Risk management methods for the market model of interest rates using semidefinite programming

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1.1 Introduction

- Option prices are a function of the underlying asset prices today and the market volatility (variance).

- Derivative pricing and hedging requires daily model calibration of that variance to option prices quoted by the market.

- Multivariate option models (on interest-rate derivatives) have a covariance matrix as their fundamental parameter.

- Current methods heavily parametrize this covariance and use Monte-Carlo estimates of option prices to calibrate the model.

- In practice however, the calibration problem can be approximated by an SDP with excellent precision.

- Both primal and dual problems have direct, intuitive interpretations.

- Robustness, smoothness, Bid-Ask spread constraints, can be included and the calibration problem is solved as a Symmetric Cone Program.

2.1 Option pricing in dimension one

- In the Black & Scholes (1973) model, the stock price dynamics S_t are given by $dS_t = \sigma S_t dW_t$ where W_t is a B.M., i.e. $\log S_T$ is Gaussian.

- The most heavily traded derivative products are European Call options which pay

$$Call_T = (S_T - K)_+$$

at a certain fixed maturity T.

The central "no arbitrage" argument in Black & Scholes (1973) and Merton (1973) shows that Calls are redundant.

- There is a *self-financing* dynamic portfolio strategy in stock and cash that *perfectly replicates* the payoff $(S_T - K)_+$ at time T.

- The option price is given by:

$$Call(S_0, K, \sigma^2 T) = E^{\mathbf{Q}}\left[(S_T - K)_+\right]$$

  where $\mathbf{Q}$ is an equivalent martingale measure

- The option is perfectly hedged by holding $\partial Call(S_t, K, \sigma)/\partial S_t$ in stock S_t and the rest in cash.

- The expectation $E^{\mathbf{Q}}\left[(S_T - K)_+\right]$ can be computed explicitly:

$$BS(S_0, K, \sigma^2 T) = S_0 N\left(\frac{\ln\left(\frac{S_0}{K}\right) + \frac{\sigma^2}{2}T}{\left(\sigma^2 T\right)^{1/2}}\right) - KN\left(\frac{\ln\left(\frac{S_0}{K}\right) - \frac{\sigma^2}{2}T}{\left(\sigma^2 T\right)^{1/2}}\right)$$

where N is the CDF of the Gaussian density.

- Because S_0 is quoted by the market today and $Call(S_0, K, \sigma^2 T)$ is strictly increasing in $\sigma^2 T$, there is a *one-to-one relationship* between Call prices and BS volatility.

- In fact, the market quotes option prices using their BS variance $\sigma^2 T$.

2.2 Multivariate option pricing

- Interest rate option pricing requires modelling the dynamics of a curve (the rate for each maturity). This is usually dicretized on a finite set of maturities.

- We now have have multiple underlying prices S_t^i for $i = 1, ..., n$ following $dS_t^i = S_t^i \sigma^i dW_t$ where $\sigma^i \in \mathbb{R}^n$ and W_t is a n dimensional B.M.

- The model is entirely parametrized by S_0^i for $i = 1, ..., n$, the value of the stocks today and by the *covariance* matrix

$$X = \left(\sigma_i^T \sigma_j\right)_{i,j=1,...,n}$$

- The simplest derivative products are European Basket Call options (Swaptions) which pay:

$$Call_T = \left(\sum_{i=1}^{n} w_i S_T^i - K \right)_+$$

- No closed form solution is available to compute the price

$$E^{\mathbf{Q}} \left[\left(\sum_{i=1}^{n} w_i S_T^i - K \right)_+ \right]$$

- The most common pricing technique is Monte-Carlo.

- The one-to-one relationship between variance and price is lost.
- The calibration is performed with a heavily parametrized (often non-convex) set of covariances.
- Monte-Carlo pricing introduces additional instability.
- Derivative desks stay perfectly hedged ($\partial Call(S_t, K, \sigma^2 T)/\partial S_t = 0$). Because a calibration is performed every day, the "numerical noise hedging" can become very costly.

