Pricing basket options

with an eye on swaptions

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Data from BNP-Paribas, London.

Introduction

Why baskets, swaptions, calibration?

Interest rate derivatives trading

- Focus on structured products activity
- Discuss stability, speed and robustness
- few stable methods for model calibration and risk-management
- How do we extract *correlation* information from market option prices?

Activity

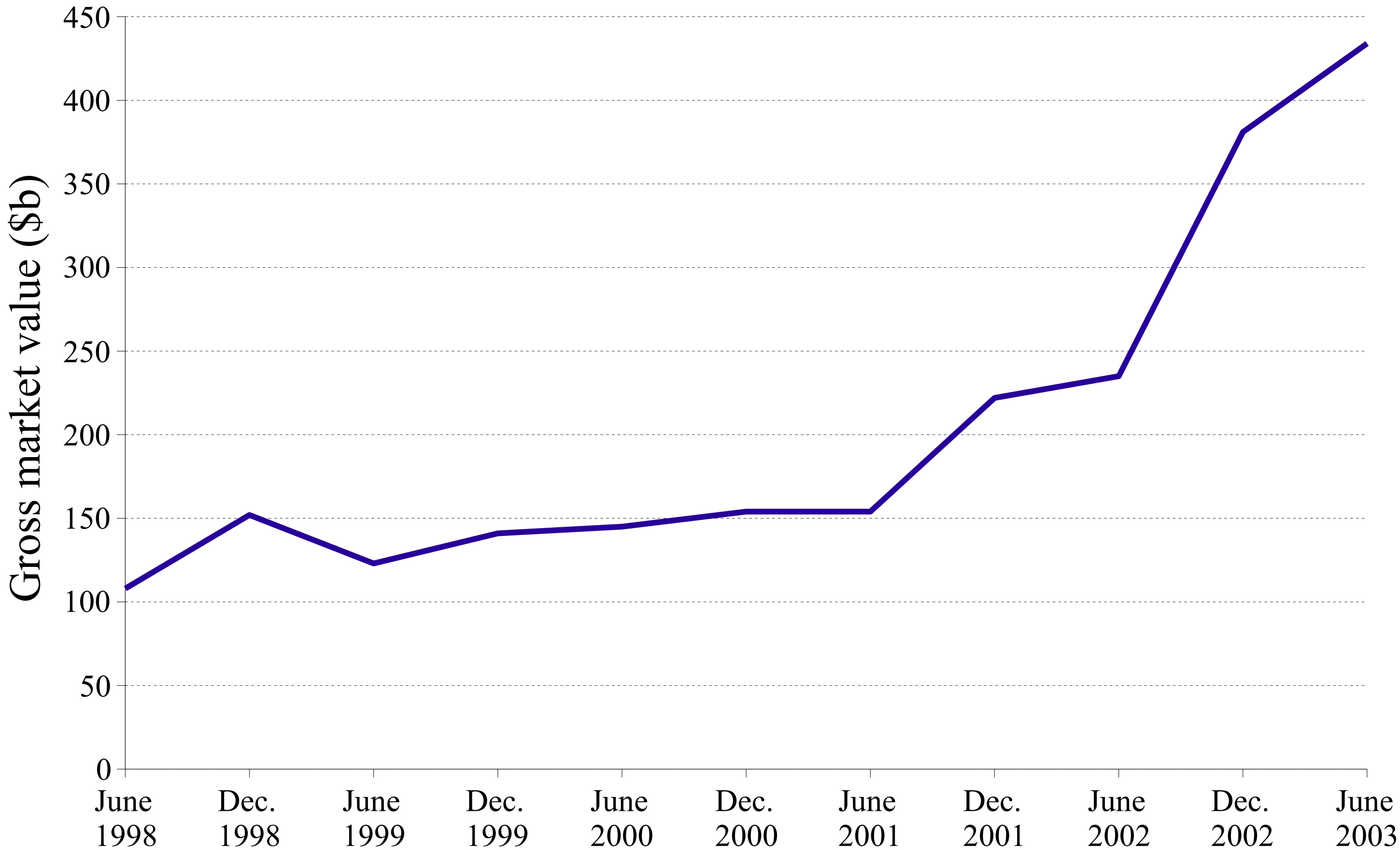

Figure 1: OTC activity in interest rate options.

Source: Bank for International Settlements.

Structured Interest Rate Products

Structured derivatives desks act as *risk brokers*

- **buy/sell** tailor made products from/to their clients
- **hedge** the resulting risk using simple options in the market
- **manage** the residual risk on the entire portfolio

P&L comes from a mix of *flow* and *arbitrage*. . .

Derivatives Production Cycle

market data

$$\Downarrow$$

model calibration

$$\Downarrow$$

pricing & hedging

$$\Downarrow$$

risk-management

Derivatives Production Cycle, *Trouble...*

market data: *illiquidity, Balkanization of the data sources*

$\Downarrow$

model calibration: *inverse problem, numerically hard*

$\Downarrow$

pricing & hedging: *American option pricing in dim.* ≥ 2

$\Downarrow$

risk-management: *all of the above...*

Objective

- However, *it works*
- Numerical difficulties creates P&L hikes, poor risk description, . . .
- Our objective here: improve *stability*, *robustness*

- Focus first on **calibration**
- Using new cone programming techniques to calibrate models and manage portfolio risk

Outline

- **Pricing baskets**
- Application to swaptions
- Cone programming, a brief introduction *(next talk)*
- IR model calibration *(next talk)*
- Risk-management *(next talk)*

Pricing baskets

Everything you ever wanted to know about basket options without ever daring to ask is in

Carmona & Durrleman (2003), in SIREV.

What this means for today:

- Either something you don't want to know
- or something you didn't know you wanted to know

Let me know. . .

Multivariate Black-Scholes

- We now look at the problem of pricing a basket in a generic Black & Scholes (1973) model with n assets F_s^i such that:

$$dF_s^i / F_s^i = \sigma_s^i dW_s$$

where $\sigma_i \in \mathbf{R}^n$ and dW_s is a n dimensional B.M.

- We study the dynamics of a basket of forwards $F_s^\omega = \sum_{i=1}^n w_i F_s^i$

- We look for an approximation to the price of a basket call:

$$\mathbf{E}\left[\left(\sum_{i=1}^n w_i F_T^i - K\right)^+\right]$$

Multivariate Black-Scholes

We can write the dynamics of the basket as:

$$\begin{cases} \frac{dF_u^\omega}{F_u^\omega} = \left(\sum_{i=1}^n \widehat{\omega}_{i,u}\sigma_u^i\right) dW_u \\ \\ \frac{d\widehat{\omega}_{i,s}}{\widehat{\omega}_{i,s}} = \left(\sum_{j=1}^n \widehat{\omega}_{j,s}\left(\sigma_s^i - \sigma_s^j\right)\right)\left(dW_s + \sum_{j=1}^n \widehat{\omega}_{j,s}\sigma_s^j ds\right) \end{cases}$$

where we have used:

$$\widehat{\omega}_{i,s} = \frac{\omega_i F_s^i}{\sum_{i=1}^n \omega_i F_s^i}$$

We notice that $0 \leq \widehat{\omega}_{i,s} \leq 1$ with $\sum_{j=1}^n \widehat{\omega}_{i,s} = 1$. We also set:

$$\tilde{\sigma}_s^i = \sigma_s^i - \sigma_s^\omega \quad \text{with} \quad \sigma_s^\omega = \sum_{j=1}^n \widehat{\omega}_{i,t}\sigma_s^j$$

note that $\sigma_s^\omega = \sum_{j=1}^n \widehat{\omega}_{i,t}\sigma_s^j$ is F_t-*measurable.*

Multivariate Black-Scholes

We can develop these dynamics around small values of $\sum_{j=1}^n \widehat{\omega}_{j,s}\tilde{\sigma}_s^j$. For some $\varepsilon > 0$, we write:

$$\begin{cases} dF_s^{\omega,\varepsilon} = F_s^{\omega,\varepsilon}\left(\sigma_s^\omega + \varepsilon \sum_{j=1}^n \widehat{\omega}_{j,s}\tilde{\sigma}_s^j\right) dW_s \\ \\ d\widehat{\omega}_{i,s}^\varepsilon = \widehat{\omega}_{i,s}^\varepsilon \left(\tilde{\sigma}_s^i - \varepsilon \sum_{j=1}^n \widehat{\omega}_{j,s}^\varepsilon \tilde{\sigma}_s^j\right)\left(dW_s + \sigma_s^\omega ds + \varepsilon \sum_{j=1}^n \widehat{\omega}_{j,s}\tilde{\sigma}_s^j ds\right) \end{cases}$$

As in Fournié, Lebuchoux & Touzi (1997) and Lebuchoux & Musiela (1999) we compute:

$$C^\varepsilon = E\left[(F_T^{\omega,\varepsilon} - k)^+ \,|\, (F_t^\omega, \widehat{\omega}_t)\right]$$

and approximate it around $\varepsilon = 0$ by:

$$C^\varepsilon = C^0 + C^{(1)}\varepsilon + o(\varepsilon)$$

Both C^0 and $C^{(1)}$ (as well as $C^{(2)}, \ldots$) can be *computed explicitly*.

Price Approximation: order zero

The order zero term can be computed directly as the solution to the limit BS PDE:

$$\begin{cases} \frac{\partial C^0}{\partial s} + \|\sigma_s^\omega\|^2 \frac{x^2}{2}\frac{\partial^2 C^0}{\partial x^2} = 0 \\ \\ C^0 = (x-K)^+ \text{ for } s = T \end{cases}$$

and we get C^0 as a Black & Scholes (1973) price with variance $\|\sigma_s^\omega\|^2$:

$$C^0 = BS(T, F_t^\omega, V_T) = F_t^\omega N(h(V_T)) - \kappa N\left(h(V_T) - \sqrt{V_T}\right)$$

with

$$h(V_T) = \frac{\left(\ln\left(\frac{F_t^\omega}{\kappa}\right) + \frac{1}{2}V_T\right)}{\sqrt{V_T}} \text{ et } V_T = \int_t^T \|\sigma_s^\omega\|^2 \, ds$$

Price Approximation: order one

- As in Fournié et al. (1997), we then look at the PDE satisfied by C^ε and differentiate it with respect to ε.

- The PDE associated with the multivariate BS dynamics is:

$$\begin{cases} L_0^\varepsilon C^\varepsilon = 0 \\ C^\varepsilon = (x-k)^+ \text{ en } s = T \end{cases}$$

Price Approximation: order one

where

$$
\begin{aligned}
L_0^\varepsilon = & \frac{\partial C^\varepsilon}{\partial s} + \left\| \sigma_s^\omega + \varepsilon \sum_{j=1}^n y_j \tilde{\sigma}_s^j \right\|^2 \frac{x^2}{2} \frac{\partial^2 C^\varepsilon}{\partial x^2} \\
& + \sum_{j=1}^n \left(\left\langle \tilde{\sigma}_s^j, \sigma_s^\omega \right\rangle + \varepsilon \sum_{k=1}^n y_k \left\langle \tilde{\sigma}_s^j - \sigma_s^\omega, \tilde{\sigma}_s^k \right\rangle - \varepsilon^2 \left\| \sum_{k=1}^n y_k \tilde{\sigma}_s^k \right\|^2 \right) x y_j \frac{\partial^2 C^\varepsilon}{\partial x \partial y_j} \\
& + \sum_{j=1}^n \left\| \tilde{\sigma}_s^j - \varepsilon \sum_{k=1}^n y_k \tilde{\sigma}_s^k \right\|^2 \frac{y_j^2}{2} \frac{\partial^2 C^\varepsilon}{\partial y_j^2} \\
& + \sum_{j=1}^n \left(\left\langle \tilde{\sigma}_s^j, \sigma_s^\omega \right\rangle + \varepsilon \sum_{k=1}^n y_k \left\langle \tilde{\sigma}_s^j - \sigma_s^\omega, \tilde{\sigma}_s^k \right\rangle - \varepsilon^2 \left\| \sum_{k=1}^n y_k \tilde{\sigma}_s^k \right\|^2 \right) y_j \frac{\partial C^\varepsilon}{\partial y_j}
\end{aligned}
$$

Price Approximation: order one

Take the limit in $\varepsilon = 0$ (mod. regularity conditions,. . .):

$$\begin{cases} L_0^0 C^{(1)} + \left(\sum_{j=1}^n y_j \left\langle \tilde{\sigma}_s^j, \sigma_s^\omega \right\rangle\right) x^2 \frac{\partial^2 C^0}{\partial x^2} = 0 \\ C^\varepsilon = 0 \text{ en } s = T \end{cases}$$

We then compute $C^{(1)}$ using the Feynmann-Kac representation:

$$C^{(1)} = F_t^\omega \int_t^T \sum_{j=1}^n \widehat{\omega}_{j,t} \left\langle \tilde{\sigma}_s^j, \sigma_s^\omega \right\rangle \exp\left(\int_t^s -\frac{1}{2} \left\| \tilde{\sigma}_u^j - \sigma_u^\omega \right\|^2 du\right)$$
$$E\left[\frac{\exp\left(\int_t^s \left(\sigma_u^\omega + \tilde{\sigma}_u^j\right) dW_u\right)}{\sqrt{V_{s,T}}} n\left(\frac{\ln \frac{F_t^\omega}{K} + \int_t^s \sigma_u^\omega dW_u - \frac{1}{2} V_{t,s} + \frac{1}{2} V_{s,T}}{\sqrt{V_{s,T}}}\right)\right] ds$$

which can be computed explicitly. The same technique produces $C^{(2)}$,. . .

Price Approximation: summary

The price of a basket call:

$$\mathbf{E}\left[\left(\sum_{i=1}^{n} w_i F_T^i - K\right)^+\right]$$

is approximated by a regular call price

$$C = BS(w^T F_t, K, T, V_T) \quad \text{with} \quad V_T = \int_t^T \mathbf{Tr}\left(\Omega_t X_s\right) ds$$

where

$$\begin{aligned}\mathbf{Tr}\left(\Omega_t X_s\right) &= \textstyle\sum_{i,j=1}^{n} \Omega_{t,i,j} X_{s,i,j} \\ &= \textstyle\sum_{i,j=1}^{n} \hat{w}_{i,t} \hat{w}_{j,t} \sigma_s^{iT} \sigma_s^j\end{aligned}$$

and

$$\Omega_t = \hat{w}_t \hat{w}_t^T \quad \text{with} \quad \hat{w}_{i,t} = \frac{w_i F_t^i}{w^T F_t}$$

Price Approximation: summary

We can get a better approximation of the price by using instead:

$$C^{\varepsilon} = C^0 + C^{(1)}$$

C^0 is given by the BS formula above:

$$C^0 = BS(w^T F_t, K, T, V_T)$$

We get $C^{(1)}$ as:

$$\begin{aligned} C^{(1)} &= w^T F_t \int_t^T \sum_{j=1}^n \widehat{w}_{j,t} \frac{\left\langle \tilde{\sigma}_s^j, \sigma_s^w \right\rangle}{V_T^{1/2}} \exp\left(2 \int_t^s \left\langle \tilde{\sigma}_u^j, \sigma_u^w \right\rangle du\right) \\ & \quad N\left(\frac{\ln \frac{w^T F_t}{K} + \int_t^s \left\langle \tilde{\sigma}_u^j, \sigma_u^w \right\rangle du + \frac{1}{2} V_T}{V_T^{1/2}}\right) ds \end{aligned}$$

Hedging Interpretation

- Suppose we are hedging the option with the approximate vol. σ_s^ω and, as in El Karoui, Jeanblanc-Picqué & Shreve (1998), we track the hedging error:

$$e_T = \frac{1}{2}\int_t^T \left(\left\| \sum_{i=1}^n \widehat{\omega}_{i,s}\sigma_s^i \right\|^2 - \left\| \sigma_s^\omega \right\|^2 \right) (F_s^\omega)^2 \frac{\partial^2 C^0(F_s^\omega, V_{t,T})}{\partial x^2} ds$$

- At the first order in $\tilde{\sigma}_s^j$, we get:

$$e_T^{(1)} = \int_t^T \sum_{i=1}^n \left\langle \tilde{\sigma}_s^i, \sigma_s^\omega \right\rangle \widehat{\omega}_{i,s} F_s^\omega \frac{n(h(V_{s,T}, F_s^\omega))}{V_{s,T}^{1/2}} ds$$

- We finally have:

$$C^{(1)} = E\left[e_T^{(1)} \right]$$

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Swaps

- The swap rate is the rate that equals the PV of a fixed and a floating leg:

$$swap(t, T_0, T_n) = \frac{B(t, T_0^{floating}) - B(t, T_{n+1}^{floating})}{level(t, T_0^{fixed}, T_n^{fixed})}$$

where

$$level(t, T_0^{fixed}, T_n^{fixed}) = \sum_{i=1}^{n+1} coverage(T_{i-1}^{fixed}, T_i^{fixed}) B(t, T_i^{fixed})$$

Swaps

- The swap rate can be expressed a basket of forward rates:

$$swap(t, T_0, T_n) = \sum_{i=0}^{n} w_i(t) K(t, T_i)$$

where $K(t, T_i)$ are the forward rates with maturities T_i, with the weights $w_i(t)$ given by

$$w_i(t) = \frac{coverage(T_i^{float}, T_{i+1}^{float}) B(t, T_{i+1}^{float})}{level(t, T_0^{fixed}, T_n^{fixed})}$$

- Empirically, these weights are *very stable* (see Rebonato (1998)).

Libor Market Model

- In the Libor Market Model, the zero coupon volatility is specified to make Libor rates
$$1 + \delta L(t,\theta) = exp\left(\int_\theta^{\theta+\delta} r(t,v)dv\right)$$
lognormal martingales under their respective measures:
$$dK(s,T_i)/K(s,T_i) = \sigma(s,T_i)dW_s^{Q^{T_i+\delta}}$$
where $\sigma(s,T_i) \in \mathbf{R}^n$ and $dW_s^{Q^{T_i+\delta}}$ is a n dimensional B.M. and
$$K(s,T_i) = L(s,T_i - s)$$

- This volatility definition, the forward curve today and the Heath, Jarrow & Morton (1992) arbitrage conditions fully specify the model.

Pricing Swaptions

- We let Q^{LVL} be the swap forward martingale probability measure given by:

$$\frac{d\mathbf{Q}^{LVL}}{d\mathbf{Q}^{T}}|_t = B(t,T)\beta(T)\sum_{i=1}^{N}\frac{\delta cvg(i,b)\beta^{-1}(T_{i+1})}{Level(t,T,T_N)}$$

- Following Jamshidian (1997), we can write the price of the Swaption with strike k as a that of a call on a swap rate:

$$Ps(t) = Level(t,T,T_N)E_t^{Q_{LVL}}\left[\left(\sum_{i=0}^{n}\omega_i(T)K(T,T_i) - k\right)^+\right]$$

- In other words, the swaption is a *call on a basket of forwards*.

A Remark on the Gaussian HJM

We can also express the price of the swaption as that of a bond put:

$$Ps(t) = B(t,T)\mathbf{E}_t^{Q_T}\left[\left(1 - B(t,T_{N+1}) - k\delta\sum_{i=i_T}^{N} B(t,T_i)\right)^+\right]$$

In the Gaussian H.J.M. model (see El Karoui & Lacoste (1992), Musiela & Rutkowski (1997) or Duffie & Kan (1996)), this expression defines the price of a swaption as that of a *put on a basket of lognormal* zero coupon prices.

Approximations

We will make two key approximations:

- We replace the weights $w_i(s)$ by their value today $w_i(t)$.
- We approximate the swap rate $\sum_{i=0}^{n} w_i(t)K(s,T_i)$ by a sum of Q^{LVL} lognormal martingales F_s^i with:

$$F_t^i = K(t, T_i)$$

and

$$dF_s^i / F_s^i = \sigma(s, T_i - s) dW_s^{LVL}$$

Swaption pricing formula

We can write the order zero price approximation for Swaptions:

$$Swaption = Level(t,T,T_N)\left(swap(t,T,T_N)N(h) - \kappa N(h - V_T^{1/2})\right)$$

with

$$h = \frac{\left(\ln\left(\frac{swap(t,T,T_N)}{\kappa}\right) + \frac{1}{2}V_T\right)}{V_T^{1/2}}$$

where

$$V_T = \int_t^T \left\| \sum_{i=1}^{N} \hat{\omega}_i(t)\sigma(s,T_i - s) \right\|^2 ds \text{ and } \hat{\omega}_i(t) = \omega_i(t)\frac{K(t,T_i)}{swap(t,T,T_N)}$$

and $dK(s,T_i) = \sigma(s,T_i - s)K(s,T_i)dW_s^{Q_{T_{i+1}}}$.

Errors

Can we quantify the error:

- What's the contribution of the weights in the swap's volatility?
- What about the drift terms coming from the forwards under Q^{LVL}?
- What is the precision of the basket price approximation?

First two questions: wait for next talk...

Price Approximation: Precision

We plot the difference between two distinct sets of swaption prices in the Libor Market Model.

- One is obtained by Monte-Carlo simulation using enough steps to make the 95% confidence margin of error always less than 1bp.
- The second set of prices is computed using the order zero approximation.

The plots are based on the prices obtained by calibrating a BGM model to EURO Swaption prices on November 6 2000, using all cap volatilities and the following swaptions: 2Y into 5Y, 5Y into 5Y, 5Y into 2Y, 10Y into 5Y, 7Y into 5Y, 10Y into 2Y, 10Y into 7Y, 2Y into 2Y, 1Y into 9Y (choice based on liquidity).

Price Approximation: Precision

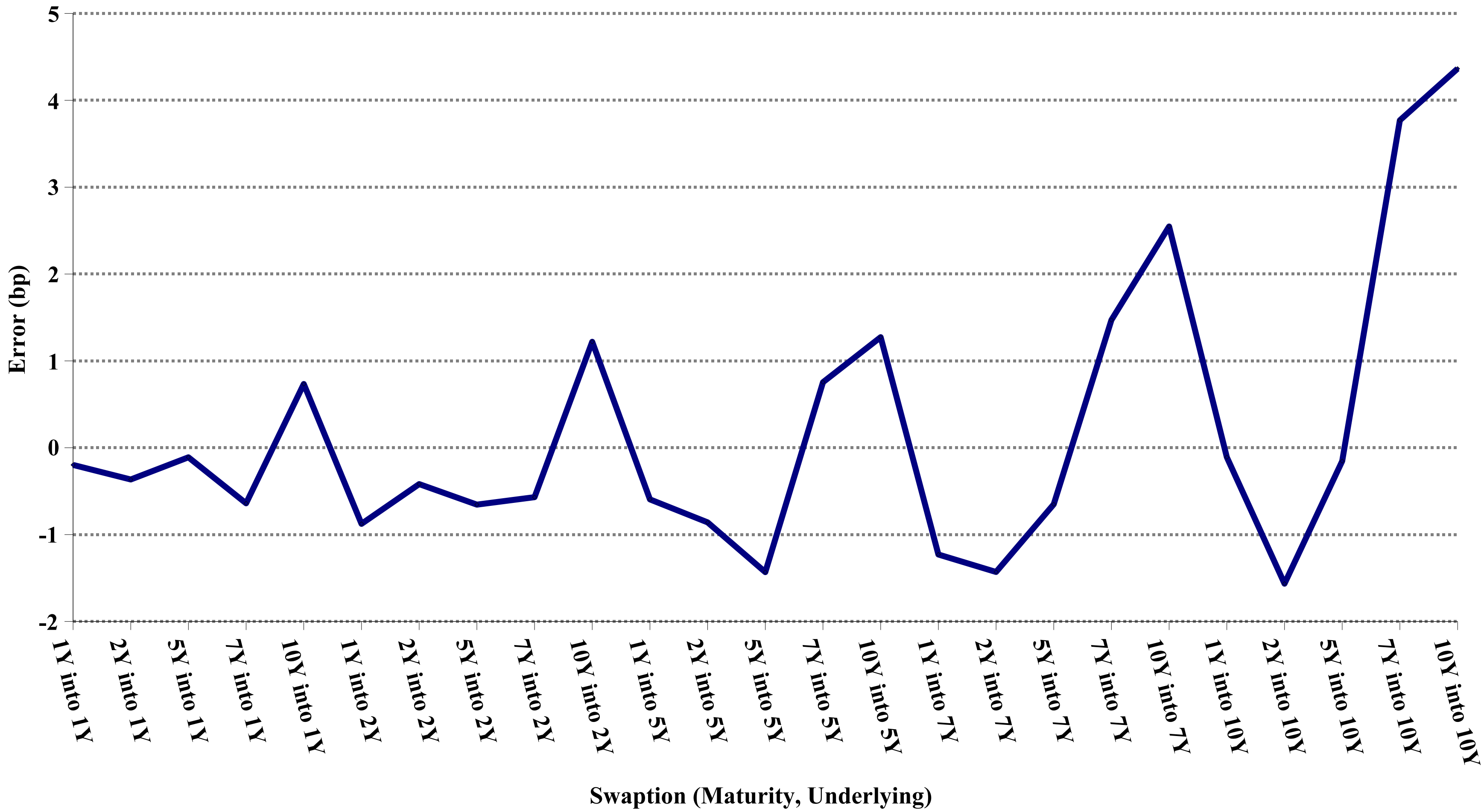

Figure 2: Error (bp) for various ATM swaptions.

Price Approximation: Precision

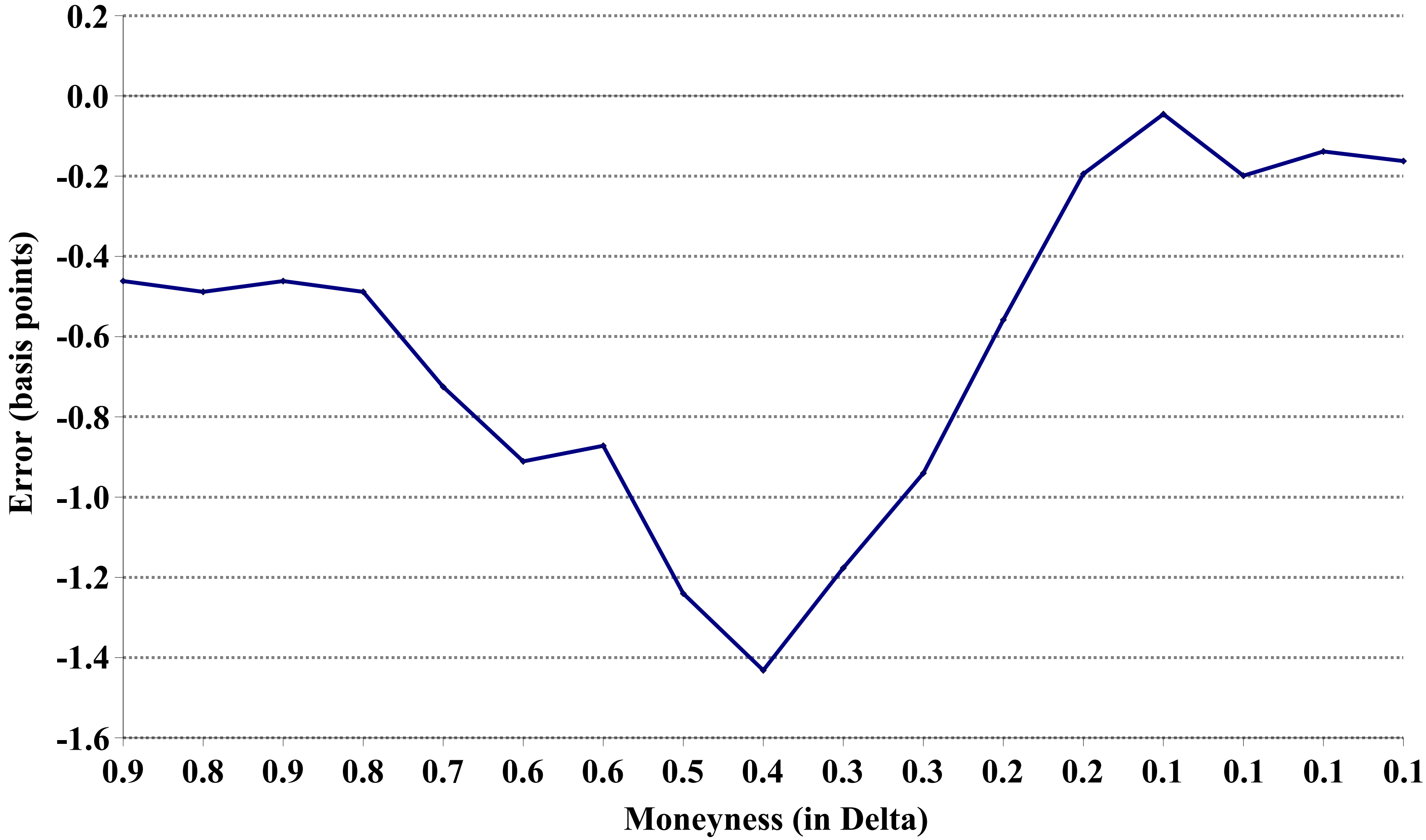

Figure 3: Error vs. moneyness, on the 5Y into 5Y

Price Approximation: Precision

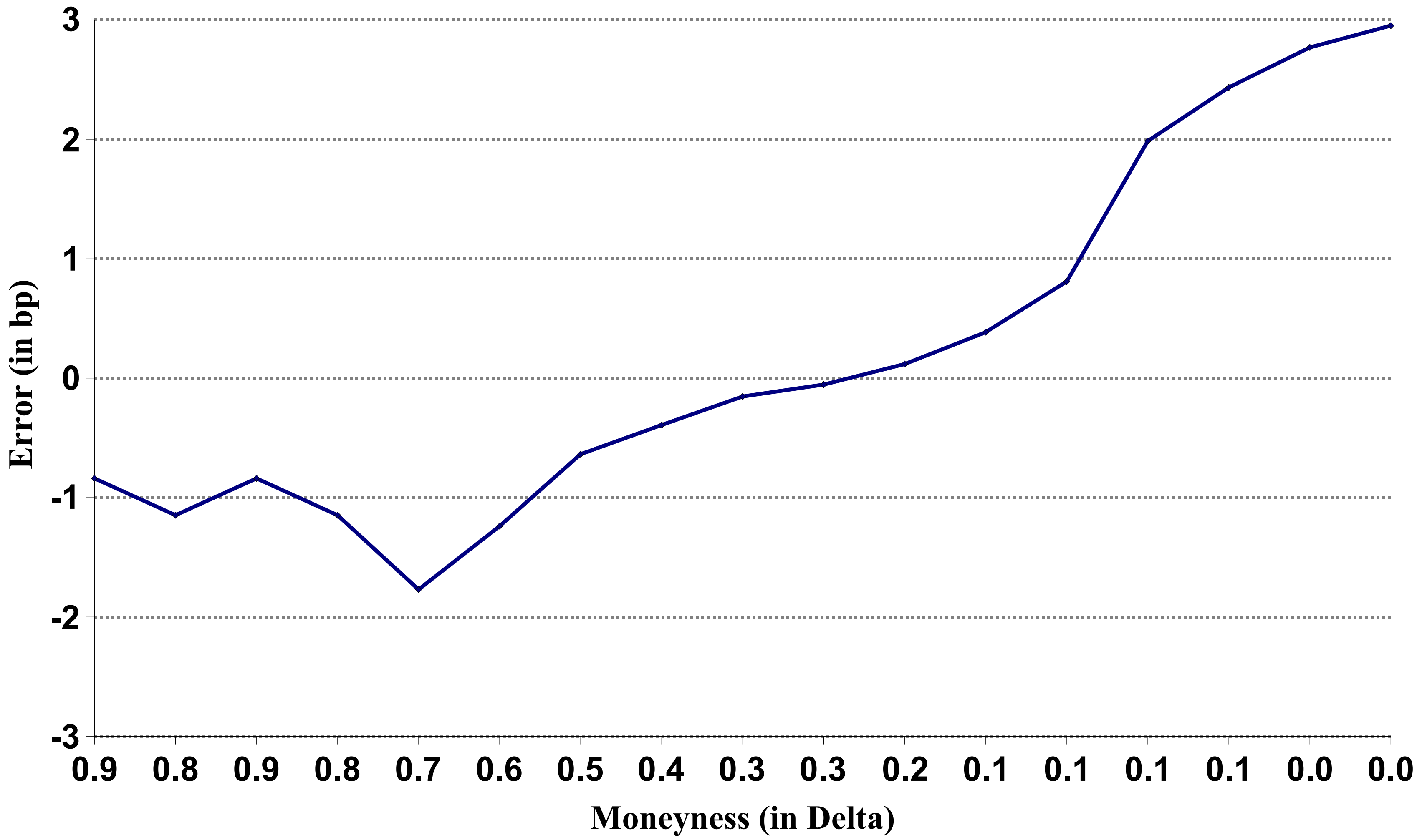

Figure 4: Error vs. moneyness on the 5Y into 10Y.

Price Approximation: Precision

- We compare again with Monte-Carlo. The model parameters are

$$
\begin{array}{rcl}
F_0^i & = & \{0.07, 0.05, 0.04, 0.04, 0.04\} \\
w_i & = & \{0.2, 0.2, 0.2, 0.2, 0.2\}
\end{array}
$$

  $T = 5$ years, the covariance matrix is:

$$
\frac{11}{100}\begin{pmatrix}
0.64 & 0.59 & 0.32 & 0.12 & 0.06 \\
0.59 & 1 & 0.67 & 0.28 & 0.13 \\
0.32 & 0.67 & 0.64 & 0.29 & 0.14 \\
0.12 & 0.28 & 0.29 & 0.36 & 0.11 \\
0.06 & 0.13 & 0.14 & 0.11 & 0.16
\end{pmatrix}
$$

- These values correspond to a $5Y$ into $5Y$ swaption.

- Our goal is to measure only the error coming from the pricing formula and not from the change of measure/martingale approximation

Price Approximation: Precision

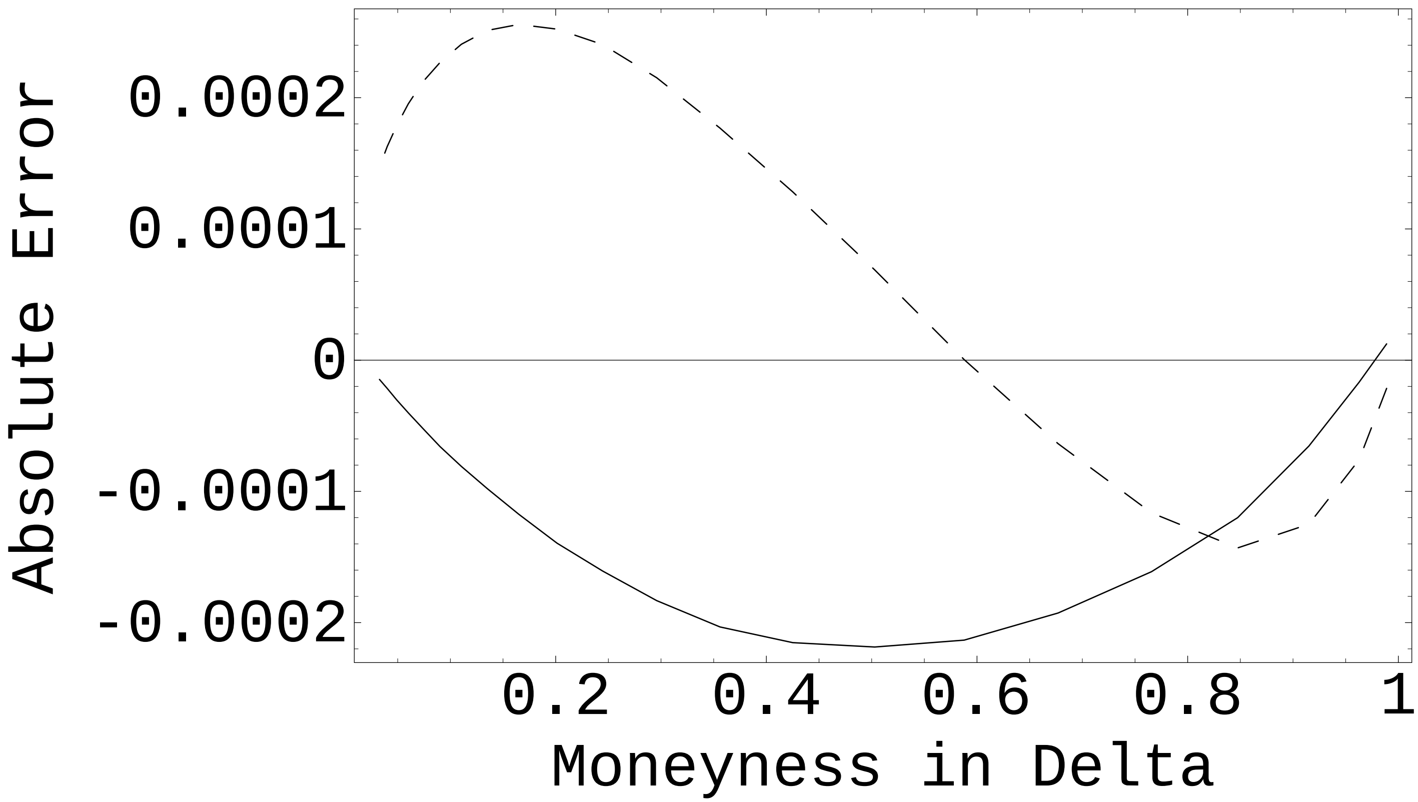

Figure 5: Order zero (dashed) and order one absolute pricing error (plain), in basis points.

Price Approximation: Precision

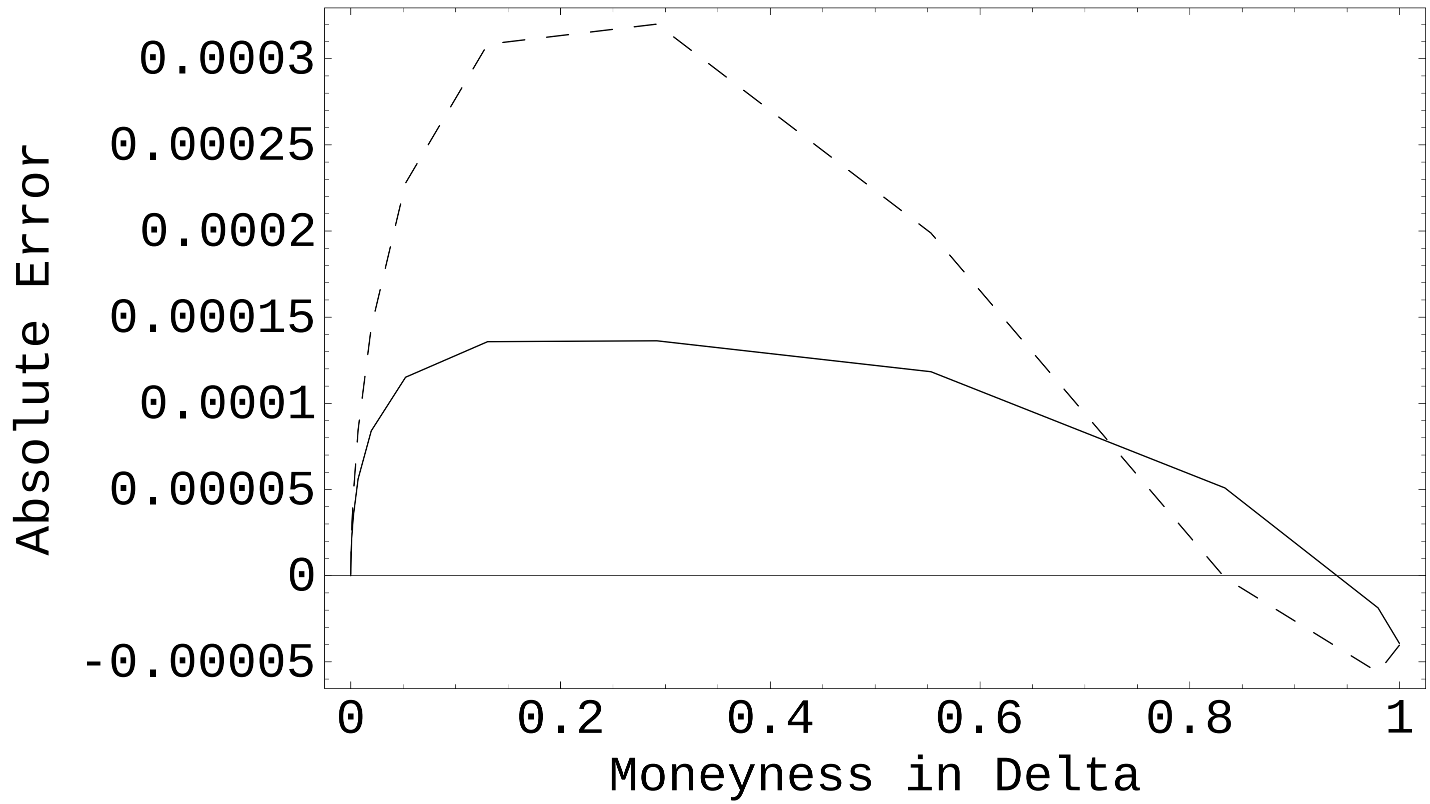

Figure 6: Order zero (dashed) and order one absolute pricing error (plain), in basis points, *zero correlation*.

Price Approximation: Precision

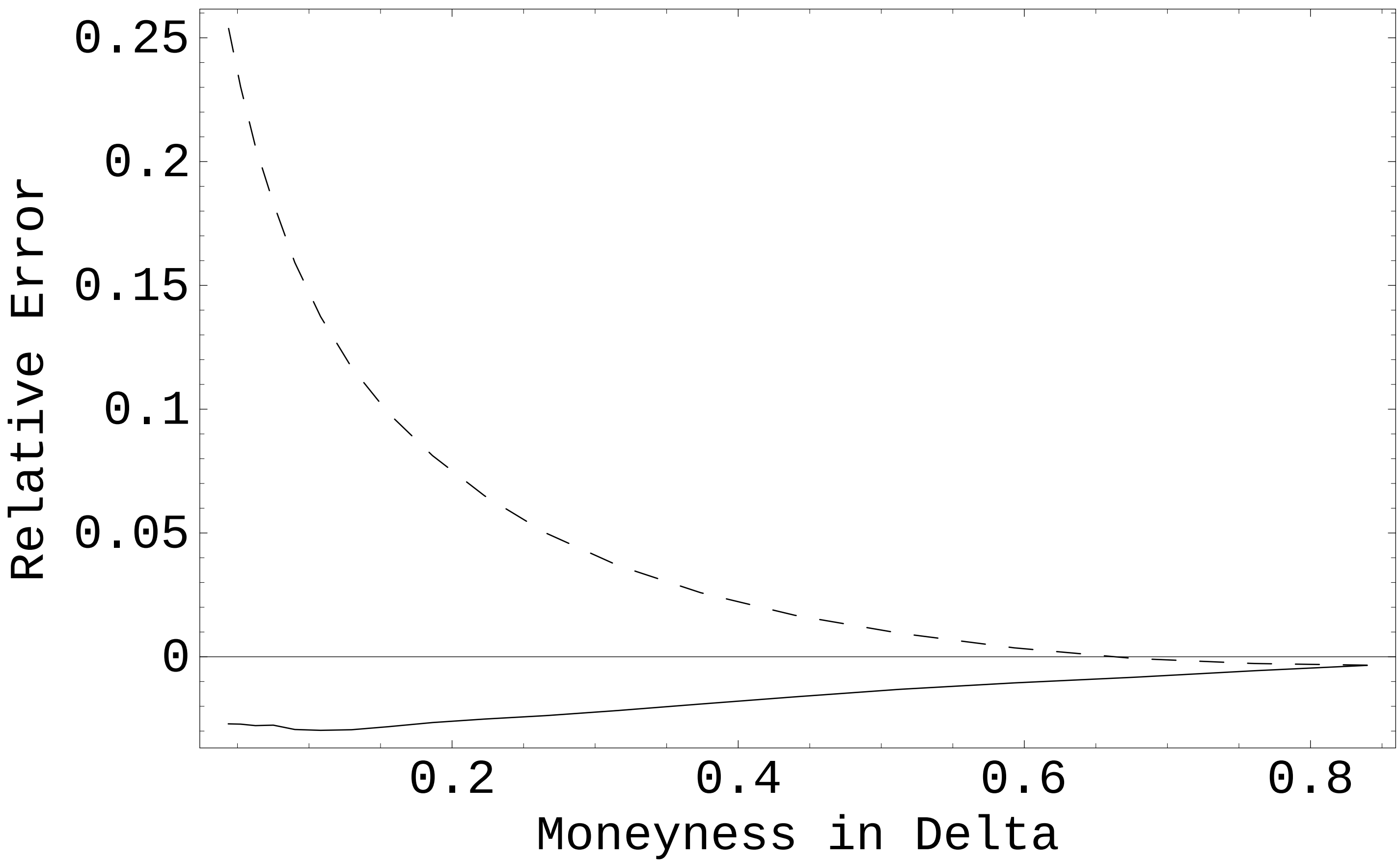

Figure 7: Order zero (dashed) and order one relative pricing error (plain), *equity case*.

Conclusion

- Order zero BS like formula sufficient for ATM swaptions
- Equity case: use order one
- Change of measure between Q^{LVL} and Q^T negligible (volatilities too low).

Next talk:

- Will discuss calibration and risk-management issues

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- **IR model calibration** ***(next talk)***
- **Risk-management** ***(next talk)***

References

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