Risk-Management Methods for the Libor Market Model Using Semidefinite Programming*

A. d'Aspremont†

February 20, 2003

Abstract

When interest rate dynamics are described by the Libor Market Model as in Brace, Gatarek & Musiela (1997), we show how some essential risk-management results can be obtained from the dual of the calibration program. In particular, if the objetive is to maximize another swaption's price, we show that the optimal dual variables describe a hedging portfolio in the sense of Avellaneda & Paras (1996). In the general case, the local sensitivity of the covariance matrix to all market movement scenarios can be directly computed from the optimal dual solution. We also show how semidefinite programming can be used to manage the Gamma exposure of a portfolio.

Keywords: Libor Market Model, Inverse problems, Semidefinite Programming, Calibration.

1 Introduction

A recent stream of works on the Libor Market Model have showed how the swap can be approximated by a basket of lognormal processes under an appropriate choice of forward measure. This, coupled with analytic European basket call pricing approximations, allows to cast the problem of calibrating the Libor Market Model to a set of European caps and swaptions as a semidefinite program, i.e. a linear program on the cone of positive semidefinite matrices (see Nesterov & Nemirovskii (1994) and Vandenberghe & Boyd (1996)). This work exploits the related duality theory to provide explicit sensitivity and hedging results based on the optimal solution to the calibration program.

The lognormal approximation for basket pricing has its origin in electrical engineering as the addition of noise in decibels (see for example Schwartz & Yeh (1981)). Its application to basket option pricing dates back to Huynh (1994) or Musiela & Rutkowski (1997). Brace, Dun & Barton (1999) tested it's empirical validity for swaption pricing and Brace & Womersley (2000) used it to study Bermudan swaptions. More recently, d'Aspremont (2002), Kawai (2002) and Kurbanmuradov, Sabelfeld & Schoenmakers (2002) obtained additional terms in the expansion and further evidence on the lognormal approximation's performance. On the calibration front, Rebonato (1998) and Rebonato (1999) highlight the importance of jointly calibrating volatilities and correlations. These works, together with Longstaff, Santa-Clara & Schwartz (2000) also detail some of the most common non-convex calibration techniques based on parametrizations of the forward rates covariance factors. The mixed static-dynamic hedging formulation of the pricing problem has its source in the works by El Karoui, Jeanblanc-Picqué & Shreve (1998), Avellaneda, Levy & Paras (1995) and Avellaneda & Paras (1996). Romagnoli & Vargiolu (2000) provide some closed-form results in the multivariate case.

Here, we show how the dual solution to the calibration program provides a complete description of the sensitivity to changes in market condition. In fact, because the algorithms used to solve the calibration

*This work has been developed under the direction of Nicole El Karoui. I am extremely grateful to her. I also benefited greatly in my work from discussions with Marco Avellaneda, Guillaume Amblard, Vlad Bally, Stephen Boyd, Jérôme Busca, Rama Cont, Darrell Duffie, Laurent El Ghaoui, Jean-Michel Lasry, Marek Musiela, Ezra Nahum, Antoon Pelsser, Yann Samuelides, Olivier Scaillet, Robert Womersley, seminar participants at the GdR FIQAM at the Ecole Polytechnique, the May 2002 Workshop on Interest Rate Models organized by Frontières en Finance in Paris, the June 2002 AFFI conference in Strasbourg and the Summer School on Modern Convex Optimization at the C.O.R.E. in U.C.L. Finally, I am very grateful to Jérôme Lebuchoux, Cyril Godart and everybody inside FIRST and the S.P.G. at Paribas Capital Markets in London, their advice and assistance has been key in the development of this work.

†CMAPX, Ecole Polytechnique, Palaiseau, France. Internet: alexandre.daspremont@polytechnique.org. Mailing address: 4076 17th #B, San Francisco, CA 94114, USA. Tel: +1 415 864 2962.

problem jointly solve the problem and its dual, *the sensitivity of the calibrated covariance matrix is readily available from the dual solution to the calibration program*. When the objective in the calibration program is another swaption's price, the dual solution also describes an approximate solution to the optimal hedging problem in Avellaneda & Paras (1996), which computes the price of a derivative product as the sum of static hedging portfolio and a dynamic strategy hedging the worst-case residual risk. We also show how semidefinite programming can be used to efficiently solve the problem of optimally managing the Gamma exposure of a portfolio using vanilla options, as posed by Douady (1995).

The results we obtain here underline the key advantages of applying semidefinite programming methods to the calibration problem: besides their radical numerical performance, they naturally provide some central results on sensitivity and risk-management. They also eliminate the numerical errors in sensitivity computations that were caused by the inherent instability of the classical non-convex calibration solutions.

The paper is organized as follows: In the next section, we quickly recall the approximate calibration program construction for the Libor Market Model. Section three shows how to compute key sensitivities from the dual solution to the calibration problem. A fourth section details how these results can be used to form hedging portfolios and finally, in the last section, we present some numerical results.

2 Model Calibration

In this section, we begin by briefly recalling the Libor Market Model construction along the lines of Brace et al. (1997) (see also Jamshidian (1997), Sandmann & Sondermann (1997) and Miltersen, Sandmann & Sondermann (1997)). We then describe how to form the calibration program.

2.1 Zero coupon dynamics

We use the Musiela parametrization of the Heath, Jarrow & Morton (1992) setup. $r(t,\theta)$ is the continuously compounded instantaneous forward rate at time t, with duration θ. To avoid any confusion, Roman letters will be used for maturity dates and Greek ones for durations. The zero-coupon is here computed as

$$B(t,T) = \exp\left(-\int_0^{T-t} r(t,\theta)d\theta\right) \tag{1}$$

All dynamics are described in a probability space $(\Omega, \{F_t; t \geq 0\}, \mathbb{P})$ where the filtration $\{F_t; t \geq 0\}$ is the $\mathbb{P}$-augmentation of the natural filtration generated by a d dimensional Brownian motion $W = \{W_t, t \geq 0\}$. The savings account is defined by:

$$\beta_t = \exp\left(\int_0^t r(s,0)ds\right)$$

and represents the amount generated at time $t \geq 0$ by continuously reinvesting 1 euro in the spot rate $r(s,0)$ during the period $0 \leq s \leq t$. As in Heath et al. (1992), the absence of arbitrage between all zero-coupons and the savings account states that:

$$\frac{B(t,T)}{\beta_t} = B(0,T)\exp\left(-\int_0^t \sigma(s,T-s)dW_s - \frac{1}{2}\int_0^t |\sigma(s,T-s)|^2 ds\right) \tag{2}$$

is a martingale under $\mathbb{P}$ for all $T > 0$, where for all $\theta \geq 0$ the zero-coupon bond volatility process $\{\sigma(t,\theta); \theta \geq 0\}$ is F_t-adapted with values in $\mathbb{R}^d$. We assume that the function $\theta \longmapsto \sigma(t,\theta)$ is absolutely continuous and the derivative $\tau(t,\theta) = \partial/\partial\theta(\sigma(t,\theta))$ is bounded on $\mathbb{R}^2 \times \Omega$.

