A Harmonic Analysis Solution to the Static Basket Arbitrage Problem

Alexandre d'Aspremont

ORFE, Princeton University

Available online at www.princeton.edu/~aspremon

Introduction

- The classic Black & Scholes (1973) option pricing is based on:
  - a *model* for the asset dynamics (geometric BM)
  - a *dynamic hedging* argument
- sensitive to liquidity, transaction costs, model risk ...
- what can we say about option prices with *much weaker assumptions*?

Static Arbitrage

Here, we rely on a *minimal set of assumptions*:

- no assumption on the asset distribution
- one period model

An arbitrage in this simple setting is a *buy and hold* strategy:

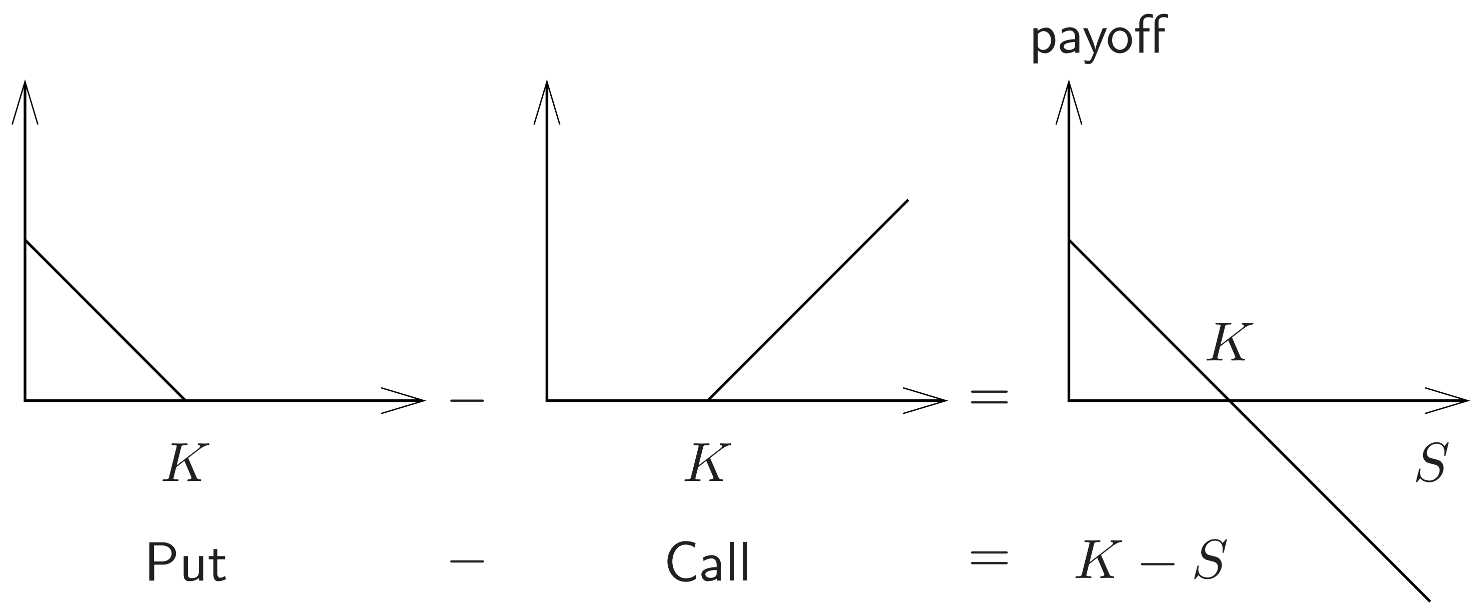
- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity

What for?

Applications:

- arbitrage free *data stripping* before calibration
- test *extrapolation* formulas
- in illiquid markets, find optimal static *hedge* or bound risk at little cost
- in particular: no capital requirements associated with *model-risk*

Simplest Example: Put Call Parity



We denote by $C(K)$ the price of the call with payoff $(S-K)^+$. If we know the forward prices, then we can deduce call prices from puts, ...

Basket Options

- A basket call payoff is given by

$$\left(\sum_{i=1}^{k} w_i S_i - K\right)^+$$

where $w_1, \ldots, w_k$ are the basket's weights and K is the option's strike price

- examples include: Index options, spread options, swaptions...

