A MARKET TEST FOR THE POSITIVITY OF ARROW-DEBREU PRICES

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Abstract

We derive tractable necessary and sufficient conditions for the absence of buy-and-hold arbitrage opportunities in a perfectly liquid, one period market. We formulate the positivity of Arrow-Debreu prices as a generalized moment problem to show that this no arbitrage condition is equivalent to the positive semidefiniteness of matrices formed by the market price of tradeable securities and their products. We apply this result to a market with multiple assets and basket call options.

1 Introduction

The fundamental theorem of asset pricing establishes the equivalence between absence of arbitrage and existence of a martingale pricing measure, and is the foundation of the Black and Scholes (1973) and Merton (1973) option pricing methodology. Option prices are computed by an arbitrage argument, as the value today of a dynamic, self-financing hedging portfolio that replicates the option payoff at maturity. This pricing technique relies on at least two fundamental assumptions: it posits a model for the asset dynamics and assumes that markets are frictionless, i.e. that continuous trading in securities is possible at no cost. Here we take the complementary approach: we do not make any assumption on the asset dynamics and we only allow trading today and at a maturity date T. In that sense, we revisit the classic result of Arrow and Debreu (1954) on the equivalence between positivity of state prices and absence of arbitrage in a one period market. In this simple market, we seek computationally *tractable* conditions for the absence of arbitrage, directly formulated in terms of *tradeable* securities.

Of course, these results are not intended to be used as a pricing framework in liquid markets. Our objective here instead is twofold. First, market data on derivative prices, aggregated from a

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very diverse set of sources, is always plagued by liquidity and synchronicity issues. Because these price data sets are used by derivatives dealers to calibrate their models, we seek a set of arbitrarily refined tests to detect unviable prices in the one period market or, in other words, detect prices which would be incompatible with *any* arbitrage free dynamic model for asset dynamics. Second, in some very illiquid markets, these conditions form simple upper or lower hedging portfolios and diversification strategies that are, by construction, immune to model misspecification and illiquidity issues.

Work on this topic starts with the Arrow and Debreu (1954) no arbitrage conditions on state prices. This was followed by a stream of works on multiperiod and continuous time extensions stating the equivalence between existence of a martingale measure and absence of dynamic arbitrage, starting with Harrison and Kreps (1979) and Harrison and Pliska (1981), with the final word probably belonging to Dalang, Morton, and Willinger (1990) and Delbaen and Schachermayer (2005). Efforts to express these conditions directly in terms of asset prices can be traced back to Breeden and Litzenberger (1978) and Friesen (1979) who derive equivalent conditions on a continuum of (possibly nontradeable) call options. Breeden and Litzenberger (1978), Jackwerth and Rubinstein (1996) and Laurent and Leisen (2000) use these results to infer information on the asset distribution from the market price of calls using a minimum entropy approach. Finally, a recent paper by Davis and Hobson (2005) provides explicit no arbitrage conditions and option price bounds in the case where only a few single asset call prices are quoted in a multiperiod market.

Given the market price of tradeable securities in a one period market, we interpret the question of testing for the existence of a state price density as a generalized moment problem. In that sense, the conditions we obtain can be seen as a direct generalization of Bochner-Bernstein type theorems on the Fourier transform of positive measures (see Bochner and Chandrasekharan (1949)). Market completeness is then naturally formulated in terms of moment determinacy. This allows us to derive equivalent conditions for the absence of arbitrage between *general payoffs* (not limited to single asset call options). We also focus on the particular case of basket calls or European call options on a basket of assets. Basket calls appear in equity markets as index options and in interest rate derivatives market as spread options or swaptions, and are key recipients of market information on correlation.

The paper is organized as follows. We begin by describing the one period market and illustrate our approach on a simple example, introducing the payoff semigroup formed by the market securities and their products. Section 2 starts with a brief primer on harmonic analysis on semigroups after which we describe the general no arbitrage conditions on the payoff semigroup. We also show how the products in this semigroup complete the market. We finish in Section 3 by a case study on spread options.