2.2 Libor diffusion process

All Heath et al. (1992) based arbitrage models are fully specified by their volatility structure and the forward rates curve today. The central assumption in the Libor Market Model is that for a given maturity δ (for ex. 3

months) the associated Libor rate process $\{L(t,\theta); t \geq 0\}$ defined by

$$1 + \delta L(t,\theta) = \exp\left(\int_{\theta}^{\theta+\delta} r(t,\nu)d\nu\right)$$

has a log-normal volatility structure:

$$dL(t,\theta) = (...)dt + L(t,\theta)\gamma(t,\theta)dW_t \tag{3}$$

where the deterministic volatility function $\gamma : \mathbb{R}_+^2 \longmapsto \mathbb{R}^d$ is bounded and piecewise continuous. Using the Ito formula combined with the dynamics detailed above, we get as in Brace et al. (1997):

$$\begin{aligned} dL(t,\theta) = &\left(\frac{\partial L(t,\theta)}{\partial \theta} + \gamma(t,\theta)\sigma(t,\theta)L(t,\theta) + \frac{\delta L(t,\theta)^2}{1+\delta L(t,\theta)}\left|\gamma(t,\theta)\right|^2\right)dt \\ &+ L(t,\theta)\gamma(t,\theta)dW_t \end{aligned}$$

and the FRA dynamics are given by:

$$dK(t,T) = \gamma(t,T-t)K(t,T)\left[\sigma(t,T-t+\delta)dt + dW_t\right] \quad \text{with } K(t,T) = L(t,T-t)$$

where $\sigma(t,T)$ is defined as:

$$\sigma(t,T_i) = \sum_{j=1}^{i-1} \frac{\delta K(t,T_j)}{1+\delta K(t,T_j)}\gamma(t,T_j-t) \tag{4}$$

and T_j is a calendar with period δ. As in Brace et al. (1997) we set $\sigma(t,\theta) = 0$ for $0 \leq \theta < \delta$.

2.3 Swaps

A swap rate is the rate that zeroes the present value of a set of periodical exchanges of fixed against floating coupons on a Libor rate of given maturity. In a representation that is central in swaption pricing approximations, we can write swaps as baskets of forwards (see for ex. Rebonato (1998)):

$$swap(t,T,T_N) = \sum_{i=i_T}^{n} \omega_i(t)K(t,T_i) \tag{5}$$

If we define i_T such that $T_{i_T} = T$, and:

$$\omega_i(t) = \frac{cvg(T_i,T_{i+1})B(t,T_{i+1})}{Level(t,T,T_N)} \tag{6}$$

Where $cvg(T_{i-1},T_i)$ is the coverage (time interval) between T_{i-1} and T_i. $Level(t,T,T_N)$ is the level payment, i.e. the sum of the discount factors for the fixed calendar of the swap weighted by their associated coverage:

$$Level(t,T,T_N) = \sum_{i=i_T}^{N} cvg(T_i,T_{i+1})B(t,T_{i+1})$$

2.4 Swaption price approximation

As in Brace & Womersley (2000), d'Aspremont (2002) and Kurbanmuradov et al. (2002), we approximate the swap dynamics by a one-dimensional lognormal process:

$$\frac{dswap(s,T,T_n)}{swap(s,T,T_n)} = \sum_{i=1}^{n} \widehat{\omega}_i\gamma(s,T_i-s)dW_t^{LVL} \tag{7}$$

where

$$\widehat{\omega}_i = \omega_i(0)\frac{K(0,T_i)}{swap(0,T,T_i)}$$

is computed from the market data today and W_t^{LVL} is a d dimensional Brownian motion under the swap martingale measure defined in Jamshidian (1997), which takes the level payment as a numéraire. We can use the order zero basket pricing approximation in Huynh (1994) and compute the price of a payer swaption starting with maturity T, written on $swap(s,T,T_N)$, with strike κ using the Black (1976) pricing formula:

$$Level(0,T,T_N)\left(swap(0,T,T_N)N(h)-\kappa N\left(h-\sqrt{V_T}\right)\right) \tag{8}$$

where

$$h=\frac{\left(\ln\left(\frac{swap(0,T,T_N)}{\kappa}\right)+\frac{1}{2}V_T\right)}{\sqrt{V_T}}$$

and $swap(0,T,T_N)$ is the value of the forward swap today with

$$\begin{aligned} V_T &= \int_0^T \left\| \sum_{i=1}^{N} \hat{\omega}_i \gamma(s,T_i-s) \right\|^2 ds = \int_t^T \left(\sum_{i,j=1}^{N} \hat{\omega}_i \hat{\omega}_j \left\langle \gamma(s,T_i-s), \gamma(s,T_j-s) \right\rangle \right) ds \\ &= \int_0^T Tr\left(\Omega_t \Gamma_s\right) ds \end{aligned} \tag{9}$$

This cumulative variance is a linear form on the forward rates covariance. Having constructed Ω_t and $\Gamma_s \in \mathbf{S}^{N-i_T}$ such that:

$$\Omega_t = \hat{\omega}\hat{\omega}^T = \left(\hat{\omega}_i\hat{\omega}_j\right)_{i,j\in[i_T,N]} \succeq 0 \text{ and } \Gamma_s = \left(\left\langle \gamma(s,T_i-s),\gamma(s,T_j-s)\right\rangle\right)_{i,j\in[i_T,N]} \succeq 0$$

where Γ_s is the covariance matrix of the forward rates (or the Gram matrix of the $\gamma(s,T_i-s)$ volatility function defined above). Here *swaptions are priced as basket options* with constant coefficients. As detailed in Brace & Womersley (2000) or d'Aspremont (2002), this simple approximation is accurate to within 1-2%. Finally, caplets are priced as one period swaptions.

2.5 The calibration program

Here, we describe the practical implementation of the calibration program using the swaption pricing approximation detailed above. This is done by discretizing in s the covariance matrix Γ_s. We note $\mathbf{S}^n$ the set of symmetric matrixes of size $n\times n$. We suppose that the calibration data set is made of m swaptions with option maturity T_{S_k} written on swaps of maturity $T_{N_k}-T_{S_k}$ for $k=1,...,m$, with market volatility given by σ_k.

