimension $K.w^*$) $A_{b^*}^{t_{b^*}} \otimes A_{b^*-1}^{t_{b^*-1}} \otimes \cdots \otimes A_{a^*}^{t_{a^*}}$ has a unique max-plus unique eigenvalue γ, then the mean throughput is

$$\lambda = \frac{T}{\gamma}.$$

G1-EXAMPLES

- Notation

$$\sigma^* = \max_{1 \le i \le K} \sigma_i \qquad \mathrm{RTT} = d_{K,0} + \sum_{i=1}^{K} (\sigma_i + d_{i-1,i}) = S$$

- (G1) detection (Little's law):

$$ACK(n) = 1, \text{ if } w_n \le \frac{\mathrm{RTT}}{\sigma^*}, \qquad 0, \text{ if } w_n > \frac{\mathrm{RTT}}{\sigma^*}.$$

- The maximal value of window is

$$b^* = \left\lfloor \frac{\mathrm{RTT}}{\sigma^*} \right\rfloor + 1 \le K + 1$$

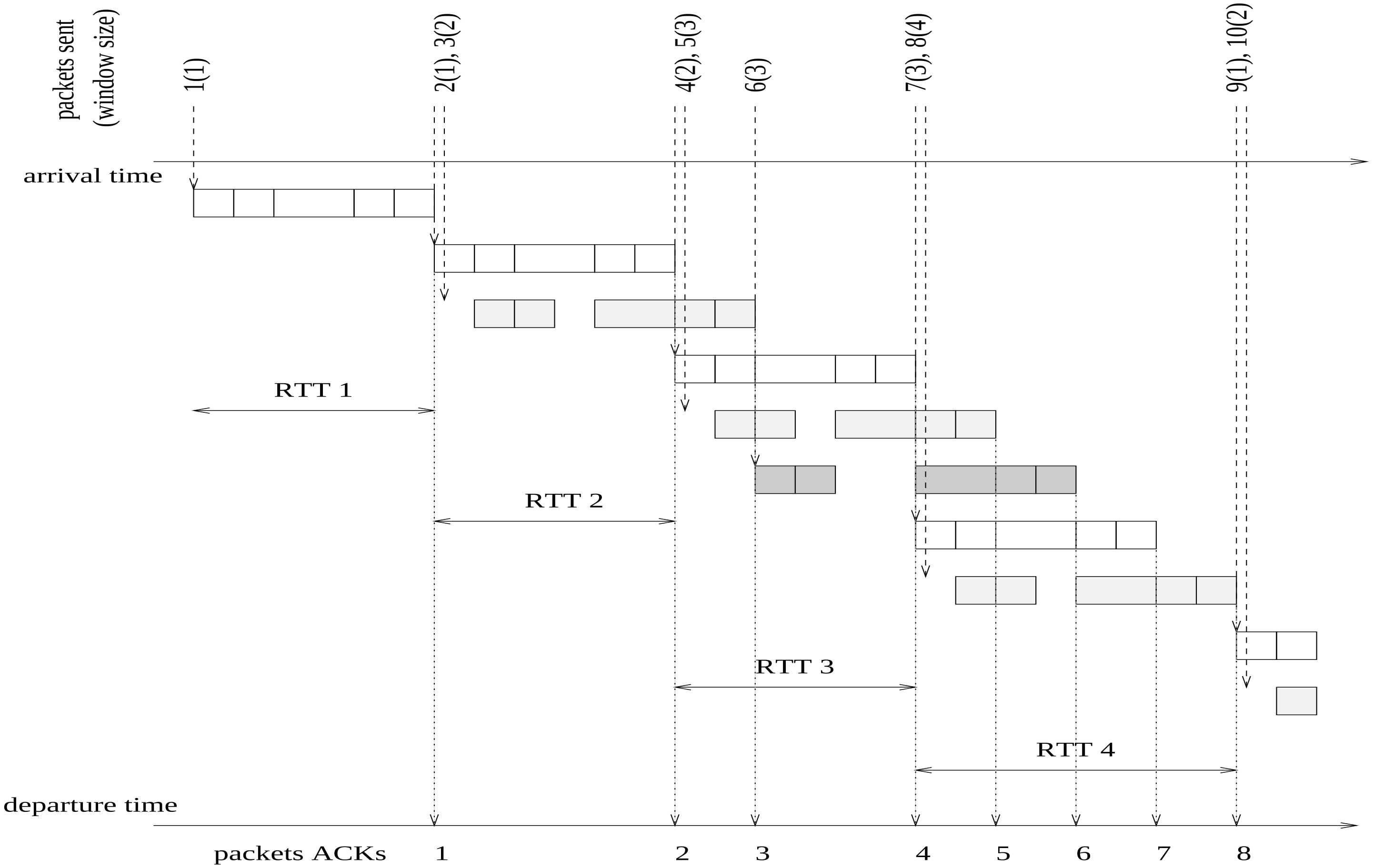
Tahoe, Deterministic Case, No Slow Start, Simplified Dynamics