1.1 One Period Model

We work in a one period model where the market is composed of n assets with payoffs at maturity T equal to x_i and price today given by p_i for $i = 1, \ldots, n$. There are also m derivative securities with payoffs $s_j(x) = s_j(x_1, \ldots, x_n)$ and price today equal to p_{n+j} for $j = 1, \ldots, m$. Finally, there is a riskless asset with payoff 1 at maturity and price 1 today and we assume, without loss of generality here, that interest rates are equal to zero (we work in the forward market). We look for conditions on p precluding arbitrage in this market, i.e. buy and hold portfolios formed at no cost today which guarantee a strictly positive payoff at maturity T.

We want to answer the following simple question: Given the market price vector p, is there a buy-and-hold arbitrage opportunity between the assets x_i and the securities $s_j(x)$? Naturally, we know from the Arrow and Debreu (1954) conditions that this is equivalent to the existence of a state price density μ with support in $\mathbf{R}_+^n$ such that:

$$\begin{array}{l} \mathbf{E}_\mu[x_i] = p_i, \quad i = 1, \ldots, n, \\ \mathbf{E}_\mu[s_j(x)] = p_{n+j}, \quad j = 1, \ldots, m. \end{array} \tag{1}$$

Bertsimas and Popescu (2002) show that this simple, fundamental problem is computationally hard (in fact NP-Hard). Even discretized on a uniform grid with L steps along each axis, this problem is still equivalent to an exponentially large linear program of size $O(L^n)$. Here, we look for a discretization that does not involve the state price density but instead formulates the no arbitrage conditions directly on the market price vector p. Of course, NP-Hardness means that we cannot reasonably hope to provide an efficient, exact solution to all instances of problem (1). Here instead, we seek an arbitrarily refined, computationally efficient relaxation for this problem and NP-Hardness means that we will have to tradeoff precision for complexity.

1.2 The Payoff Semigroup

To illustrate our approach, let us begin here with a simplified case were $n = 1$, i.e. there is only one forward contract with price p_1, and the derivative payoffs $s_j(x)$ are monomials with $s_j(x) = x^j$ for $j = 2, \ldots, m$. In this case, conditions (1) on the measure μ are written:

$$\begin{array}{l} \mathbf{E}_\mu[x^j] = p_j, \quad j = 2, \ldots, m, \\ \mathbf{E}_\mu[x] = p_1, \end{array} \tag{2}$$

with the implicit constraint that the support of μ be included in $\mathbf{R}_+$. We recognize (2) as a Stieltjes moment problem (see Stieltjes (1894)). For $x \in \mathbf{R}_+$, let us form the column vector $v_m(x) \in \mathbf{R}^{m+1}$ as follows:

$$v_m(x) \triangleq (1, x, x^2, \ldots, x^m)^T.$$

For each value of x, the matrix $P_m(x)$ formed by the outer product of the vector $v_m(x)$ with itself is given by:

$$P_m(x) \triangleq v_m(x) v_m(x)^T = \begin{pmatrix} 1 & x & \ldots & x^m \\ x & x^2 & & x^{m+1} \\ \vdots & & \ddots & \vdots \\ x^m & x^{m+1} & \ldots & x^{2m} \end{pmatrix}$$

$P_m(x)$ is a *positive semidefinite* matrix (it has only one nonzero eigenvalue equal to $\|v_m(x)\|^2$). If there is no arbitrage and there exists a state price density μ satisfying the price constraints (2), then there must be a symmetric moment matrix $M_m \in \mathbf{R}^{(m+1)\times(m+1)}$ such that:

$$M_m \triangleq \mathbf{E}_\mu[P_m(x)] = \begin{pmatrix} 1 & p_1 & \dots & p_m \\ p_1 & p_2 & & \mathbf{E}_\mu[x^{m+1}] \\ \vdots & & \ddots & \vdots \\ p_m & \mathbf{E}_\mu[x^{m+1}] & \dots & \mathbf{E}_\mu[x^{2m}] \end{pmatrix}$$

and, as an average of positive semidefinite matrices, M_m must be positive semidefinite. In other words, the existence of a positive semidefinite matrix M_m whose first row and columns are given by the vector p is a necessary condition for the absence of arbitrage in the one period market. In fact, positivity conditions of this type are also *sufficient* (see Vasilescu (2002) among others). Testing for the absence of arbitrage is then equivalent to solving a *linear matrix inequality*, i.e. finding matrix coefficients corresponding to $\mathbf{E}_\mu[x^j]$ for $j = m+1, \dots, 2m$ that make the matrix $M_m(x)$ positive semidefinite.

This paper's central result is to show that this type of reasoning is not limited to the unidimensional case where the payoffs $s_j(x)$ are monomials but extends to arbitrary payoffs. Instead of looking only at monomials, we will consider the *payoff semigroup* $\mathbb{S}$ generated by the payoffs 1, x_i and $s_j(x)$ for $i = 1, \dots, n$ and $j = 1, \dots, m$ and their products (in graded lexicographic order):

$$\mathbb{S} \triangleq \left\{1, x_1, \dots, x_n, s_1(x), \dots, s_m(x), x_1^2, \dots, x_i s_j(x), \dots, s_m(x)^2, \dots\right\} \tag{3}$$

In the next section, we will show that the no arbitrage conditions (1) are equivalent to positivity conditions on matrices formed by the prices of the assets in $\mathbb{S}$. We also detail under which technical conditions the securities in $\mathbb{S}$ make the one period market complete.

1.3 Semidefinite Programming

The key incentive for writing the no arbitrage conditions in terms of linear matrix inequalities is that the later are *tractable*. The problem of finding coefficients that make a particular matrix positive semidefinite can be written as:

$$\begin{array}{ll} \text{find} & y \\ \text{such that} & C + \sum_{k=1}^m y_k A_k \succeq 0 \end{array} \tag{4}$$

in the variable $y \in \mathbf{R}^m$, with parameters $C,\ A_k \in \mathbf{R}^{n\times n}$, for $k = 1, \dots, m$, where $X \succeq 0$ means X positive semidefinite. This problem is convex and is also known as a semidefinite feasibility problem. Reasonably large instances can be solved efficiently using the algorithms detailed in Nesterov and Nemirovskii (1994) or Boyd and Vandenberghe (2004) for example.

2 No Arbitrage Conditions

In this section, we begin with an introduction on harmonic analysis on semigroups, which generalizes the moment conditions of the previous section to arbitrary payoffs. We then state our main result on the equivalence between no arbitrage in the one period market and positivity of the price matrices for the products in the payoff semigroup $\mathbb{S}$ defined in (3).

2.1 Harmonic analysis on semigroups

We start by a brief primer on harmonic analysis on semigroups (based on Berg, Christensen, and Ressel (1984) and the references therein). Unless otherwise specified, all measures are supposed to be positive.

A function $\rho(s) : \mathbb{S} \to \mathbf{R}$ on a semigroup $(\mathbb{S}, \cdot)$ is called a *semicharacter* if and only if it satisfies $\rho(st) = \rho(s)\rho(t)$ for all $s, t \in \mathbb{S}$ and $\rho(1) = 1$. The dual of a semigroup $\mathbb{S}$, i.e. the set of semicharacters on $\mathbb{S}$, is written $\mathbb{S}^*$.

Definition 1 A function $f(s) : \mathbb{S} \to \mathbf{R}$ is a *moment function on* $\mathbb{S}$ if and only if $f(1) = 1$ and $f(s)$ can be represented as:

$$f(s) = \int_{\mathbb{S}^*} \rho(s) d\mu(\rho), \quad \text{for all } s \in \mathbb{S}, \tag{5}$$

where μ is a Radon measure on $\mathbb{S}^*$.

When $\mathbb{S}$ is the semigroup defined in (3) as an enlargement of the semigroup of monomials on $\mathbf{R}^n$, its dual $\mathbb{S}^*$ is the set of applications $\rho_x(s) : \mathbb{S} \to \mathbf{R}$ such that $\rho_x(s) = s(x)$ for all $s \in \mathbb{S}$ and all $x \in \mathbf{R}^n$. Hence when $\mathbb{S}$ is the payoff semigroup, to each point $x \in \mathbf{R}^n$ corresponds a semicharacter that evaluates a payoff at that point. In this case, the measure μ is a probability measure on $\mathbf{R}^n$ and the representation (5) becomes:

$$f(s) = \int_{\mathbf{R}^n} s(x) d\mu(x) = \mathbf{E}_\mu [s(x)], \quad \text{for all payoffs } s \in \mathbb{S}. \tag{6}$$

This means that when $\mathbb{S}$ is the semigroup defined in (3) and there is no arbitrage, a moment function is a function that for each payoff $s \in \mathbb{S}$ returns its *price* $f(s) = \mathbf{E}_\mu [s(x)]$. Testing for no arbitrage is then equivalent to testing for the existence of a moment function f on $\mathbb{S}$ that matches the market prices in (1).

Definition 2 A function $f(s) : \mathbb{S} \to \mathbf{R}$ is called *positive semidefinite* if and only if for all finite families $\{s_i\}$ of elements of $\mathbb{S}$, the matrix with coefficients $f(s_i s_j)$ is positive semidefinite.

We remark that moment functions are necessarily positive semidefinite. Here, based on results by Berg, Christensen, and Ressel (1984), we exploit this property to derive necessary and sufficient conditions for representation (6) to hold.

The central result in Berg, Christensen, and Ressel (1984, Th. 2.6) states that the set of exponentially bounded positive semidefinite functions $f(s) : \mathbb{S} \to \mathbf{R}$ such that $f(1) = 1$ is a Bauer simplex whose extreme points are given by the semicharacters in $\mathbb{S}^*$. Hence a function f is positive semidefinite and exponentially bounded if and only if it can be represented as $f(s) = \int_{\mathbb{S}^*} \rho d\mu(\rho)$ with the support of μ included in some compact subset of $\mathbb{S}^*$. Bochner' theorem on the Fourier transform of positive measures and Berstein's corresponding theorem for the Laplace transform are particular cases of this representation result. In what follows, we use it to derive tractable necessary and sufficient conditions for the function $f(s)$ to be represented as in (6).

2.2 Main Result: No Arbitrage Conditions

We assume that asset prices are bounded and that $\mathbb{S}$ is the payoff semigroup defined in (3), this means that without loss of generality, we can assume that the payoffs $s_j(x)$ are positive.

Theorem 3 *There is no arbitrage in the one period market and there exists a state price measure μ such that:*

$$\begin{array}{l}\mathbf{E}_\mu[x_i] = p_i, \quad i = 1, \ldots, n, \\ \mathbf{E}_\mu[s_j(x)] = p_{n+j}, \quad j = 1, \ldots, m,\end{array}$$

if and only if the function $f(s) : \mathbb{S} \rightarrow \mathbf{R}$ defined as $f(s) = \mathbf{E}_\mu[s(x)]$ satisfies:

(i) $f(s)$ *is a positive semidefinite function of* $s \in \mathbb{S}$,

(ii) $f(e_i s)$ *is a positive semidefinite function of* $s \in \mathbb{S}$ *for* $i = 1, \ldots, n+m$,

(iii) $\left(\beta f(s) - \sum_{i=1}^{n+m} f(e_i s)\right)$ *is a positive semidefinite function of* $s \in \mathbb{S}$,

(iv) $f(1) = 1$ *and* $f(e_i) = p_i$ *for* $i = 1, \ldots, n+m$,

for some (large) constant $\beta > 0$.

Proof. To simplify notations, we define the functions $e_i(x)$ for $i = 1, \ldots, m+n$ and $x \in \mathbf{R}_+^n$ such that $e_i(x) = x_i$ for $i = 1, \ldots, n$ and $e_{n+j}(x) = s_j(x)$ for $j = 1, \ldots, m$. By scaling $e_i(x)$ we can assume without loss of generality that $\beta = 1$. For s, u in $\mathbb{S}$, we note E_s the shift operator such that for $f(s) : \mathbb{S} \rightarrow \mathbf{R}$, we have $E_u(f(s)) \triangleq f(su)$ and we let $\mathcal{E}$ be the commutative algebra generated by the shift operators on $\mathbb{S}$. The family of shift operators $\tau = \left\{\{E_{e_i}\}_{i=1,\ldots,n+m}, \left(I - \sum_{i=1}^{n+m} E_{e_i}\right)\right\} \subset \mathcal{E}$ is such that $I - T \in \text{span}^+\tau$ for each $T \in \tau$ and $\text{span}\ \tau = \mathcal{E}$, hence τ is linearly admissible in the sense of Berg and Maserick (1984) or Maserick (1977), which states that (ii) and (iii) are equivalent to f being τ-positive. Then, Maserick (1977, Th. 2.1) means that f is τ-positive if and only if there is a measure μ such that $f(s) = \int_{\mathbb{S}^*} \rho(s) d\mu(\rho)$, whose support is a compact subset of the τ-positive semicharacters. This means in particular that for a semicharacter $\rho_x \in \text{supp}(\mu)$ we must have $\rho_x(e_i) \geq 0$, for $i = 1, \ldots, n$ hence $x \geq 0$. If ρ_x is a τ-positive semicharacter then we must have $\{x \geq 0 : \|x\|_1 \leq 1\}$, hence f being τ-positive is equivalent to f admitting a representation of the form $f(s) = \mathbf{E}_\mu\left[s(x)\right]$, for all $s \in \mathbb{S}$ with μ having a compact support in a subset of the unit simplex. ■

We have assumed that the asset distribution has a compact support, but as this can be made arbitrarily large, we do not loose much generality from a numerical point of view. Similar but more technical results hold in the non compact case and are detailed in the companion preprint d'Aspremont (2003).

2.3 Market Completeness

As we will see below, under technical conditions on the asset prices, the moment problem is determinate and there is a one-to-one correspondence between the price $f(s)$ of the assets in $s \in \mathbb{S}$ and the state price measures μ, in other words, the payoffs in $\mathbb{S}$ make the market *complete*.

Here, we suppose that there is no arbitrage in the one period market. Theorem 3 shows that there is at least one measure μ such that $f(s) = \mathbf{E}_\mu[s(x)]$, for all payoffs $s \in \mathbb{S}$. In fact, we show below that when asset prices are bounded, this pricing measure is unique.

Theorem 4 *Suppose that the asset prices x_i for $i = 1, \ldots, n$ are bounded, then for each set of prices $f(s)$ there is a unique state price measure μ with compact support satisfying:*

$$f(s) = \mathbf{E}_\mu[s(x)], \quad \textit{for all payoffs } s \in \mathbb{S}.$$

Proof. If there is no arbitrage and asset prices x_i for $i = 1, \ldots, n$ are bounded, then the prices $f(s) = \mathbf{E}_\mu[s(x)]$, for $s \in \mathbb{S}$ are exponentially bounded in the sense of Berg, Christensen, and Ressel (1984, §4.1.11) and Berg, Christensen, and Ressel (1984, Th. 6.1.5) shows that the measure μ associated to the market prices $f(s)$ is unique. ■

This result shows that the securities in $\mathbb{S}$ make the market complete in the bounded case. Again, without this boundedness assumption, testing for market completeness is a much more difficult problem and we refer the reader to d'Aspremont (2003) for a discussion.

2.4 Implementation

The conditions in theorem 3 involve testing the positivity of infinitely large matrices and are of course not directly implementable. In practice, we can get a reduce set of conditions by only considering elements of $\mathbb{S}$ up to a certain (even) degree $2d$:

$$\mathbb{S}_d \triangleq \left\{1, x_1, \ldots, x_n, s_1(x), \ldots, s_m(x), x_1^2, \ldots, x_i s_j(x), \ldots, s_m(x)^2, x_1^3, \ldots, s_m(x)^{2d}\right\} \tag{7}$$

We look for a moment function f satisfying conditions (i) through (iv) in Theorem 3 for all elements s in the reduced semigroup $\mathbb{S}_d$. Conditions (i)-(iii) now amount to testing the positivity of matrices of size $N_d = \binom{n+m+2d}{n+m}$ or less. Condition (i) for example is written:

$$\begin{pmatrix}
1 & p_1 & \cdots & p_{m+n} & f(x_1^2) & \cdots & f\left(s_m(x)^{\frac{N_d}{2}}\right) \\
p_1 & f(x_1^2) & \cdots & f(x_1 s_m(x)) & f(x_1^3) & \cdots & f\left(x_1 s_m(x)^{\frac{N_d}{2}}\right) \\
\vdots & \vdots & \ddots & & & & \\
p_{m+n} & f(x_1 s_m(x)) & & & \vdots & & \\
f(x_1^2) & f(x_1^3) & & \cdots & f(x_1^4) & & \\
\vdots & \vdots & & & & & \vdots \\
f\left(s_m(x)^{\frac{N_d}{2}}\right) & f\left(x_1 s_m(x)^{\frac{N_d}{2}}\right) & & & & \cdots & f\left(s_m(x)^{N_d}\right)
\end{pmatrix} \succeq 0,$$

because the market price conditions in (1) impose $f(x_i) = p_i$ for $i = 1, \ldots, n$ and $f(s_j(x)) = p_{n+j}$ for $j = 1, \ldots, m$. Condition (ii) stating that $f(x_1 s)$ be a positive semidefinite function of s is then written as:

$$\begin{pmatrix} p_1 & f\left(x_1^2\right) & f\left(x_1 x_2\right) & \cdots & f\left(x_1 s_m(x)^{\frac{N_d}{2}-1}\right) \\ f\left(x_1^2\right) & f\left(x_1^4\right) & f\left(x_1^3 x_2\right) & & \\ f\left(x_1 x_2\right) & f\left(x_1^3 x_2\right) & f\left(x_1^2 x_2^2\right) & & \\ \vdots & & & \ddots & \vdots \\ f\left(x_1 s_m(x)^{\frac{N_d}{2}-1}\right) & & & \cdots & f\left(x_1^2 s_m(x)^{N_d-2}\right) \end{pmatrix} \succeq 0,$$

and the remaining linear matrix inequalities in conditions (ii) and (iii) are handled in a similar way. These conditions are a finite subset of the full conditions in theorem 3 and form a set of linear matrix inequalities in the values of $f(s)$ (see §1.3). The exponential growth of N_d with n and m means that only small problem instances can be solved using current numerical software. This is partly because most interior point based semidefinite programming solvers are designed for small or medium scale problems with high precision requirements. Here instead, we need to solve large problems which don't require many digits of precision.

2.5 Multi-Period Models

Suppose now that the products have multiple maturities $T_1, \ldots, T_q$. We know from Harrison and Kreps (1979) and Harrison and Pliska (1981) that the absence of arbitrage in this dynamic market is equivalent to the existence of a martingale measure on the assets $x_1, \ldots, x_n$. Theorem 3 gives conditions for the existence of *marginal* state price measures μ_i at each maturity T_i and we need conditions guaranteeing the existence of a martingale measure whose marginals match these distributions μ_i at each maturity date T_i. A partial answer is given by the theorem below, which can be traced to Blackwell, Stein, Sherman, Cartier, Meyer and Strassen.

Theorem 5 *If μ and ν are any two probability measures on a fininte set $A = \{a_1, \ldots, a_N\}$ in $\mathbf{R}^N$ such that $\mathbf{E}_\mu[\phi] \geq \mathbf{E}_\nu[\phi]$ for every continuous concave function ϕ defined on the convex hull of A, then there is a martingale transition matrix Q such that $\mu Q = \nu$.*

Finding *tractable* conditions for the existence of a martingale measure with given marginals, outside of the particular case of European call options considered in Davis and Hobson (2005) for example, remains an open problem.

3 Case Study: Spread Options

To illustrate the results of section 2, we explicitly treat the case of a one period market with two assets x_1, x_2 with positive, bounded payoff at maturity T and price p_1, p_2 today. European call options with payoff $(x - K_i)^+$ for $i = 1, 2$, are also traded on each asset with prices p_3 and p_4. We are interested in computing bounds on the price of a spread option with payoff $(x_1 - x_2 - K)^+$ given the prices of the forwards and calls.

We first notice that the complexity of the problem can be reduced by considering straddle options with payoffs $|x_i - K_i|$ instead of calls. Because a straddle can be expressed as a combination of calls, forwards and cash:

$$|x_i - K_i| = (K_i - x_i) + 2(x_i - K_i)^+.$$

The advantage of using straddles is that the square of a straddle is a polynomial in the payoffs x_i, $i = 1, 2$. Using straddles instead of calls very significantly reduces the number of elements in the semigroup $\mathbb{S}_d$: when k option prices are given on 2 assets, this number is $(k+1)\binom{2+2d}{2}$, instead of $\binom{2+k+2d}{n+k}$. The payoff semigroup $\mathbb{S}_d$ is now:

$$\mathbb{S}_d = \left\{1, x_1, x_2, |x_1 - K_1|, |x_2 - K_2|, |x_1 - x_2 - K|, x_1^2, x_1x_2, x_1|x_1 - K_1|, \ldots, x_2^{2d}\right\}$$

By sampling the conditions in theorem 3 on $\mathbb{S}_d$ as in section 2.4, we can compute a lower bound on the minimum (resp. an upper bound on the maximum) price for the spread option compatible with the absence of arbitrage. This means that we get an upper bound on the solution of:

$$\begin{array}{ll} \text{maximize} & \mathbf{E}_\mu[|x_1 - x_2 - K|] \\ \text{subject to} & \mathbf{E}_\mu[|x_i - K_i|] = p_{i+2} \\ & \mathbf{E}_\mu[x_i] = p_i, \quad i = 1, 2 \end{array}$$

by solving the following program:

$$\begin{array}{lll} \text{maximize} & f(|x_1 - x_2 - K|) & \\ \text{subject to} & \begin{pmatrix} 1 & p_1 & \cdots & f\left(x_2^d\right) \\ p_1 & f\left(x_1^2\right) & & \\ \vdots & & \ddots & \vdots \\ f\left(x_2^d\right) & & \cdots & f\left(x_2^{2d}\right) \end{pmatrix} & \succeq 0 \\ & & \vdots \\ & \begin{pmatrix} f(b(x)) & f(b(x)x_1) & \cdots & f\left(b(x)x_2^{d-1}\right) \\ f(b(x)x_1) & f\left(b(x)^2x_1^2\right) & & \\ \vdots & & \ddots & \vdots \\ f\left(b(x)x_2^{d-1}\right) & & \cdots & f\left(b(x)^2x_2^{2(d-1)}\right) \end{pmatrix} & \succeq 0 \end{array}$$

where

$$b(x) = \beta - x_1 - x_2 - |x_1 - K_1| - |x_2 - K_2| - |x_1 - x_2 - K|,$$

which is a semidefinite program (see §1.3) in the values of $f(s)$ for $s \in \mathbb{S}_d$.

4 Conclusion

By interpreting the Arrow and Debreu (1954) no arbitrage conditions as a moment problem, we have derived equivalent conditions directly written on the price of tradeable assets instead of state prices. This also shows how allowing trading in the products of market payoffs completes the market.

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