2.5.1 A simple example

Let M be the maximum number of periods covered by all the input instruments and $S=\max_{k=1,\dots,m} S_k$. In the simple case where the volatility of the forwards is of the form $\gamma(s,T-s)=\gamma(T-s)$ with γ piecewise constant over intervals of length δ, the calibration problem becomes:

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & Tr(\Omega_k X)=\sigma_k^2 T_{S_k} \text{ for } k=1,...,m \\ & X\succeq 0 \end{array} \tag{10}$$

which is a semidefinite feasibility problem in the covariance matrix $X\in\mathbf{S}^M$ ($X\succeq 0$ meaning X p.s.d.). Again, $\sigma_k^2 T_k\in\mathbb{R}_+$ is the Black (1976) cumulative variance of swaption k written on $swap(0,T_{S_k},T_{N_k})$ and $\Omega_k=\sum_{j=1}^{S_k}\delta\varphi_{k,j}$ with $\varphi_{k,j}\in\mathbf{S}^M$ the rank one matrix with submatrix $\hat{\omega}_k\hat{\omega}_k^T$ starting at element (j,j) and all other blocks equal to zero. Note that $\hat{\omega}_k$ is here the vector of weights associated to swaption k with $\hat{\omega}_k=\left(\hat{\omega}_{i,k}\right)_{i=S_k,\dots,N_k}$.

2.5.2 The general case

Here we show that for general volatilities $\gamma(s, T-s)$, the format of the calibration problem remains similar to that of the simple example above, except that X becomes block-diagonal. In the general non-stationary case where γ is of the form $\gamma(s, T-s)$ and piecewise constant on intervals of size δ, the expression of the market cumulative variance becomes

$$\sigma_k^2 T_{S_k} = \sum_{i=0}^{T_{S_k}} \delta Tr\left(\Omega_{k,i} X_i\right)$$

where $\Omega_{k,i} \in \mathbf{S}^{M-i}$ is a block-matrix with submatrix $\hat{\omega}_k \hat{\omega}_k^T$ starting at element $(S_k - i, S_k - i)$ and all other blocks equal to zero if $S_k - i \geq 0$, and is zero otherwise. Here X_i is the Gram matrix of the vectors $\gamma(T_i, T_j - T_i)$. Calibrating the model to the swaptions $k = (1, ..., m)$ can then be written as the following semidefinite feasibility problem.:

$$\begin{array}{ll} \text{find} & X_i \quad i = i, ..., T_M \\ \text{s.t.} & \sum_{i=0}^{T_{S_k}} \delta_i Tr\left(\Omega_{i,k} X_i\right) = \sigma_k^2 T_k \text{ for } k = 1, ..., m \\ & X_i \succeq 0 \text{ for } i = 0, ..., T_S \end{array}$$

and the variables here are the matrixes $X_i \in \mathbf{S}^{M-i}$. We can write this general problem in the same format used in the simple stationary case. Let X be the block matrix

$$X = \left[\begin{array}{cccc} X_1 & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & 0 \\ 0 & . & 0 & X_{T_M} \end{array}\right]$$

the calibration program can be written as in (10):

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & Tr(\bar{\Omega}_k X) = \sigma_k^2 T_k \text{ for } k = 1, ..., m \\ & X \succeq 0, X \text{ band-diagonal} \end{array} \tag{11}$$

except that $\bar{\Omega}_k$ and $X \in \mathbf{S}^{M-i}$ are here "block-diagonal". We can also replace the equality constraints in (10) with Bid-Ask spreads. The new calibration problem is then written as the L.M.I.:

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & \sigma_{Bid,k}^2 T_{S_k} \leq Tr(\Omega_k X) \leq \sigma_{Ask,k}^2 T_{S_k} \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

in the variable $X \in \mathbf{S}^M$, with parameters Ω_k, $\sigma_{Bid,k}^2$, $\sigma_{Ask,k}^2$, T_{S_k}. Again, we can rewrite this program as a standard form L.M.I.:

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & Tr\left(\left[\begin{array}{ccc} \Omega_k & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{array}\right]\left[\begin{array}{ccc} X & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & U_2 \end{array}\right]\right) = \sigma_{Ask,k}^2 T_{S_k} \\ & Tr\left(\left[\begin{array}{ccc} \Omega_k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{array}\right]\left[\begin{array}{ccc} X & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & 0 & U_2 \end{array}\right]\right) = \sigma_{Bid,k}^2 T_{S_k} \text{ for } k = 1, ..., m \\ & X, U_1, U_2 \succeq 0 \end{array}$$

which can be summarized as

$$\begin{array}{ll} \text{find} & \tilde{X} \\ \text{s.t.} & Tr(\tilde{\Omega}_{Ask,k} \tilde{X}) = \sigma_{Ask,k}^2 T_{S_k} \\ & Tr(\tilde{\Omega}_{Bid,k} \tilde{X}) = \sigma_{Bid,k}^2 T_{S_k} \text{ for } k = 1, ..., m \\ & \tilde{X} \succeq 0, \tilde{X} \text{ block-diagonal} \end{array} \tag{12}$$

with $\tilde{X}, \tilde{\Omega}_k \in \mathbf{S}^{3M}$. Because of these transformations and to simplify the analysis, we will always handle the stationary case with equality constraints in the following section, knowing that all results can be directly extended to the general case (non-stationary covariance with Bid-Ask constraints) by embedding them in a larger, block-diagonal semidefinite program.

3 Sensitivity analysis

In this section, we begin by a brief description of Lagrangian duality for semidefinite programs. We then show how the dual optimal solution can be exploited for computing solution sensitivities with minimal numerical cost.

3.1 Semidefinite duality

We very briefly summarize here the duality theory for semidefinite programming. We refer the reader to Nesterov & Nemirovskii (1994) or Vandenberghe & Boyd (1996) for a complete analysis. As we have seen in the previous section, the calibration problem can be written as a standard form primal semidefinite program:

$$\begin{array}{ll} \text{minimize} & Tr(CX) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{S_k} \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array} \tag{13}$$

in the variable $X \in \mathbf{S}^M$ with parameters $\Omega_k, C \in \mathbf{S}^M$ and $\sigma_k^2 T_{S_k} \in \mathbb{R}_+$. For $X \succeq 0$, $y \in \mathbb{R}^m$, we form the following Lagrangian:

$$\begin{aligned} L(X, y) &= Tr(CX) + \sum_{k=1}^{m} y_k \left(Tr(\Omega_k X) - \sigma_k^2 T_{S_k} \right) \\ &= Tr\left(\sum_{k=1}^{m} (y_k \Omega_k + C) X \right) - \sum_{k=1}^{m} y_k \sigma_k^2 T_{S_k} \end{aligned}$$

and because the semidefinite cone is self-dual, we find that $L(X, y)$ is bounded below in $X \succeq 0$ iff:

$$0 \preceq \sum_{k=1}^{m} y_k \Omega_k + C$$

hence the dual semidefinite problem becomes:

$$\begin{array}{ll} \text{maximize} & -\sum_{k=1}^{m} y_k \sigma_k^2 T_{S_k} \\ \text{s.t.} & 0 \preceq \left(\sum_{k=1}^{m} y_k \Omega_k + C \right) \end{array} \tag{14}$$

All modern solvers (see for example Sturm (1999)) produce both primal and dual solutions to this problem as well as a certificate of optimality for the solution in the form of the associated duality gap:

$$\mu = Tr\left(X \left(\sum_{k=1}^{m} y_k \Omega_k - C \right) \right)$$

which is an upper bound on the absolute error. We now show how this dual solution can be used for risk-management purposes.