- basket option prices are used to gather information on *correlation*

We denote by $C(w, K)$ the price of such an option, can we get conditions to test for the presence of an arbitrage in basket price data?

No Arbitrage Conditions

Suppose we are given market prices q_i for basket calls with weights w_i and strike prices K_i:

- *Fundamental theorem of asset pricing*: there is no arbitrage in the static market if and only if there is a probability measure π such that:

$$\mathbf{E}_\pi[(w_i^T x - K_i)^+] = q_i$$

- Except in dimension one: Bertsimas & Popescu (2002) show that this problem is *NP-Hard*.

We look for *tractable necessary conditions*.

Necessary Conditions

- Consider a one period market with securities paying $s_i(x)$, $i = 0, \ldots, m$, at maturity T.

- For each value of form a rank one, symmetric, moment matrix:

$$M(x) = \begin{pmatrix} 1 & s_0(x) & s_1(x) & \ldots \\ s_0(x) & s_0^2(x) & s_0(x)s_1(x) & \\ s_1(x) & s_0(x)s_1(x) & s_1^2(x) & \\ \vdots & & & \ddots \end{pmatrix}$$

- This matrix is of the form vv^T where v is a vector, hence must be positive semidefinite.

Necessary Conditions

- Suppose there is no arbitrage, and a pricing measure π
- Taking expectations, we form:

$$P := \mathbf{E}_\pi[M(x)] = \begin{pmatrix} 1 & p_0 & p_1 & \ldots \\ p_0 & P_{22} & P_{23} & \\ p_1 & P_{32} & P_{33} & \\ \vdots & & & \ddots \end{pmatrix}$$

  where $p_i = \mathbf{E}_\pi[s_i(x)]$, $i = 1, \ldots, m$.

- P must also be positive semidefinite and its first row and column are *market prices*. . .

Necessary Conditions

There is *no arbitrage* between the asset prices p_i and there is a pricing measure π

$$\Downarrow$$

There are coefficients $P_{ij} \in \mathbf{R}$ such that the matrix:

$$P := \mathbf{E}_\pi[M(x)] = \begin{pmatrix} 1 & p_0 & p_1 & \cdots \\ p_0 & P_{22} & P_{23} & \\ p_1 & P_{32} & P_{33} & \\ \vdots & & & \ddots \end{pmatrix}$$

is *positive semidefinite*.

Harmonic Analysis on Semigroups

Some quick definitions...

- a pair $(\mathbb{S}, \cdot)$ is called a *semigroup* iff:
  - if $s, t \in \mathbb{S}$ then $s \cdot t$ is also in $\mathbb{S}$
  - there is a neutral element $e \in \mathbb{S}$ such that $e \cdot s = s$ for all $s \in \mathbb{S}$
- the *dual* $\mathbb{S}^*$ of $\mathbb{S}$ is the set of *semicharacters*, $i.e.$ applications $\chi : \mathbb{S} \to \mathbf{R}$ such that
  - $\chi(s)\chi(t) = \chi(s \cdot t)$ for all $s, t \in \mathbb{S}$
  - $\chi(e) = 1$, where e is the neutral element in $\mathbb{S}$
- a function α is called an *absolute value* on $\mathbb{S}$ iff
  - $\alpha(e) = 1$
  - $\alpha(s \cdot t) \leq \alpha(s)\alpha(t)$, for all $s, t \in \mathbb{S}$

Harmonic Analysis on Semigroups

last definitions (honest)...

- a function $f : \mathbb{S} \to \mathbf{R}$ is *positive semidefinite* iff for every family $\{s_i\} \subset \mathbb{S}$ the matrix with elements $f(s_i \cdot s_j)$ is positive semidefinite

- a function f is *bounded* with respect to the absolute value α iff there is a constant $C > 0$ such that

$$|f(s)| \leq C\alpha(s), \quad s \in \mathbb{S}$$

- f is *exponentially bounded* iff it is bounded with respect to an absolute value

Harmonic Analysis on Semigroups: Central Result

Main result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded *positive definite functions* is a *Bauer simplex* whose extreme points are the bounded semicharacters...