3.1 Semidefinite Programming formulation

In practice, we can approximate the price of a basket option by:

$$BS\left(\sum_{i=1}^{n} w_i S_0^i, K, \sigma_w^2 T\right)$$

where:

$$\sigma_w^2 = \left\|\sum_{i=1}^{n} \hat{w}_i \sigma_i\right\|^2 \quad \text{with} \quad \hat{w}_i = \frac{w_i S_0^i}{\sum_{j=1}^{n} w_j S_0^j}$$

which can be rewritten:

$$\sigma_w^2 = \mathbf{Tr}\left(\Omega X\right)$$

where $X = \left(\sigma_i^T \sigma_j\right)_{i,j=1,\ldots,n}$ and $\Omega = \hat{w}\hat{w}^T$.

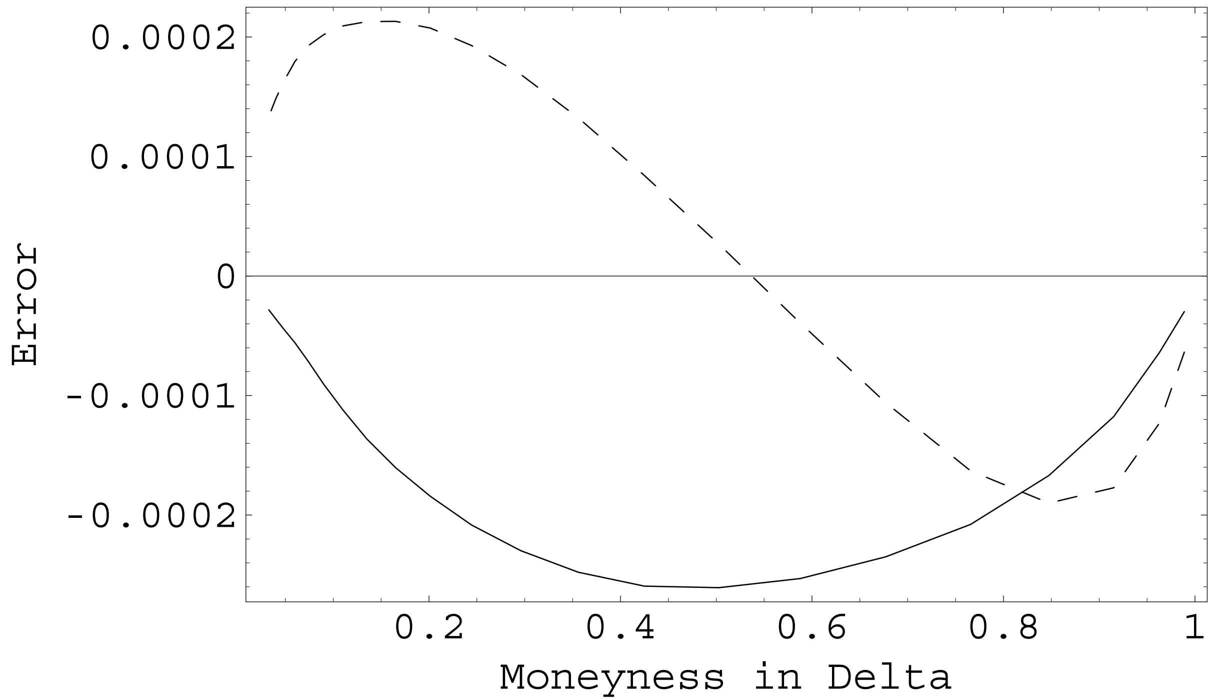

Figure 1: Order zero (dashed) and order one (plain) absolute approximation error versus the multidimensional Black-Scholes basket prices obtained by simulation for various strikes.

- The approximation of the basket price as a Black-Scholes price with variance given by $\sigma_w^2 = \mathbf{Tr}\,(\Omega X)$ defines the calibration set as the feasible set of an SDP.