3.2 Computing sensitivities

Let us suppose that we have solved both the primal and the dual calibration problems above with market constraints $\sigma_k^2 T_{S_k}$ and let X^{opt} and y^{opt} be the optimal solutions. Suppose also that the market swaption price constraints are modified by a small amount $u \in \mathbb{R}^m$. The new calibration problem becomes:

$$\begin{array}{ll} \text{maximize} & Tr(CX) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{S_k} + u_k \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

in the variable $X \in \mathbf{S}^M$ with parameters $\Omega_k, C \in \mathbf{S}^M$ and $\sigma_k^2 T_{S_k} \in \mathbb{R}_+$ and if we note $p^{opt}(u)$ the optimal solution to the revised problem, we get (at least formally for now) the sensitivity of the solution to a change in market condition as:

$$\frac{\partial p^{opt}(0)}{\partial u_k} = -y_k^{opt} \tag{15}$$

where y^{opt} is the optimal solution to the dual problem. As we will see in this section and the next one, this has various interpretations depending on the objective function. here, we want to study the variation in the solution matrix X^{opt}, given a small change u in the market conditions. Let us suppose that we have solved the general calibration problem:

$$\begin{array}{ll} \text{maximize} & Tr(CX) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{S_k} + u_k \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

$X \in \mathbf{S}^M$ with parameters $\Omega_k, C \in \mathbf{S}^M$ and $\sigma_k^2 T_{S_k} \in \mathbb{R}_+$. Here C is, for example, an historical estimate of the covariance matrix.

Notation 1 *Let us call X^{opt} and y^{opt} the primal and dual solutions to the above problem with $u = 0$. We note*

$$Z^{opt} = \left(C - \sum_{k=1}^{m} y_k^{opt} \Omega_k \right)$$

the dual solution matrix. As in Alizadeh, Haeberly & Overton (1998), we also define the symmetric Kronecker product as:

$$(P \circledast Q)\, K := \frac{1}{2}\left(PKQ^T + QKP^T\right)$$

We note A and A^*, the linear operators defined by:

$$\begin{array}{l} A : \mathbf{S}^M \longrightarrow \mathbb{R}^m \\ X \longmapsto AX := \left(Tr\left(A_i X\right)\right)_{i=1,\dots,m} \end{array} \quad \textit{and its dual} \quad \begin{array}{l} A^* : \mathbb{R}^m \longrightarrow \mathbf{S}^M \\ y \longmapsto A^* y := \sum_{i=1}^m y_i \Omega_i \end{array}$$

The results in Todd & Yildirim (1999) compute the impact ΔX on the solution of a small change in the market price data $(u_k)_{k=1,..,m}$, i.e. given u small enough we compute the next Newton step ΔX. Each solver implements one particular search direction to compute this step and some most common ones are the A.H.O. search direction based on the work by Alizadeh et al. (1998), the H.K.M. direction by Helmberg, Rendl, Vanderbei & Wolkowicz (1996), Kojima, Shindoh & Hara (1997) and Monteiro (1997). Depending on the choice of the search direction, we define a matrix m such that:

- $M = I$ for the A.H.O. direction
- $M = Z^{opt}$ for the H.K.M. direction.

We also define the linear operators:

$$E = Z^{opt} \circledast M \;\text{ and }\; F = MX^{opt} \circledast I$$

and their adjoints

$$E^* = Z^{opt} \circledast M \;\text{ and }\; X^{opt} M \circledast I$$

We remark that if $A, B \in \mathbf{S}^M$commute, with eigenvalues $\alpha, \beta \in \mathbb{R}^M$ and common eigenvectors v_i for $i = 1, ..., M$, then $A \circledast B$ has eigenvalues $(\alpha_i \beta_j + \alpha_j \beta_i)$ for $i, j = 1, ..., M$ and eigenvectors $v_i v_i^T$ if $i = j$

and $(v_i v_j^T + v_j v_i^T)$ if $i \neq j$ for $i, j = 1, ..., M$. Provided the strict feasibility and nonsingularity conditions in §3 of Todd & Yildirim (1999) hold, we can compute the Newton step ΔX as:

$$\Delta X = E^{-1} F A^* \left[\left(A E^{-1} F A^* \right)^{-1} u \right] \tag{16}$$

and this will lead to a feasible point $X^{opt} + \Delta X \succeq 0$ iff the market variation movement u is such that:

$$\left\| \left(X^{opt} \right)^{-\frac{1}{2}} \left(E^{-1} F A^* \left[\left(A E^{-1} F A^* \right)^{-1} u \right] \right) \left(X^{opt} \right)^{-\frac{1}{2}} \right\|_2 \leq 1 \tag{17}$$

The intuition behind this formula is that semidefinite programming solvers are based on the Newton method and condition (17) ensures that the solution X^{opt} remains in the region of quadratic convergence of the Newton algorithm. This means that only one Newton step is required to produce the new optimal solution $X^{opt} + \Delta X$ and (16) simply computes this step. The matrix in (16) produces a *direct method for updating* X which we can now use to compute price sensitivities for any given portfolio. This illustrates how a semidefinite programming based calibration allows to test various *realistic scenarios* at a minimum numerical cost and improves on the classical non-convex methods that either had to "bump the market data and recalibrate" the model for every scenario with the risk of jumping from one local optimum to the next, or simulate unrealistic market movements by directly adjusting the covariance matrix. One key remaining question however is that of stability: the calibration program in (10) has a unique solution, but this optimum can be very unstable and the matrix in (16) badly conditioned. In the spirit of the work by Cont (2001) on volatility surfaces, we now look for a way to stabilize the calibration result.

3.3 Robustness

The previous sections were focused on how to compute the impact of a change in market conditions. Here we will focus on how to anticipate those variations and make the calibrated matrix optimally robust to a given set of scenarios. Depending on the way the perturbations are modeled, this problem can remain convex and be solved very efficiently. Let us suppose here that we want to solve the calibration problem on a set of market Bid-Ask spreads data defined by the following L.M.I.:

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & \sigma^2_{Bid,k} T_{S_k} \leq Tr(\Omega_k X) \leq \sigma^2_{Ask,k} T_{S_k} \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

in the variable $X \in \mathbf{S}^M$ with parameters $\Omega_k, C \in \mathbf{S}^M$ and $\sigma^2_{Bid,k} T_{S_k}$, $\sigma^2_{Ask,k} T_{S_k} \in \mathbb{R}_+$. In the absence of any information on the uncertainty in the market data, we can simply maximize the distance between the solution and the market bounds to ensure that it remains valid in the event of a small change in the market variance input. As the robustness objective is equivalent to a distance maximization between the solution and the constraints (or Chebyshev centering), the input of assumptions on the movement structure is equivalent to a choice of norm. Without any particular structural information on the volatility market dynamics, we can use the l_∞ norm and the calibration problem becomes:

$$\begin{array}{ll} \text{maximize} & t \\ \text{s.t.} & \sigma^2_{Bid,k} T_{S_k} + t \leq Tr(\Omega_k X) \leq \sigma^2_{Ask,k} T_{S_k} - t \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

or, using the l_1 norm instead:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^m t_k \\ \text{s.t.} & \sigma^2_{Bid,k} T_{S_k} + t_k \leq Tr(\Omega_k X) \leq \sigma^2_{Ask,k} T_{S_k} - t_k \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

The problems above optimally center the solution within the Bid-Ask spreads, which makes it robust to a change in market conditions given no particular information on the nature of that change. In the same vein, Ben-Tal, El Ghaoui & Lebret (1998) also show how to design a program that is robust to a change in the matrixes Ω_k. However, because the matrixes Ω_k are computed from ratios of zero-coupon bonds, their variance is negligible compared to that of σ_k^2.

Suppose now that V is a statistical estimate of the daily covariance of $\sigma_k^2 T_{S_k}$ (the mid-market volatilities in this case) and let us assume that these volatilities have a Gaussian distribution. We adapt the method used by Lobo, Vandenberghe, Boyd & Lebret (1998) for robust L.P. We suppose that the matrix V has full rank. We can then center the model with respect to this information:

$$\begin{array}{ll} \text{maximize} & Tr(CX) \\ \text{s.t.} & Tr\left(\Omega_k X\right) - \sigma_k^2 T_{S_k} = v_k \ \text{ for } k = 1, ..., m \\ & \left\| V^{-\frac{1}{2}} v \right\|_\infty \leq \Phi^{-1}(\mu) \\ & X \succeq 0 \end{array}$$

where $\|\cdot\|_\infty$ is the l_∞ norm and $\Phi(x)$ is given by

$$\Phi(x) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \exp(-u^2/2) du$$

There is no guarantee that this program is feasible and we can solve instead for the best confidence level by forming the following program:

$$\begin{array}{ll} \text{minimize} & t \\ \text{s.t.} & Tr\left(\Omega_k X\right) - \sigma_k^2 T_{S_k} = v_k \ \text{ for } k = 1, ..., m \\ & \left\| V^{-\frac{1}{2}} v \right\|_\infty \leq t \\ & X \succeq 0 \end{array}$$

The optimal confidence level is then $\eta = \Phi(t)$ and "centers" the calibrated matrix with respect to the uncertainty in $\sigma_k^2 T_{S_k}$. This is a symmetric cone program, i.e. a program mixing LP, second-order and semidefinite cone constraints, and can be solved very efficiently using the code by Sturm (1999) for example.

4 Hedging

In this section, we show how the calibration programs can be used to build a superreplicating portfolio, approximating the upper and lower hedging prices defined in El Karoui & Quenez (1991) and El Karoui & Quenez (1995). An efficient technique for computing those price bounds with general non-convex payoffs on a single asset with univariate dynamics was introduced in Avellaneda et al. (1995) and recent work on this topic by Romagnoli & Vargiolu (2000) provided closed-form solutions for the prices of exchange options and options on the geometrical mean of two assets. Gozzi & Vargiolu (2000) applied the same technique to caps and floors.

4.1 Approximate solution

Here, using the approximation in (8), we first compute arbitrage bounds on the price of a basket by adapting the method used by Avellaneda et al. (1995) in the one-dimensional case. We then provide approximate (to within 1-2%) closed-form solutions for these arbitrage bounds on swaptions and show how one can use the dual solution to build an optimal hedging portfolio in the sense of Avellaneda & Paras (1996), using derivative securities taken from the calibration set.

As in Avellaneda & Paras (1996), the price here is derived from a mixed static-dynamic representation:

$$\textbf{Price} = \text{Min}\left\{\text{Value of } \textbf{static hedge} + \text{Max}\left(\text{PV of } \textbf{residual liability}\right)\right\} \tag{18}$$

where the static hedge is a portfolio composed of the calibration assets and the maximum residual liability is computed as in El Karoui et al. (1998) or Avellaneda et al. (1995). Because of the sub-additivity of the above program with respect to payoffs, we expect this diversification of the volatility risk to bring down the total cost of hedging. Let $K(t) = (K(t, T_i))_{i=1,\ldots,M}$ and suppose we have a set of market prices $C_i, i = 1, \ldots, m$, with corresponding market volatilities σ_k and payoffs

$$h_{\omega_k,K_k}(K(T)) = \left(\sum_{j=S_k}^{N_k} \omega_{j,k} K(T_{S_k}, T_j)\right)^+$$

for basket options with coefficient matrixes $\Omega_k \in \mathbf{S}^M$ for $k = 1, \ldots, m$. To approximate the optimal hedging portfolio in the sense of Avellaneda & Paras (1996) we can form a portfolio composed of a static part with λ_k^{opt} basket k, where λ_k^{opt} is given by:

$$\lambda_k^{opt} = -y_k^{opt} \frac{\partial BS_0\left(Tr(\Omega_0 X^{opt})\right)/\partial v}{\partial BS_k\left(Tr(\Omega_k X^{opt})\right)/\partial v} \text{ for } k = 1, ..., m \tag{19}$$

where, for simplicity, we have noted $BS_k(v)$ the price of basket k as a function of the cumulative variance v, computed as in (8) with $X^{opt} \in \mathbf{S}^M$ and $y_k^{opt} \in \mathbb{R}^m$ are the primal and dual solutions to the semidefinite program:

$$\begin{array}{ll} \text{maximize} & \sigma_{\max}^2 T = Tr(\Omega_0 X) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_{S_k} \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

This is because as in Avellaneda & Paras (1996) we can write the price as:

$$\inf_{\lambda \in \mathbb{R}^m} \left\{ \sum_{k=1}^m \lambda_k C_k + \left(\sup_P E^P \left[\beta(T)^{-1} h_{\omega,K_o}(K(T)) - \sum_{k=1}^m \lambda_k \beta(T)^{-1} h_{\omega_k,K_k}(K(T)) \right] \right) \right\} \tag{20}$$

where P varies within the set of equivalent martingale measure and $\beta(T)$ is the value of the savings account in T. We can rewrite the above problem as:

$$\inf_\lambda \left\{ \sup_P \left(E^P \left[\beta(T)^{-1} h_{\omega_0,K_0}(K(T)) \right] - \sum_{i=1}^m \lambda_k \left(E^P \left[\beta(T)^{-1} h_{\omega_k,K_k}(K(T)) \right] - C_k \right) \right) \right\}$$

where we recognize the optimum hedging portfolio problem as the dual of the maximum price problem above:

$$\begin{array}{ll} \text{maximize} & E^P \left[\beta(T)^{-1} h_{\omega_0,K_0}(K(T)) \right] \\ \text{s.t.} & E^P \left[\beta(T)^{-1} h_{\omega_k,K_k}(K(T)) \right] = C_k \text{ for } k = 1, ..., m \end{array}$$

Using (8), we get an approximate solution by solving the following problem:

$$\begin{array}{ll} \text{maximize} & BS_0(Tr(\Omega_0 X)) \\ \text{s.t.} & BS_k\left(Tr(\Omega_k X)\right) = C_k \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

and its dual:

$$\inf_\lambda \left\{ \sup_{X \succeq 0} \left(BS(Tr(\Omega_0 X)) - \sum_{k=1}^m \lambda_k \left(BS\left(Tr(\Omega_k X)\right) - C_k \right) \right) \right\}$$

The primal problem, after we write it in terms of variance, becomes the following semidefinite program:

$$\begin{array}{ll} \text{maximize} & \sigma_{\max}^2 T = Tr(\Omega_0 X) \\ \text{s.t.} & Tr(\Omega_k X) = \sigma_k^2 T_k \text{ for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

Again, we note $y^{opt} \in \mathbb{R}^m$ the solution to the dual of this last problem:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^m y_k \sigma_k^2 T_k \\ \text{s.t.} & 0 \preceq \sum_{k=1}^m y_k \Omega_k - \Omega_0 \end{array}$$

The KKT optimality conditions on the primal-dual semidefinite program pair above (see Vandenberghe & Boyd (1996) for example) can be written:

$$\begin{cases} 0 \preceq \sum_{k=1}^{m} y_k \Omega_k - \Omega_0 \\ 0 = \sum_{k=1}^{m} y_k \Omega_k X - \Omega_0 X \\ Tr(\Omega_k X) = \sigma_k^2 T_k \text{ for } k = 1, ..., m \\ 0 \preceq X \end{cases}$$

and we can compare those to the KKT conditions for the price maximization problem:

$$\begin{cases} Z = \frac{\partial BS_0(Tr(\Omega_0 X))}{\partial v} \Omega_0 + \sum_{k=1}^{m} \lambda_k \frac{\partial BS_k(Tr(\Omega_k X))}{\partial v} \Omega_k \\ XZ = 0 \\ BS_k\left(Tr(\Omega_k X)\right) = C_i \text{ for } k = 1, ..., m \\ 0 \preceq X, Z \end{cases}$$

with dual variables $\lambda \in \mathbb{R}^m$ and $Z \in \mathbf{S}_n$. An optimal dual solution for the price maximization problem can then be constructed from y^{opt}, the optimal dual solution of the semidefinite program on the variance, as:

$$\lambda_k^{opt} = -y_k^{opt} \frac{\partial BS_0\left(Tr(\Omega_0 X)\right)/\partial v}{\partial BS_k\left(Tr(\Omega_k X)\right)/\partial v}$$

which corresponds to the composition in the baskets k of the optimal static hedging portfolio (18) .

4.2 The exact problem

The bounds found in the section above are only approximate solutions to the superreplicating problem. Although the relative error in this approximation is known to be about 1-2%, it is interesting to notice that although it does not remain completely tractable (as a dynamic program in multiple dimensions), the exact problem shares the same optimization structure as the approximate one. Let us recall the results in Romagnoli & Vargiolu (2000). If, as above, we note $C(K(t), t)$ the superreplicating price of a basket option, then $C(K(t), t)$ is the solution to the a multidimensional Black-Scholes-Barenblatt equation We can create a superreplicating strategy by dynamically trading in a portfolio composed of $\Delta_t^i = \frac{\partial C}{\partial x_i}(t, K(t, T_i))$ in each asset. The BSB equation in (Romagnoli & Vargiolu (2000)) can be rewritten it in a format that is similar to that of the approximate problem above, to become:

$$\begin{cases} \frac{\partial C(x,t)}{\partial t} + \frac{1}{2} \max_{\Gamma \in \Lambda} Tr\left(diag(x) \frac{\partial^2 C(x,t)}{\partial x^2} diag(x) \Gamma\right) = 0 \\ \\ C(x, T) = \left(\sum_{i=1}^{n} \omega_i x_i - k\right)^+ \end{cases}$$

where $diag(x)$ is the diagonal matrix formed with the components of x and $\Gamma = \gamma\gamma^T$ is the model covariance matrix. If the set Λ is given by the intersection of the semidefinite cone (the covariance matrix has to be p.s.d.) with a polyhedron (for example approximate price constraints, sign constraints or bounds on the matrix coefficients, ...), then the embedded optimization problem in (Romagnoli & Vargiolu (2000)) becomes a semidefinite program:

$$\text{maximize }_{\Gamma \in \Lambda} Tr\left(\Gamma diag(x) \frac{\partial^2 C(x,t)}{\partial x^2} diag(x)\right)$$

on the feasible set Λ. Hence we recover the same optimization problem as in the approximate solution found in the section above, the only difference being here that the solution to the exact general problem might not be equal to a Black-Scholes price. This gives a very straightforward interpretation of the embedded optimization problem in the BSB equation developed in Romagnoli & Vargiolu (2000).

4.3 Optimal Gamma Hedging

For simplicity here, we work in a pure equity framework and, along the lines of Douady (1995), we study the problem of optimally adjusting the Gamma of a portfolio using only options on single assets. This problem is essentially motivated by a difference in liquidity between the vanilla and basket option markets, which makes it impractical to use some baskets to adjust the Gamma of a portfolio. Suppose we have an initial portfolio with a Gamma sensitivity matrix given by Γ in a market with underlying assets $x_i, i = 1, \ldots n$. We want to hedge (imperfectly) this position with y_i vanilla options on each single asset x_i with Gamma given by γ_i. We assume that the portfolio is maintained delta-neutral, hence a small perturbation of the stock price will induce a change in the portfolio price given by:

$$\Delta P(X + \Delta X) = P(X) + \frac{1}{2}\Delta S^T \Gamma(y) \Delta S$$

where $\Gamma(y) = \Gamma + diag(\gamma)y$, with $diag(\gamma)$ the diagonal matrix with components γ_i. As in Douady (1995), our objective is to minimize in y the maximum possible perturbation given by:

$$\max_{\Delta S \in E} \left|\Delta S^T \Gamma(y) \Delta S\right|$$

where E is the ellipsoid defined by

$$E = \left\{X \in \mathbf{R}^n | X^T \Sigma X = 1\right\} \text{ with } \Sigma = (cov(x_i, x_j))_{i,j=1,\ldots,n}$$

the covariance matrix of the underlying assets. This amounts to minimizing the maximum eigenvalue of the matrix $\Sigma\Gamma(y)$ and can be solved by the following semidefinite program:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & -tI \preceq \Sigma\Gamma + \Sigma diag(\gamma)y \preceq tI \end{array}$$

We can also introduce constraints on the cost of hedging. Suppose that there are proportional transaction costs associated with trading in the vanilla option on x_i given by $k_i |y_i|$ for some $k_i \geq 0$. The problem becomes a symmetric cone program:

$$\begin{array}{ll} \text{minimize} & t_0 + \alpha \sum_{i=1}^n k_i t_i \\ \text{subject to} & -t_0 I \preceq \Sigma\Gamma + \Sigma diag(\gamma)y \preceq t_0 I \\ & -t_i \leq y_i \leq t_i \end{array}$$

and can be solved using the code by Sturm (1999). The parameter α describes the relative importance of minimizing hedging cost compared to minimizing the gamma.