- this means that we have the following representation for positive definite functions on $\mathbb{S}$:

$$f(s) = \int_{\mathbb{S}^*} \chi(s) d\mu(\chi)$$

  where μ is a Radon measure on $\mathbb{S}^*$

Harmonic Analysis on Semigroups: Simple Examples

- *Berstein's theorem* for the Laplace transform

$$\mathbb{S} = (\mathbf{R}_+, +),\ \chi_x(t) = e^{-xt} \ \text{ and } \ f(t) = \int_{\mathbf{R}_+} e^{-xt} d\mu(x)$$

- with involution, *Bochner's theorem* for the Fourier transform

$$\mathbb{S} = (\mathbf{R}, +),\ \chi_x(t) = e^{2\pi ixt} \ \text{ and } \ f(t) = \int_{\mathbf{R}} e^{2\pi ixt} d\mu(x)$$

- *Hamburger's solution* to the unidimensional moment problem

$$\mathbb{S} = (\mathbf{N}, +),\ \chi_x(k) = x^k \ \text{ and } \ f(k) = \int_{\mathbf{R}} x^k d\mu(x)$$

The Option Pricing Problem

- the basket option payoffs $(w^T x - K)^+$ are not ideal in this setting
- solution, use *straddles*: $|w^T x - K|$
- as straddles are just the *sum of a call and a put*, their price can be computed from that of the corresponding call and forward by call-put parity
- the fact that $|w^T x - K|^2$ is a polynomial keeps the complexity low

Payoff Semigroup

- the fundamental semigroup $\mathbb{S}$ is here the multiplicative *payoff semigroup* generated by the cash, the forwards and the straddles:

$$\mathbb{S} = \{1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots\}$$

- the *semicharacters* are the functions $\chi_x : \mathbb{S} \rightarrow \mathbf{R}$ which evaluate the payoffs at a certain point x

$$\chi_x(s) = s(x), \quad \text{for all } s \in \mathbb{S}$$

The Option Pricing Problem Revisited

- the original static arbitrage problem can be reformulated as

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & f(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m \\ & f(s) = \mathbf{E}_\pi[s], \quad s \in \mathbb{S} \quad \text{(f moment function)} \end{array}$$

- the variable is now $f : \mathbb{S} \to \mathbf{R}$, a function that associates to each payoff s in $\mathbb{S}$, its price $f(s)$

- the representation result in Berg et al. (1984) shows when a (price) function $f : \mathbb{S} \to \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\pi[s]$$

Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in $\mathbf{R}^n_+$, and note e_i for $i = 1, \ldots, n+m$ the forward and option payoff functions we get:

A function $f(s) : \mathbb{S} \to \mathbf{R}$ *can be represented as*

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \textit{for all } s \in \mathbb{S},$$

for some measure ν *with compact support, iff for some* $\beta > 0$*:*

(i) $f(s)$ *is positive semidefinite*

(ii) $f(e_i s)$ *is positive semidefinite for* $i = 1, \ldots, n+m$

(iii) $\left(\beta f(s) - \sum_{i=1}^{n+m} f(e_i s)\right)$ *is positive semidefinite*

this turns the basket arbitrage problem into a *semidefinite program*

Semidefinite Programming

A *semidefinite program* is written:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, CX \\ \text{subject to} & \mathbf{Tr}\, A_i X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0, \end{array}$$

in the variable $X \in \mathbf{S}^n$, with parameters $C, A_i \in \mathbf{S}^n$ and $b_i \in \mathbf{R}$ for $i = 1, \ldots, m$. Its *dual* is given by:

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & C - \sum_{i=1}^m \lambda_i A_i \succeq 0, \end{array}$$

in the variable $\lambda \in \mathbf{R}^m$.

A recent extension of interior point techniques for linear programming shows how to solve these convex programs *very efficiently* (see Nesterov & Nemirovskii (1994), Sturm (1999) and Boyd & Vandenberghe (2004)).

Feasibility Problems

Of course, the related feasibility problems:

$$
\begin{array}{ll}
\text{find} & X \\
\text{such that} & \mathbf{Tr}\, A_i X = b_i, \quad i = 1, \ldots, m \\
& X \succeq 0,
\end{array}
$$

and

$$
\begin{array}{ll}
\text{find} & \lambda \\
\text{such that} & C - \sum_{i=1}^{m} \lambda_i A_i \succeq 0,
\end{array}
$$

can be solved as efficiently (setting for example $C = I$ or $b = \mathbf{1}$ in the previous programs).