$$\{w_1, w_2 \ldots\} = \{1, 2, 2, 3, 3, 3, \ldots, \underbrace{b^* - 1, .., b^* - 1}_{b^*-1 \text{ times}}, b^*, 1, 2, 2, ..\}.$$

Let $T = \frac{b^*(b^*-1)}{2} + 1$:

$$Z(n \times T + 1) = \left(A_{b^*} \otimes A_{b^*-1}^{b^*-1} \otimes \ldots \otimes A_2^2 \otimes A_1\right)^n \otimes A_1 \otimes Z(0).$$

Tahoe, Deterministic Case, No Slow Start, Simplified Dynamics (*continued*)



Example: $K = 5$, $d \equiv 0$, $(\sigma_1, \cdots, \sigma_5) = (1, 1, 2, 1, 1)$, $b^* = 4$.

Tahoe, Deterministic Case, No Slow Start, Simplified Dynamics (*continued*)

Theorem Tahoe w_n becomes periodic with period T and the throughput γ is $\frac{T}{\lambda}$ where λ is the unique eigenvalue of the max-plus matrix

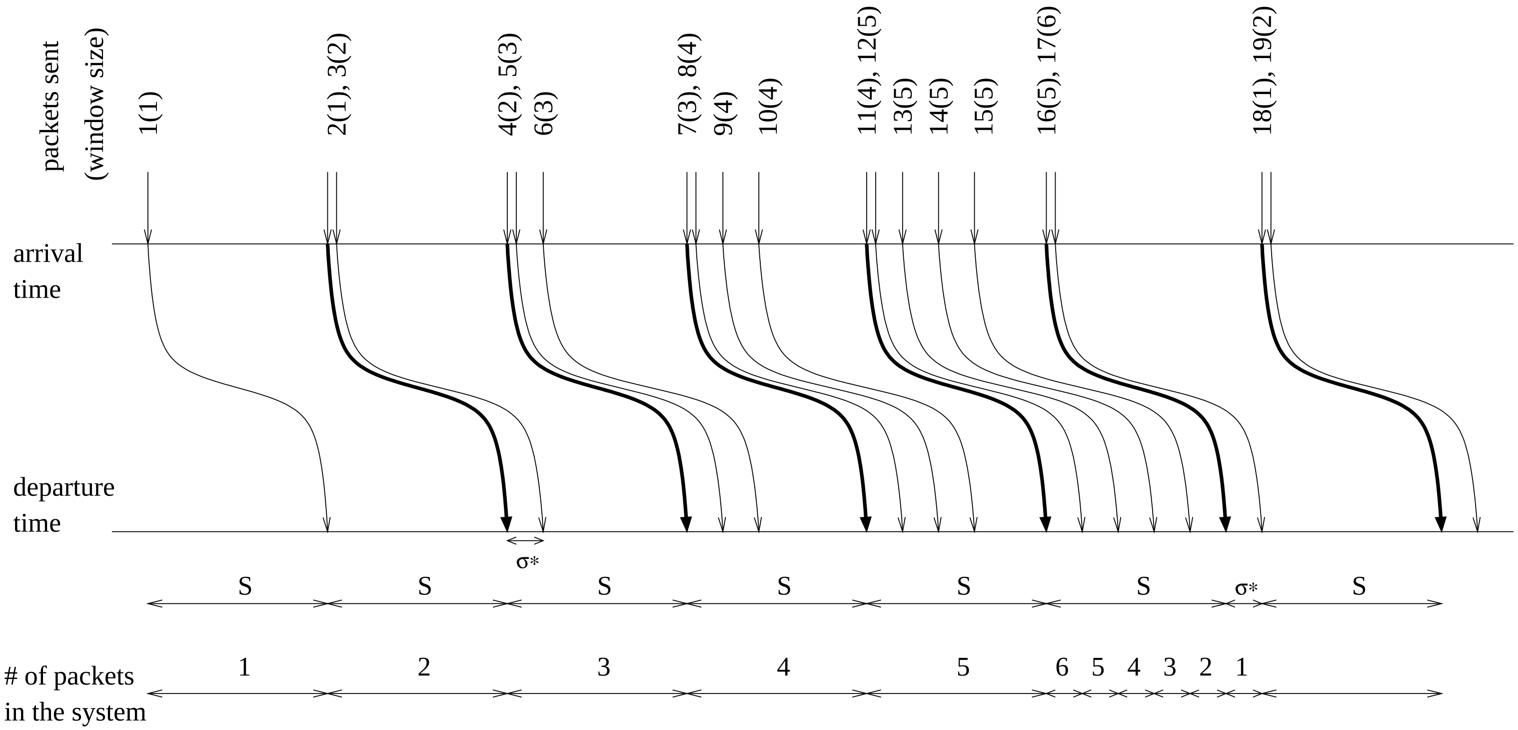
$$\left(A_{b^*} \otimes A_{b^*-1}^{b^*-1} \otimes \cdots \otimes A_1\right) \in I\!R_{\max}^{K.b^*,K.b^*}$$