- Given market prices σ_k^2 for $k = 1, \ldots, m$ on a set of options (Ω_k, T_k), the calibration problem becomes:

$$\begin{array}{ll} \text{find} & X \\ \text{s.t.} & \mathbf{Tr}\,(\Omega_k X) = \sigma_k^2 \ \text{ for } k = 1, \ldots, m \\ & X \succeq 0 \end{array}$$

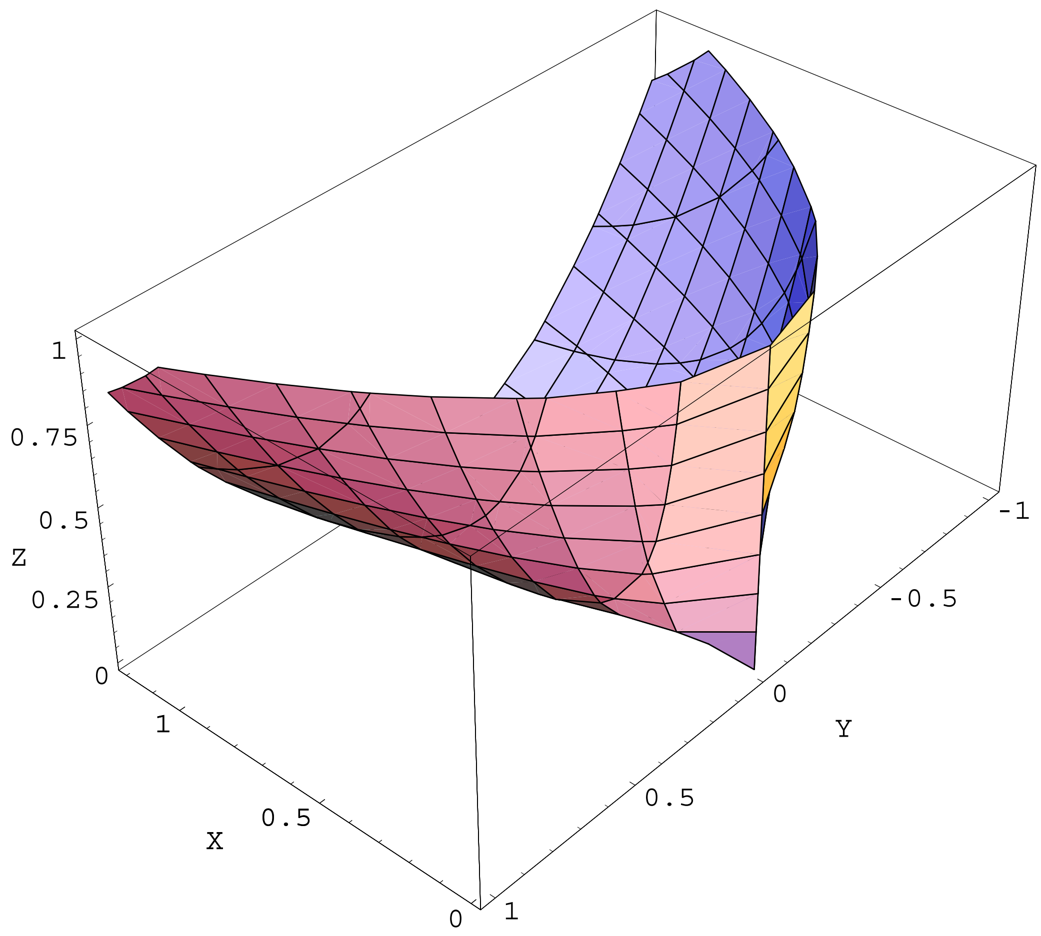

Figure 2: The semidefinite cone in dimension 3.

3.2 Smoothness

We can minimize the surface of the solution matrix with:

$$S = \sum_{i,j \in [2,n]} \left\| \Delta_{i,j} X \right\|^2$$

where

$$\Delta_{i,j} X = \begin{pmatrix} X_{i,j} - X_{i-1,j} \\ X_{i,j} - X_{i,j-1} \end{pmatrix}$$

The calibration program becomes:

$$\begin{array}{ll} \text{min} & t \\ \text{subject to} & \sum_{i,j \in [2,n]} \left\| \Delta_{i,j} X \right\|^2 \leq t \\ & \sigma^2_{Bid,k} T_k \leq \mathbf{Tr}(\Omega_k X) \leq \sigma^2_{Ask,k} T_k, \quad k = 1, \ldots, M \\ & X \succeq 0 \end{array}$$

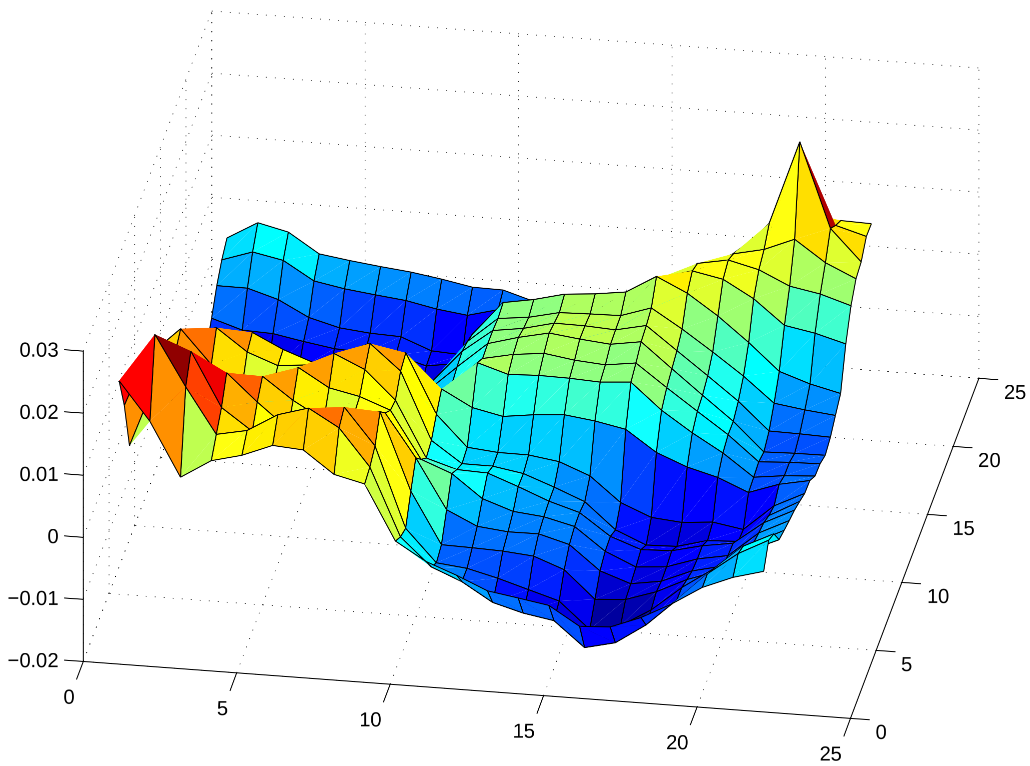

Figure 3: Solution to the calibration problem with smoothness constraints

We can look at the eigenvectors of this purely market implied matrix to compare them with classical PCA results.

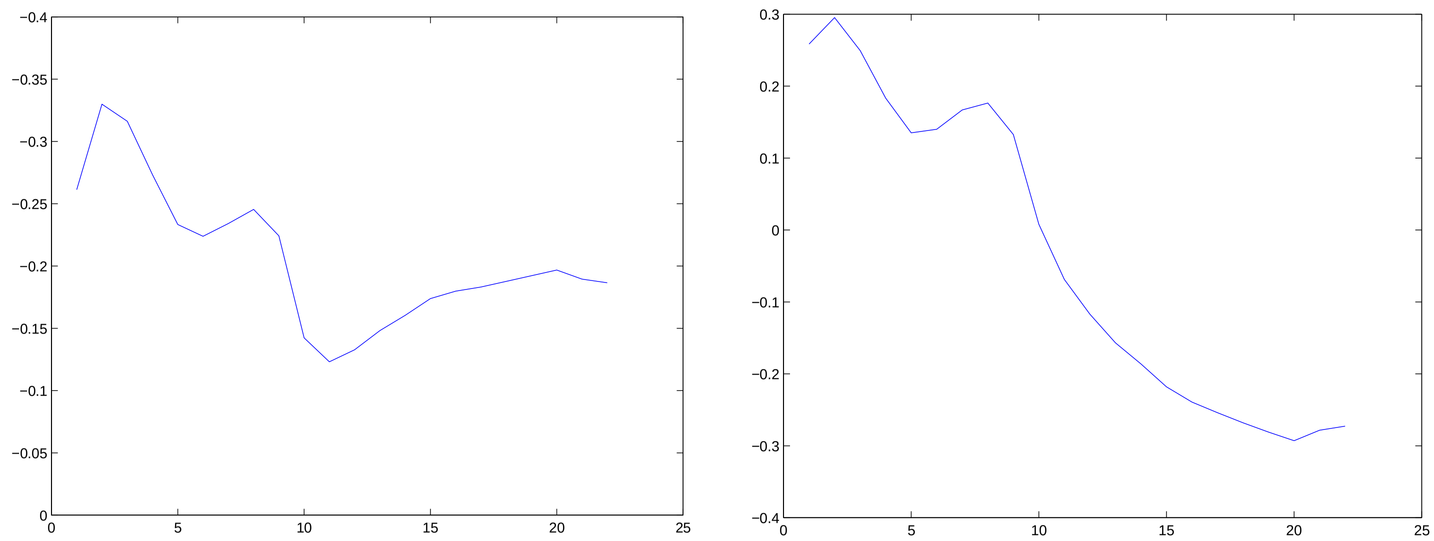

Figure 4: First eigenvector "level", second eigenvector "spread".

3.3 Robustness

We can make the solution (uniformly) robust to a change in market conditions $\sigma^2_{Bid,k}$ and $\sigma^2_{Ask,k}$ by solving:

$$\begin{array}{ll}\text{maximize} & t \\ \text{s.t.} & \sigma^2_{Bid,k} T_k + t \leq Tr(\Omega_k X) \leq \sigma^2_{Ask,k} T_k - t \quad \text{for } k = 1, ..., m \\ & X \succeq 0\end{array}$$

3.4 The dual problem

Let Ω_0 be the matrix assocaited with ap particular target option. The program

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}\,(\Omega_0 X) \\ \text{s.t.} & \mathbf{Tr}\,(\Omega_k X) = \sigma_k^2 \;\; \text{for } k = 1, ..., m \\ & X \succeq 0 \end{array}$$

will compute an upper arbitrage bound on the price of Ω_0. The dual, in this case:

$$\begin{array}{ll} \text{minimize} & \sum_{k=1}^m y_k \sigma_k^2 T_k \\ \text{s.t.} & \Omega_0 \preceq \sum_{k=1}^m y_k \Omega_k \end{array}$$

will give the coefficients of the associated hedging portfolio:

$$\lambda_k = -y_k \frac{\partial BS_0\left(Tr(\Omega_0 X)\right)/\partial v}{\partial BS_k\left(Tr(\Omega_k X)\right)/\partial v}$$

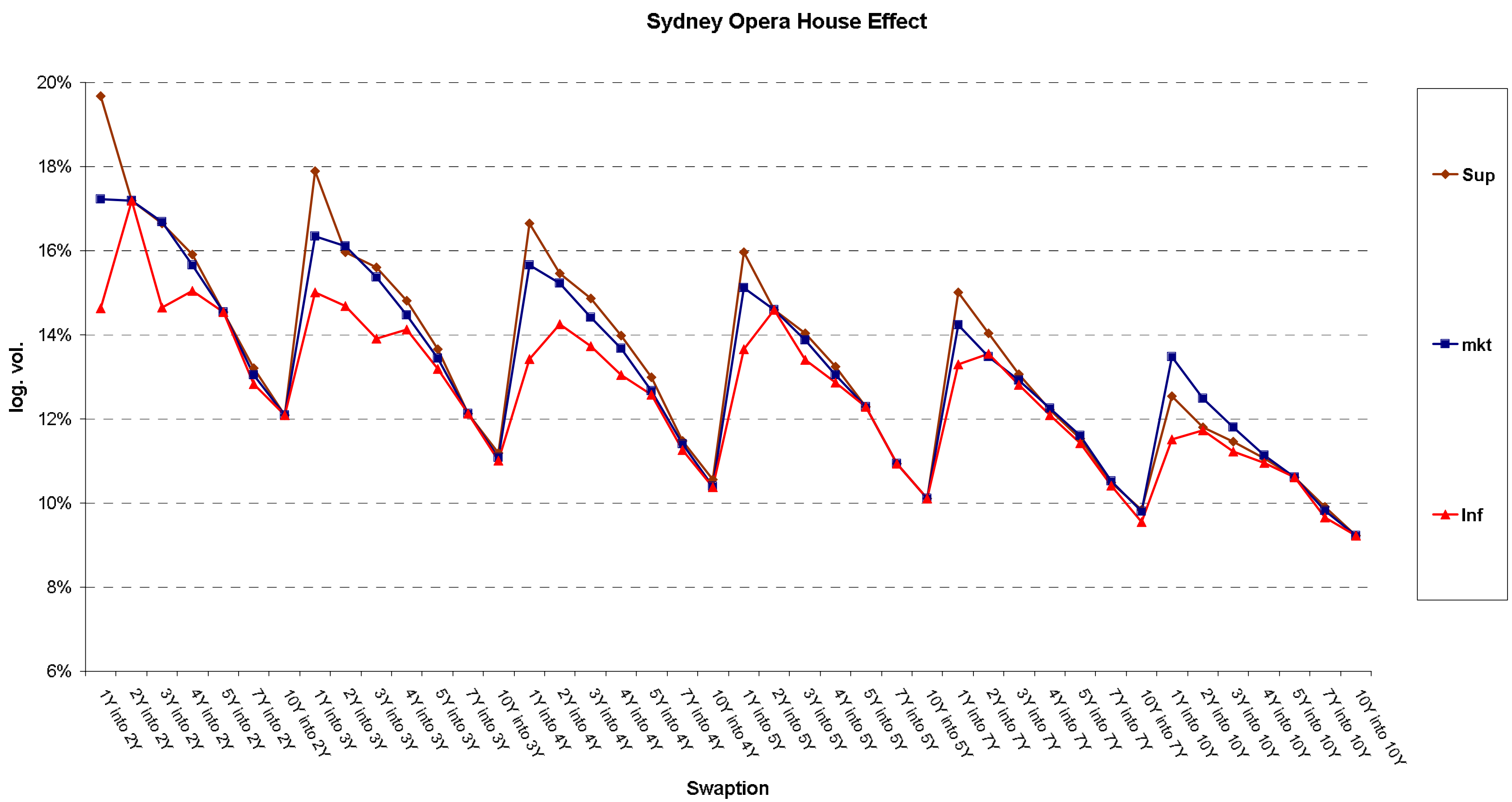

Figure 5: Calibration result and price bounds.

3.5 The rank issue

- American option pricing is usually done by dynamic programming and a low rank solution is desirable.

- Very good heuristical methods exist.

- Monte-Carlo pricing of American options is making progress...

3.6 Conclusion

- Multivariate derivative models calibration is a very intuitive, direct application of semidefinite programming.

- Increased flexibility and stability should significantly improve the pricing and hedging performance.

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