5 Numerical results

We use a data set from Nov. 6 2000 and we plot in figure (1) the upper and lower bounds obtained by maximizing (resp. minimizing) the volatility of a given swaption provided that the Libor covariance matrix remains positive semidefinite and that it matches the calibration instruments. We calibrate by fitting all caplets up to 20 years plus the following set of swaptions: 2Y into 5Y, 5Y into 5Y, 5Y into 2Y, 10Y into 5Y, 7Y into 5Y, 10Y into 2Y, 10Y into 7Y, 2Y into 2Y. This choice of swaptions was motivated by liquidity (where all swaptions on underlying and maturity in 2Y, 5Y, 7Y, 10Y are meant to be liquid). Table (1) details the market caplet volatilities, while table (2) shows the swaption volatilities and the corresponding $\hat{\omega}_i$ weights (data courtesy of BNP Paribas, London). For simplicity, all frequencies are annual.

Quite surprisingly considering the simplicity of the model (stationarity of the *sliding* Libor dynamics $L(t, \theta)$), figure (1) shows that all swaptions seem to fit reasonably well in the bounds imposed by the model, except for the 10Y underlying. This is in line with the findings of Longstaff et al. (2000). In table (3) and (4), we show the market volatility movement vector with largest impact on the covariance matrix (first vector in the singular value decomposition of the sensitivity matrix in 16), computed in the A.H.O. case, using the same dataset above and a minimum trace objective.

6 Conclusion

The results above have showed how semidefinite programming based calibration methods can provide integrated calibration and risk-management results with guaranteed numerical performance, the dual program having a very natural interpretation in terms of hedging intruments and sensitivity.

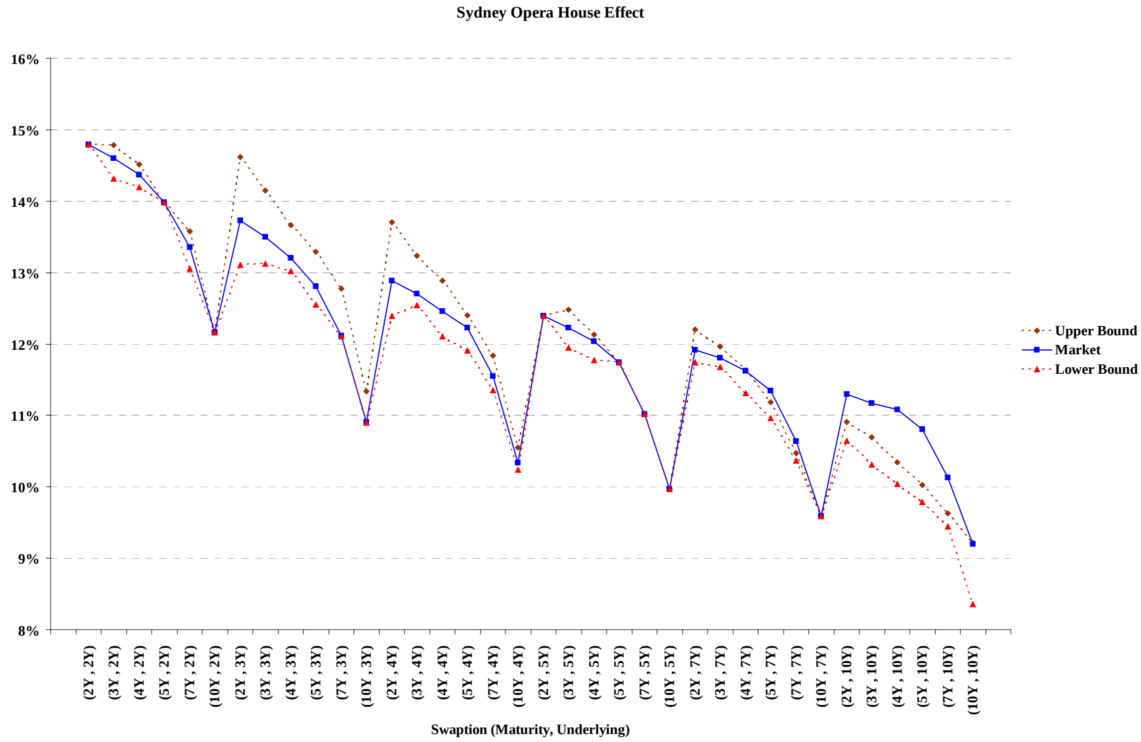

Figure 1: Calibration result and price bounds on a "Sydney opera house" set of swaptions.

Caplet Vols (%, 1Y to 10Y)	14.3	15.6	15.4	15.1	14.8	14.5	14.2	14.0	13.9	13.3
Caplet Vols (%, 11Y to 20Y)	13.0	12.7	12.4	12.2	12.0	11.9	11.8	11.8	11.7	12.0

Table 1: Caplet volatilities.

Swaption	**Vol (%)**	$\hat{\omega}_i$						
2Y into 5Y	12.4	0.22	0.20	0.20	0.19	0.18		
5Y into 5Y	11.7	0.22	0.21	0.20	0.19	0.18		
5Y into 2Y	14.0	0.51	0.49					
10Y into 5Y	10.0	0.22	0.21	0.20	0.19	0.18		
7Y into 5Y	11.0	0.23	0.21	0.20	0.19	0.18		
10Y into 2Y	12.2	0.51	0.49					
10Y into 7Y	9.6	0.17	0.16	0.15	0.14	0.13	0.13	0.12
2Y into 2Y	14.8	0.52	0.48					

Table 2: Swaption volatilities and weights.

Caplet (1Y to 10Y)	0.00	-0.01	0.01	0.21	-0.25	-0.08	0.04	-0.17	-0.09	0.18
Caplet (11Y to 20Y)	0.15	-0.01	0.19	-0.18	0.29	0.04	0.09	-0.52	0.09	0.00

Table 3: First sensitivity factor, coefficients corresponding to caplets.