$$\lambda = \frac{1}{2}\frac{b^*(b^*-1)+2}{(b^*-1)\mathrm{RTT}+\sigma^*}.$$

Graphical Interpretation of the Eigenvalue

Evolution of entrance ($y_0(n)$) and departure time ($y_K(n)$) of pack. $\# \, n$

$$\lambda = (b^* - 1)\mathrm{RTT} + \sigma^*$$



Tahoe, Deterministic Case, Slow Start

With ideal slow start sequence (slow start in 0 time): periodic sequence:

$$\{1,2,3,4,\ldots,w_s-1,\underbrace{w_s,\ldots,w_s}_{w_s \text{ times}},\ldots\underbrace{b^*-1,..,b^*-1}_{b^*-1 \text{ times}},b^*,1,2,3,..\}$$

with $w_s = \left\lfloor \frac{b^*}{2} \right\rfloor$.

Theorem Tahoe with ideal slow start w_n becomes periodic and

$$\lambda = \frac{1}{2}\frac{b^*(b^*-1) - \left\lfloor \frac{b^*}{2} \right\rfloor \left(\left\lfloor \frac{b^*}{2} \right\rfloor - 3 \right)}{(b^*+1-\lfloor \frac{b^*}{2} \rfloor)\mathrm{RTT} + (\lfloor \frac{b^*}{2} \rfloor - 1)\sigma^*}.$$

Exact Tahoe Dynamics

- When moving from the simplified dynamics to the exact one, the window size that should be used for Tahoe without slow start is:

$$\{v_1, v_2 \ldots\} = \{1, 1, 2, 2, 2, 3, 3, 3, 3, \ldots, \underbrace{b^* - 1, .., b^* - 1}_{b^* \text{ times}}, b^*, 1, 2, 2, ..\}.$$

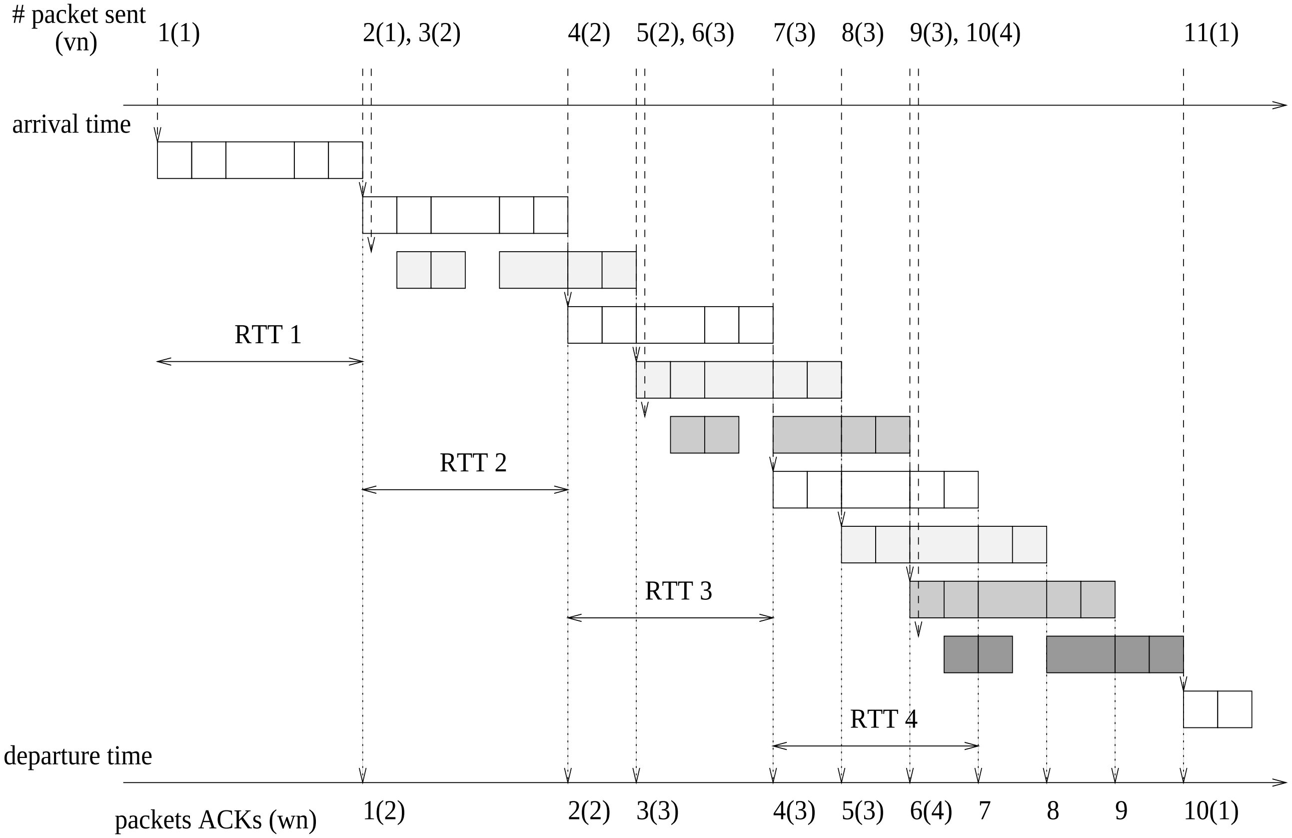
- The throughput is given by:

$$\lambda = \frac{1}{2} \frac{b^*(b^* + 1)}{b^*\mathrm{RTT} + (b^* - 1)\sigma^*}.$$

- For the same model with slow start, the sequence is

$$\{1, 1, 2, 2, .., \theta\text{-}1, \theta\text{-}1, \underbrace{\theta, .., \theta}_{\theta+1 \text{ times}}, \ldots, \underbrace{b^*\text{-}1, .., b^*\text{-}1}_{b^* \text{ times}}, b^*, 1, 1, 2, 2, ..\}.$$

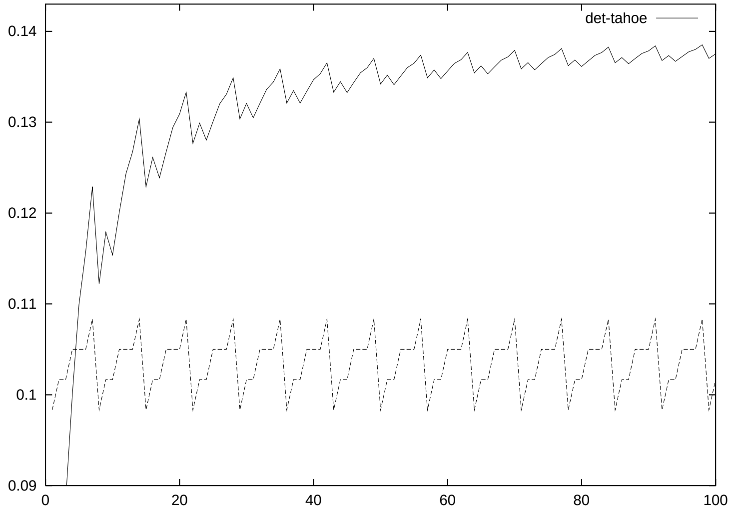
Exact Tahoe Dynamics (*continued*)



Example: $K = 5$, $w^* = 4$ and $(\sigma_1, \cdots, \sigma_5) = (1, 1, 2, 1, 1)$, no slow start.

Exact Tahoe Dynamics (*continued*)

- example Take TCP Tahoe without slow-start phase over 4 tandem queues with $\sigma_1 = 3.2$, $\sigma_2 = 4.61$, $\sigma_3 = 2.7$, $\sigma_4 = 4.61$. $b^* = 4$, $w_n \in \{1, 2, 3, 4\}$.



Simplified dynamics: 0.140084; Exact dynamics: 0.134571.