Also, because most solvers produce both primal and dual solution, we also get a Farkas type *certificate of infeasibility* or a *proof of optimality* in the duality gap.

Option Pricing: a Semidefinite Program

Relaxation: sample the elements of $\mathbb{S}$ up to a certain degree, the variable is then the vector $f(s)$ with

$$e = (1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

testing for the absence of arbitrage is then a *semidefinite program*:

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$

Price Bounds

We can also consider the related problem of finding *bounds* on the price of a straddle, given prices of other similar options:

$$\begin{array}{ll} \text{max./min.} & \mathbf{E}_\pi(|w_0^T x - K_0|) \\ \text{subject to} & \mathbf{E}_\pi(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m, \end{array}$$

which, using the previous result becomes the following semidefinite program:

$$\begin{array}{ll} \text{max./min.} & f(e_0) \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$.

Duality

- the price maximization program is:

$$\begin{array}{ll} \text{maximize} & \int_{\mathbf{R}_+^n} (w_0^T x - K_0)^+ \pi(x) dx \\ \text{subject to} & \int_{\mathbf{R}_+^n} (w_i^T x - K_i)^+ \pi(x) dx = p_i, \quad i = 1, \ldots, m \\ & \int_{\mathbf{R}_+^n} \pi(x) dx = 1, \end{array}$$

  in the variable $\pi \in \mathcal{K}$.

- the dual is a *portfolio problem*:

$$\begin{array}{ll} \text{minimize} & \lambda^T p + \lambda_0 \\ \text{subject to} & \sum_{i=1}^m \lambda_i (w_i^T x - K_i)^+ + \lambda_0 \geq \psi(x) \text{ for every } x \in \mathbf{R}_+^n \end{array}$$

  in the variable $\lambda \in \mathbf{R}^{m+1}$.

very intuitive, but intractable. . .

Conic Duality

let $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and $\mathcal{P}$ the set of positive semidefinite sequences on $\mathbb{S}$

- instead of the conic duality between *probability measures* and *positive portfolios*

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between *positive semidefinite sequences* $\mathcal{P}$ and *sums of squares polynomials* Σ

$$p \in \Sigma \Leftrightarrow \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \to \mathbf{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$

Option Pricing: Dual

- the dual of the price maximization problem

$$\begin{array}{ll} \text{maximize} & f(e_0) \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

- now becomes...

$$\begin{array}{ll} \text{minimize} & \sum_{j=1}^{n+m} p_j \lambda_j + \lambda_{n+m+1} \\ \text{subject to} & \sum_{j=1}^{n+m} \lambda_j e_j(x) + \lambda_{n+m+1} - |w_0^T x - K_0| \\ & = q_0(x) + \sum_{j=1}^{n+m} q_j(x) e_j(x) + (\beta - \sum_{k=0}^{n+m} e_k(x)) q_{n+1}(x) \end{array}$$

in the variables $\lambda \in \mathbf{R}^{n+m+1}$ and $q_j \in \Sigma$ for $j = 0, \ldots, (n+1)$

Option Pricing: Caveats

- *Size*: grows exponentially with the number of assets: no free lunch, even in numerical complexity. . .

- Some numerical difficulties: conditioning, etc

Conclusion

- Testing for the absence of arbitrage between basket options (swaptions, etc) is NP-hard but good relaxations can be found
- We get a set of *relaxed conditions* for the *absence of arbitrage*
- Small scale problems are tractable in practice as *semidefinite programs*

References

Berg, C., Christensen, J. P. R. & Ressel, P. (1984), *Harmonic analysis on semigroups : theory of positive definite and related functions*, Vol. 100 of *Graduate texts in mathematics*, Springer-Verlag, New York.

Bertsimas, D. & Popescu, I. (2002), 'On the relation between option and stock prices: a convex optimization approach', *Operations Research* **50**(2), 358–374.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–659.

Boyd, S. & Vandenberghe, L. (2004), *Convex Optimization*, Cambridge University Press.

Nesterov, Y. & Nemirovskii, A. (1994), *Interior-point polynomial algorithms in convex programming*, Society for Industrial and Applied Mathematics, Philadelphia.

Sturm, J. F. (1999), 'Using sedumi 1.0x, a matlab toolbox for optimization over symmetric cones', *Optimization Methods and Software* **11**, 625–653.