Swaption	(2Y,5Y)	(5Y,5Y)	(5Y,2Y)	(10Y,5Y)	(7Y,5Y)	(10Y,2Y)	(10Y,7Y)	(2Y,2Y)
Sensitivity	-0.05	0.16	0.28	0.17	-0.18	-0.38	0.15	-0.08

Table 4: First sensitivity factor, coefficients corresponding to swaptions.

References

Alizadeh, F., Haeberly, J. A. & Overton, M. L. (1998), 'Primal–dual interior–point methods for semidefinite programming: Convergence rates, stability and numerical results', *SIAM Journal on Optimization* **8**, 746–768.

Avellaneda, M., Levy, A. & Paras, A. (1995), 'Pricing and hedging derivative securities in markets with uncertain volatilities', *Applied Mathematical Finance* **2**, 73–88.

Avellaneda, M. & Paras, A. (1996), 'Managing the volatility risk of portfolios of derivative securities: the lagrangian uncertain volatility model', *Applied Mathematical Finance* **3**, 21–52.

Ben-Tal, A., El Ghaoui, L. & Lebret, H. (1998), Robust semidefinite programming, *in* S. Sastry, H. Wolkowitcz & L. Vandenberghe, eds, 'Handbook on Semidefinite Programming', Vol. 27 of *International Series in Operations Research and Management Science*.

Black, F. (1976), 'The pricing of commodity contracts.', *Journal of Financial Economics* **3**, 167–179.

Brace, A., Dun, T. & Barton, G. (1999), 'Towards a central interest rate model', *Working Paper. FMMA* .

Brace, A., Gatarek, D. & Musiela, M. (1997), 'The market model of interest rate dynamics', *Mathematical Finance* **7**(2), 127–155.

Brace, A. & Womersley, R. S. (2000), 'Exact fit to the swaption volatility matrix using semidefinite programming', *Working paper, ICBI Global Derivatives Conference* .

Cont, R. (2001), 'Inverse problems in financial modeling: theoretical and numerical aspects of model calibration.', *Lecture Notes, Princeton University.* .

d'Aspremont, A. (2002), Interest Rate Model Calibration and Risk-Management Using Semidefinite Programming, PhD thesis, Ecole Polytechnique.

Douady, R. (1995), 'Optimisation du gamma d'une option sur panier ou sur spread en l'absence d'options croisées.', *Working paper* .

El Karoui, N., Jeanblanc-Picqué, M. & Shreve, S. E. (1998), 'On the robustness of the black-scholes equation', *Mathematical Finance* **8**, 93–126.

El Karoui, N. & Quenez, M. (1991), 'Programmation dynamique et évaluation des actifs contingents en marchés incomplets.', *Comptes Rendus de l'Académie des Sciences de Paris, Série I* **313**, 851–854.

El Karoui, N. & Quenez, M. (1995), 'Dynamic programming and pricing of contingent claims in an incomplete market', *Siam Journal of Control and Optimization* **33**, 29–66.

Gozzi, F. & Vargiolu, T. (2000), 'On the superreplication approach for european interest rates derivatives', *Proceedings of the Ascona '99 Seminar on Stochastic Analysis, Random Fields and Applications* .

Heath, D., Jarrow, R. & Morton, A. (1992), 'Bond pricing and the term structure of interest rates: A new methodology', *Econometrica* **61**(1), 77–105.

Helmberg, C., Rendl, F., Vanderbei, R. J. & Wolkowicz, H. (1996), 'An interior–point method for semidefinite programming', *SIAM Journal on Optimization* **6**, 342–361.

Huynh, C. B. (1994), 'Back to baskets', *Risk* **7**(5).

Jamshidian, F. (1997), 'Libor and swap market models and measures', *Finance and Stochastics* **1**(4), 293–330.

Kawai, A. (2002), 'Analytical and monte-carlo swaption pricing under the forward swap measure', *Journal of Computational Finance* **6**(1), 101–111.

Kojima, M., Shindoh, S. & Hara, S. (1997), 'Interior-point methods for the monotone semidefinite linear complementarity problem in symmetrix matrixes', *SIAM Journal on Optimization* **7**, 86–125.

Kurbanmuradov, O., Sabelfeld, K. & Schoenmakers, J. (2002), 'Lognormal approximations to libor market models', *Journal of Computational Finance* **6**(1), 69–100.

Lobo, M., Vandenberghe, L., Boyd, S. & Lebret, H. (1998), 'Applications of second-order cone programming', *Linear Algebra and its Applications* **284**, 193–228. Special Issue on Linear Algebra in Control, Signals and Image Processing.

Longstaff, F. A., Santa-Clara, P. & Schwartz, E. S. (2000), 'The relative valuation of caps and swaptions: Theory and empirical evidence.', *Working Paper, The Anderson School at UCLA.* .

Miltersen, K., Sandmann, K. & Sondermann, D. (1997), 'Closed form solutions for term structure derivatives with log-normal interest rates', *Journal of Finance* **52**(1), 409–430.

Monteiro, R. (1997), 'Primal-dual path following algorithms for semidefinite programming', *SIAM Journal on Optimization* **7**, 663–678.

Musiela, M. & Rutkowski, M. (1997), *Martingale methods in financial modelling*, Vol. 36 of *Applications of mathematics*, Springer, Berlin.

Nesterov, I. E. & Nemirovskii, A. S. (1994), *Interior-point polynomial algorithms in convex programming*, Society for Industrial and Applied Mathematics, Philadelphia.

Rebonato, R. (1998), *Interest-Rate Options Models*, Financial Engineering, Wiley.

Rebonato, R. (1999), 'On the simultaneous calibration of multi-factor log-normal interest rate models to black volatilities and to the correlation matrix.', *Working paper* .

Romagnoli, S. & Vargiolu, T. (2000), 'Robustness of the black-scholes approach in the case of options on several assets', *Finance and Stochastics* **4**, 325–341.

Sandmann, K. & Sondermann, D. (1997), 'A note on the stability of lognormal interest rate models and the pricing of eurodollar futures', *Mathematical Finance* **7**, 119–128.

Schwartz, S. C. & Yeh, Y. S. (1981), 'On the distribution function and moments of power sums with log-normal components.', *The Bell System Technical Journal* **61**(7), 1441–1462.

Sturm, J. F. (1999), 'Using sedumi 1.0x, a matlab toolbox for optimization over symmetric cones.', *Working paper, Department of Quantitative Economics, Maastricht University, The Netherlands.* .

Todd, M. & Yildirim, E. A. (1999), 'Sensitivity analysis in linear programming and semidefinite programming using interior-points methods.', *Working paper, School of Operation Research and Industrial Engineering, Cornell University.* .

Vandenberghe, L. & Boyd, S. (1996), 'Semidefinite programming', *SIAM Review* **38**, 49–95.