Reno, Deterministic Case, No Slow Start

Theorem Reno without slow start w_n becomes periodic and

$$\lambda = \frac{1}{2}\frac{b^*(b^*-1) - \lfloor \frac{b^*}{2} \rfloor \left(\lfloor \frac{b^*}{2} \rfloor - 1 \right) + 2}{(b^* - \lfloor \frac{b^*}{2} \rfloor)\mathrm{RTT} + \sigma^*}.$$

Asymptotic Relation with Loss Probability

$$\text{Tahoe without SlSt} \; : \; p_{loss} = \frac{2}{b^*(b^*-1)+2}, \quad \sqrt{p_{loss}} \sim \frac{\sqrt{2}}{b^*},$$

$$\text{Tahoe with SlSt} \; : \; p_{loss} = \frac{2}{b^*(b^*-1) - \lfloor \frac{b^*}{2} \rfloor \left(\lfloor \frac{b^*}{2} \rfloor - 1 \right) + 2}, \quad \sqrt{p_{loss}} \sim \sqrt{\frac{8}{3}\frac{1}{b^*}},$$

$$\text{Reno} \; : \; p_{loss} = \frac{2}{b^*(b^*-1) - \lfloor \frac{b^*}{2} \rfloor \left(\lfloor \frac{b^*}{2} \rfloor - 3 \right)}, \quad \sqrt{p_{loss}} \sim \sqrt{\frac{8}{3}\frac{1}{b^*}},$$

when $b^* \to \infty$. One finds back the values for c_o in

$$D_{th} = \frac{c_o}{RTT\sqrt{p_{loss}}}$$

case 1: $c_o = \frac{1}{\sqrt{2}} \simeq 0.71$; cases 2 and 3: $c_o = \sqrt{\frac{3}{2}} \simeq 1.22$.

G2-EXAMPLE

- Reno without slow start
- For all $a \leq b$ and $i \leq b$ let

$$\mathcal{A}(a,b,i) = A_b^i \otimes A_{b-1}^{b-1} \otimes \cdots \otimes A_{a+1}^{a+1} \otimes A_a^a.$$

- Search for periodic regimes which are made of a single increasing phase followed by a loss.

If there exists such a regime with $a^* = w$ and $b^* = 2w$, then there exists $i \leq 2w$ s.t. the matrix $\mathcal{A}(w, 2w, i)$ has an eigenpair (ρ, X) verifying:

1. $\overline{X} \in \mathcal{S}$
2. for all ($n < 2w$ and $j \leq n$) or ($n = 2w$ and $j < i$) $\overline{\mathcal{A}(w,n,j)X} \notin \mathcal{S}$.

Deterministic Services, Random Feedbacks

- The sequence $\{(W(n), \overline{Z}(n))\}$ forms a Markov chain with finite state space Δ.
- If this Markov chain is irreducible, with stationary probability π, then

$$\lambda^{-1} = \gamma = \sum_{(w\overline{z})\in\Delta} \alpha(\overline{z})\pi(w, \overline{z}),$$

with $\alpha(\overline{z}) = \overline{z}_K - \overline{z}_{2K}$

G1–EXAMPLE

THEOREM

Under (G1), if $p > 0$, $\{W(n)\}$ is an irreducible Markov chain on the integer interval $[1, b^*]$, with $b^* = \left\lfloor \frac{\text{RTT}}{\sigma^*} \right\rfloor + 1$, and the throughput depends on service times only through RTT and σ^*.

The sequence $\{y_K(n+1) - y_K(n)\}_{n \in \mathbb{N}}$ takes its values in the set $\Phi = \{\text{RTT} - (k-1)\sigma^*, k = 1, .., b^* - 1\} \cup \{\sigma^*\}$.

Example: Markov TCP Tahoe without Slow Start

- Under (G1) the throughput of Tahoe without slow start is:

$$\lambda = \frac{1}{\sigma^* + \Sigma_{k=1}^{b^*-1} \left[\mathrm{RTT} - k\sigma^*\right] q(k)}, \quad \text{with} \quad q(k) = \frac{p(1-p)^{\frac{k(k+1)}{2}-1}}{1-(1-p)^{\frac{b^*(b^*-1)}{2}+1}}.$$

- When $b^* \to \infty$, the asymptotic throughput takes a simple form if $p \sim \frac{2}{(b^*)^2}$:

$$\lambda \sim \left(\frac{1-e^{-1}}{2\int_0^1 e^{-t^2} dt}\right) \frac{1}{\sigma^*} \simeq 0.42 \frac{1}{\sigma^*}.$$

The Impact of Random Losses

- Losses due to random perturbations are preponderant compared to losses due to a too high sendrate

$$\mu(b^*) + p(1 - \mu(b^*)) = \mu(1),$$

$\mu(b^*)$ is the loss due to congestion
$p(1 - \mu(b^*))$ is the loss due to random perturbations.

The Impact of Random Losses (*continued*)

- For $\mu(1)$ fixed, the throughput decreases in p: