Restarting Frank-Wolfe.

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Introduction

Frank-Wolfe. Classical first order methods for solving

$$\begin{array}{ll}\text{minimize} & f(x)\\ \text{subject to} & x \in C,\end{array}$$

in $x \in \mathbb{R}^n$, with $C \subset \mathbb{R}^n$ convex.

Assumes that the **linear minimization oracle**

$$\begin{array}{ll}\text{minimize} & d^T x\\ \text{subject to} & x \in C\end{array}$$

can be solved efficiently for any $d \in \mathbb{R}^n$.

Franke-Wolfe

Algorithm 1 Franke-Wolfe **(FW)**

```
Inputs: $x_0 \in C$.
for $k = 1, \ldots, k^{max}$ do
    Solve the linear minimization oracle
        $x_d := \text{argmin} \quad x^T \nabla f(y_k)$
               $\text{subject to} \quad x \in C$
    Update the current point
        $x_{k+1} = x_k + \frac{2}{k+2}(x_d - x_k)$
end for
Output: approximate solution $\hat{x}$
```

Note that all iterates are feasible.

Franke-Wolfe

Complexity.

- Assume that f is differentiable. Define the curvature C_f of the function $f(x)$ as
$$C_f \triangleq \sup_{\substack{s,x\in\mathcal{M},\ \alpha\in[0,1],\\ y=x+\alpha(s-x)}} \frac{1}{\alpha^2}(f(y)-f(x)-\langle y-x,\nabla f(x)\rangle).$$
- The basic Frank-Wolfe algorithm will then produce an ϵ solution after at most
$$N_{\max}=\frac{4C_f}{\epsilon}$$
iterations.

Franke-Wolfe

Stopping criterion. At each iteration, we get a lower bound on the optimum as a byproduct of the affine minimization step.

- If x_d minimizes $\nabla f(x_k)^T x_d$ over C, we have by convexity

$$f(x_k) + \nabla f(x_k)^T (x_d - x_k) \leq f(x), \quad \text{for all } x \in C$$

- Calling f^* the optimal value of problem, we then get

$$f(x_k) - f^* \leq \nabla f(x_k)^T (x_k - x_d).$$

This allows us to bound the suboptimality of iterate at no additional cost.

Franke-Wolfe

Machine Learning Applications. See [Jaggi, 2013].

- When C is an atomic norm ball, each vertex is an atom and FW naturally produces "sparse" solutions.
- Linear minimization oracle is often easy to solve.
  - Complexity $O(n)$ for $\|\cdot\|_q$-balls.
  - Just an SVD for classical matrix norms (matrix completion, etc.)
  - Also works for structured atomic norms.
  - Idem for structured prediction [Lacoste-Julien et al., 2012].
- For some combinatorial polytopes with an exponential number of vertices, the linear minimization oracle is tractable, while projection is hard.

A Faster Franke-Wolfe

Faster convergence.

- **Linear convergence** with away steps when the optimum is inside the set [Guélat and Marcotte, 1986].
- **Linear convergence** for away step variants when the function is strongly convex [Garber and Hazan, 2013, Lacoste-Julien and Jaggi, 2015].
- Various **extensions** further improved upon these results for special cases, e.g. [Lacoste-Julien et al., 2013, Freund and Grigas, 2016, Garber and Meshi, 2016, Braun et al., 2017, Lan et al., 2017, Bashiri and Zhang, 2017, Garber et al., 2018, Kerdreux et al., 2018b, Braun et al., 2018],
- See also Joulin et al. [2014], Shah et al. [2015], Osokin et al. [2016], Freund et al. [2017], Miech et al. [2017] for applications of Frank-Wolfe to **machine learning problems**.

A Faster Franke-Wolfe

Today.

- Restarting accelerated gradient methods gives significantly improved performance.
- Complexity gains controlled by the “sharpness” of the optimum.
- Can we do the same for Frank-Wolfe?

Introduction

“Templates for convex cone problems with applications to sparse signal recovery.” (TFOCS) by [Becker, Candès, and Grant, 2011].

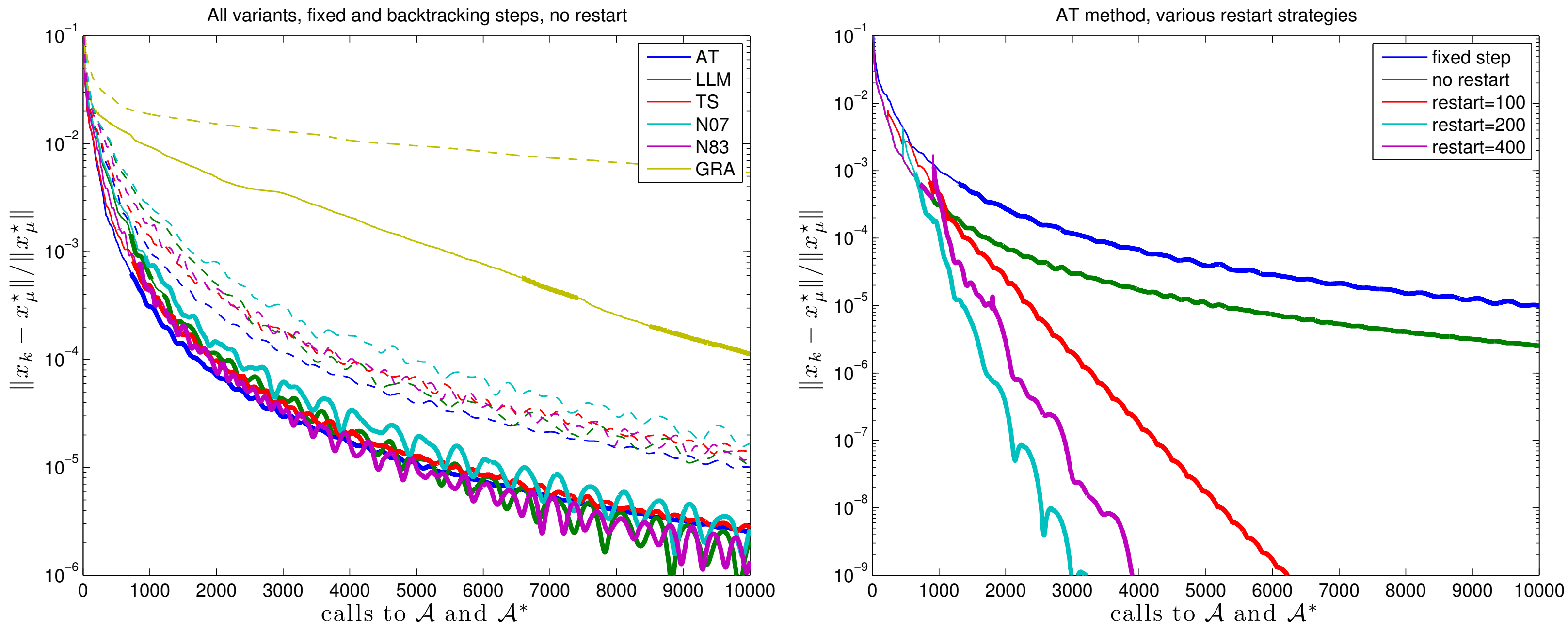


Figure 6: Comparing first order methods applied to a smoothed Dantzig selector model. Left: comparing all variants using a fixed step size (dashed lines) and backtracking line search (solid lines). Right: comparing various restart strategies using the AT method.

Restarting fast gradient methods yields linear convergence. . .

Outline

Today.

- Introduction
- **Sharpness & Łojasiewicz's growth condition**
- Optimal restart schemes for gradient methods
- Restarting Frank-Wolfe
- Numerical results

Sharpness

Consider

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in Q \end{array}$$

where $f(x)$ is a **convex** function, $Q \subset \mathbb{R}^n$.

- Assume ∇f **is Hölder continuous,**

$$\|\nabla f(x) - \nabla f(y)\|_* \leq L\|x-y\|^{s-1}, \quad \text{for every } x, y \in \mathbb{R}^n,$$

- Assume **"sharpness"**, i.e. the following local growth condition holds

$$\mu d(x, X^*)^r \leq f(x) - f^*, \quad \text{for every } x \in K,$$

where f^* is the minimum of f, $K \subset \mathbb{R}^n$ is a compact set, $d(x, X^*)$ the distance from x to the set $X^* \subset K$ of minimizers of f, and $r \geq 1$, $\mu > 0$ are constants.

Sharpness, Restart

Strong convexity is a particular case of sharpness.

$$\mu d(x, X^*)^2 \leq f(x) - f^*$$

If f is also **smooth**, an optimal gradient method (ignoring strong convexity), will produce a point x satisfying

$$f(x) - f^* \leq \frac{cL}{t^2} d(x_0, X^*)^2,$$

after t iterations.

- Restarting the algorithm, we thus get

  $$f(x_{k+1}) - f^* \leq \frac{cL}{\mu t_k^2} \left(f(x_k) - f^* \right), \quad k = 1, \ldots, N$$

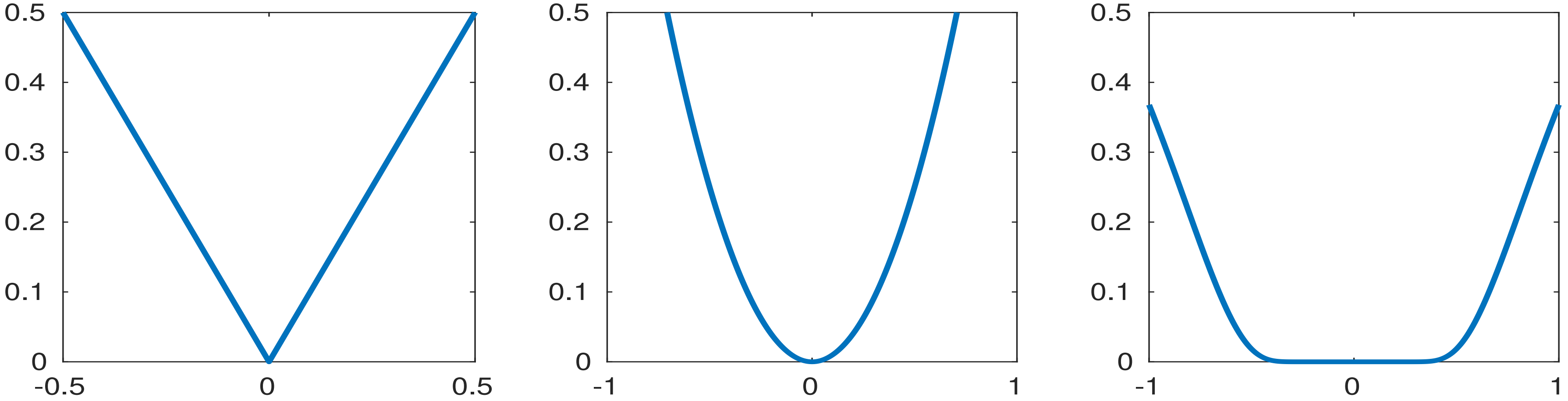
  at each outer iteration, after t_k inner iterations.
- Restart proves **linear convergence**.

Sharpness

Smoothness is classical [Nesterov, 1983, 2005], sharpness less so. . .

$$\mu d(x, X^*)^r \leq f(x) - f^*, \quad \text{for every } x \in K.$$

- Real analytic functions all satisfy this locally, a result known as Łojasiewicz's inequality [Lojasiewicz, 1963].
- Generalizes to a much wider class of non-smooth functions [Lojasiewicz, 1993, Bolte et al., 2007]
- Conditions of this form are also known as **sharp minimum**, **Hölderian error bound**, etc. [Polyak, 1979, Burke and Ferris, 1993, Burke and Deng, 2002].



The functions $|x|$, x^2 and $\exp(-1/x^2)$.

Sharpness & Smoothness

- Gradient ∇f Hölder continuous ensures

  $$f(x) - f^* \leq \frac{L}{s} d(x, X^*)^s,$$

  an **upper bound** on suboptimality.

- If in addition f sharp on a set K with parameters (r, μ), we have

  $$\frac{s\mu}{rL} \leq d(x, X^*)^{s-r}$$

  hence $s \leq r$.

In the following, we write

$$\kappa \triangleq L^{\frac{2}{s}} / \mu^{\frac{2}{r}} \qquad \text{and} \qquad \tau \triangleq 1 - \frac{s}{r}$$

If $r = s = 2$, κ matches the classical condition number of the function.

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- **Optimal restart schemes for gradient methods**
- Restarting Frank-Wolfe
- Numerical results

Sharpness & Complexity

- Restart schemes were studied for strongly or uniformly convex functions [Nemirovskii and Nesterov, 1985, Nesterov, 2007, Iouditski and Nesterov, 2014, Lin and Xiao, 2014]
- In particular, Nemirovskii and Nesterov [1985] link sharpness with (optimal) faster convergence rates using restart schemes.
- Weaker versions of this strict minimum condition used more recently in restart schemes by [Renegar, 2014, Freund and Lu, 2015].
- Several heuristics [O'Donoghue and Candes, 2015, Su et al., 2014, Giselsson and Boyd, 2014] studied adaptive restart schemes to speed up convergence.
- The robustness of restart schemes was also studied by Fercoq and Qu [2016] in the strongly convex case.
- Sharpness used to prove linear converge matrix games by Gilpin et al. [2012].

Restart schemes

Algorithm 2 Scheduled restarts for smooth convex minimisation **(RESTART)**

```
Inputs : $x_0 \in \mathbb{R}^n$ and a sequence $t_k$ for $k = 1, \ldots, R$.
for $k = 1, \ldots, R$ do
    $x_k := \mathcal{A}(x_{k-1}, t_k)$
end for
Output : $\hat{x} := x_R$
```

Here, the number of inner iterations t_k satisfies

$$t_k = Ce^{\alpha k}, \quad k = 1, \ldots, R.$$

for some $C > 0$ and $\alpha \geq 0$ and will ensure

$$f(x_k) - f^* \leq \nu e^{-\gamma k}.$$

Restart schemes

Proposition [Roulet and A., 2017]

Restart. *Let f be a smooth convex function with parameters $(2, L)$, sharp with parameters (r, μ) on a set K. Restart with iteration schedule $t_k = C^*_{\kappa,\tau} e^{\tau k}$, for $k = 1, \ldots, R$, where $C^*_{\kappa,\tau} \triangleq e^{1-\tau}(c\kappa)^{\frac{1}{2}}(f(x_0) - f^*)^{-\frac{\tau}{2}}$, with $c = 4e^{2/e}$ here. The precision reached at the last point $\hat{x}$ is given by,*

$$f(\hat{x}) - f^* \leq e^{-2e^{-1}(c\kappa)^{-\frac{1}{2}}N}(f(x_0) - f^*) = O\left(\exp(-\kappa^{-\frac{1}{2}}N)\right), \quad \textit{when } \tau = 0,$$

while,

$$f(\hat{x}) - f^* \leq \frac{f(x_0) - f^*}{\left(\tau e^{-1}(f(x_0) - f^*)^{\frac{\tau}{2}}(c\kappa)^{-\frac{1}{2}}N + 1\right)^{\frac{2}{\tau}}} = O\left(N^{-\frac{2}{\tau}}\right), \quad \textit{when } \tau \in (0, 1],$$

where $N = \sum_{k=1}^{R} t_k$ is the total number of iterations.

Adaptation

The sharpness constant μ and exponent r in

$$\mu d(x, X^*)^r \leq f(x) - f^*, \quad \text{for every } x \in K.$$

are of course **never observed.** Can we make restart schemes **adaptive?** Otherwise, sharpness is useless. . .

- **Yes:** Grid of size $(\log_2 N)^2$ on restart parameters suffices. Fully adaptive if primal gap is known. See [Roulet and A., 2017].
- Can we prove something similar for Frank-Wolfe?

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Restarting Frank-Wolfe

Strong Wolfe gap.

- Let f be a smooth convex function, $\mathcal{C}$ a polytope and let $x \in \mathcal{C}$ be arbitrary. Then the **strong Wolfe gap** $w(x)$ over $\mathcal{C}$ is defined as

$$w(x) \triangleq \left(\min_{S \in \mathcal{S}_x} \max_{y \in S, z \in \mathcal{C}} \nabla f(x)^T (y - z) \right)_+ \tag{1}$$

  where $x \in \mathbf{Co}(S)$ and $S_x = \{S \mid S \subset \mathbf{Ext}(\mathcal{C}), x \in \mathbf{Co}(S), |S| \text{ finite}\}$.

- We also write

$$w(x, S) \triangleq \left(\max_{y \in S, z \in \mathcal{C}} \nabla f(x)^T (y - z) \right)_+ \tag{2}$$

  given $S \in \mathcal{S}_x$.

Gap: $w(x)$ and $w(x, S)$ equal zero if and only if x is an optimal solution.

Restarting Frank-Wolfe

Definition [Kerdreux, A., and Pokutta, 2018a]

Strong Wolfe Primal Bound. *For any compact subset K of $\mathcal{C}$ such that $x^* \in K$, there are $\mu > 0$ and $r > 0$ such that*

$$f(x) - f^\star \leq \mu w(x)^r, \quad \textit{for } x \in K \tag{3}$$

[Lacoste-Julien and Jaggi, 2015, Theorem 6] shows $r = 2$ when f is strongly convex and $\mathcal{C}$ is a polytope.

Restarting Frank-Wolfe

Definition

Scaling inequality. *For all $x \in \mathcal{C} \setminus X^*$ and all differentiable convex function f,*

$$w(x) \geq \delta(\mathcal{C}) \max_{x^* \in X^*} \langle \nabla f(x); \frac{x - x^*}{||x - x^*||} \rangle. \qquad \text{(Scaling)}$$

- A convex polytope satisfies the δ-scaling inequality with $\delta(\mathcal{C}) = PWidth(\mathcal{C})$ [Lacoste-Julien and Jaggi, 2015].
- Łojasiewicz's factorization lemma then shows that the **strong Wolfe primal bound holds** when the scaling inequality holds.

Restarting Frank-Wolfe

Fractional Away-Step Frank-Wolfe Algorithm.

```
Given a smooth convex function $f$ with curvature $C_f^{\mathcal{A}}$. Starting point $x_0 = \sum_{v\in\mathcal{S}'} \alpha_0^v v \in \mathcal{C}$ with support $\mathcal{S}_0 \subset \mathbf{Ext}(\mathcal{C})$ and schedule parameter $\gamma > 0$.
Set $t := 0$
while $w(x_t, \mathcal{S}_t) > e^{-\gamma} w(x_0, \mathcal{S}_0)$ do
    $v_t := LP_{\mathcal{C}}(\nabla f(x_t))$ and $d_t^{FW} \triangleq v_t - x_t$
    $s_t := LP_{S_t}(-\nabla f(x_t))$ with $S_t$ current active set and $d_t^{Away} \triangleq x_t - s_t$
    if $-\nabla f(x_t)^T d_t^{FW} > e^{-\gamma} w(x_0, \mathcal{S}_0)/2$ then
        $d_t := d_t^{FW}$ with $\eta_{\text{max}} = 1$
    else
        $d_t := d_t^{Away}$ with $\eta_{\text{max}} = \frac{\alpha_t^{s_t}}{1-\alpha_t^{s_t}}$
    end if
    $x_{t+1} := x_t + \eta_t d_t$ with $\eta_t \in [0, \eta_{\text{max}}]$ via line-search
    Update active set $\mathcal{S}_{t+1}$ and coefficients $\{\alpha_{t+1}^v\}_{v\in\mathcal{S}_{t+1}}$
    $t := t + 1$
end while
Output: $x_t \in \mathcal{C}$ such that $w(x_t, \mathcal{S}_t) \leq e^{-\gamma} w(x_0, \mathcal{S}_0)$
```

Restarting Frank-Wolfe

Proposition [Kerdreux, A., and Pokutta, 2018a]

FAFW convergence. *Let f be a globally subanalytic, smooth convex function with away curvature C_f^A, satisfying the strong Wolfe primal bound on a compact set K for some $r \geq 1$ and $\mu > 0$. Let $\gamma > 0$ and assume $x_0 \in K$ is such that $e^{-\gamma} w(x_0)/2 \leq C_f^A$. The algorithm above outputs $x_T \in K$ such that*

$$w(x_T, \mathcal{S}_T) \leq w(x_0, \mathcal{S}_0) e^{-\gamma}$$

after at most

$$T \leq |\mathcal{S}_0| - |\mathcal{S}_T| + 16 e^{2\gamma} C_f^A \mu w(x_0, \mathcal{S}_0)^{r-2}$$

iterations, where $\mathcal{S}_0$ and $\mathcal{S}_T$ are the supports of respectively x_0 and x_T.

Restarting Frank-Wolfe

Proposition [Kerdreux, A., and Pokutta, 2018a]

FAFW with restarts. *Let f be a globally subanalytic, smooth convex function with away curvature C_f^A, satisfying the strong Wolfe primal bound on a compact set K with $r \geq 1$ and $\mu > 0$. Let $\gamma > 0$ and assume $x_0 \in K$ is such that $e^{-\gamma} w(x_0, \mathcal{S}_0)/2 \leq C_f^A$. With $\gamma_k = \gamma$, the output of FAFW with restarts satisfies*

$$\begin{cases} f(x_T) - f(x^\star) \leq w_0 \dfrac{1}{\left(1 + \tilde{T} C_\gamma^r\right)^{\frac{1}{2-r}}} & \text{when } 1 \leq r < 2 \\[2ex] f(x_T) - f(x^\star) \leq w_0 \exp\left(-\dfrac{\gamma}{e^{2\gamma}} \dfrac{\tilde{T}}{8 C_f^A \mu}\right) & \text{when } r = 2\ , \end{cases} \tag{4}$$

with $w_0 = w(x_0, \mathcal{S}_0)$ and $\tilde{T} \triangleq T - (|\mathcal{S}_0| - |\mathcal{S}_T|)$, where $C_\gamma^r \triangleq \frac{e^{\gamma(2-r)} - 1}{8 e^{2\gamma} C_f^A \mu w(x_0, \mathcal{S}_0)^{r-2}}$.

Restarting Frank-Wolfe

Proposition [Kerdreux, A., and Pokutta, 2018a]

Robustness in γ. *Suppose f satisfies strong Wolfe primal bound for $r > 0$ and write $\gamma^*(r)$ the optimal choice of $\gamma > 0$ in the complexity bound. Running FAFW with $\gamma = 1/2$ yields $\hat{x}$ satisfying*

$$f(\hat{x}) - f^* \leq \sqrt{\frac{e}{e-1}} \frac{w(x_0, \mathcal{S}_0)}{\left(1 + \tilde{T} C^r_{\gamma^*(r)}\right)^{\frac{1}{2-r}}} \quad \text{when } 0 \leq r < 2 \; . \tag{5}$$

When $r = 2$, we have $\gamma^*(r) = 1/2$.

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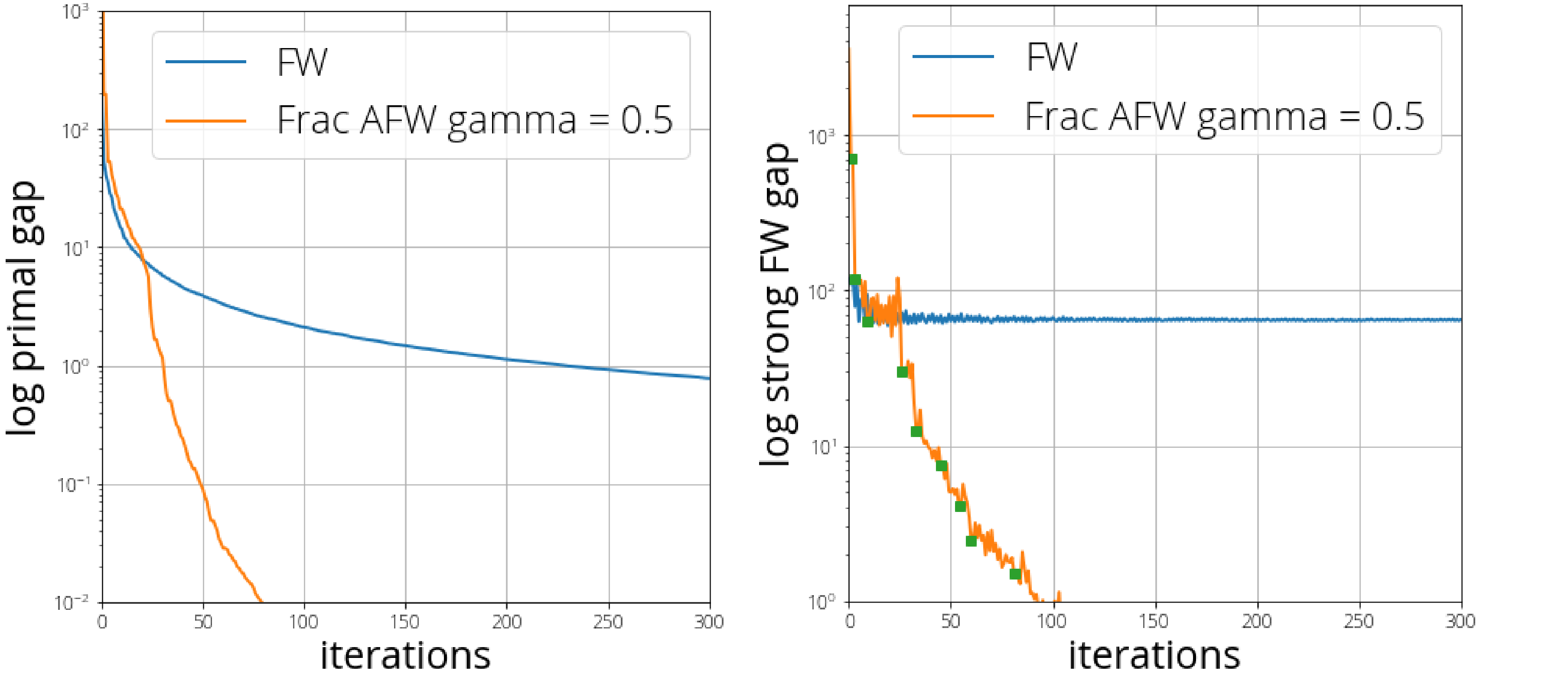
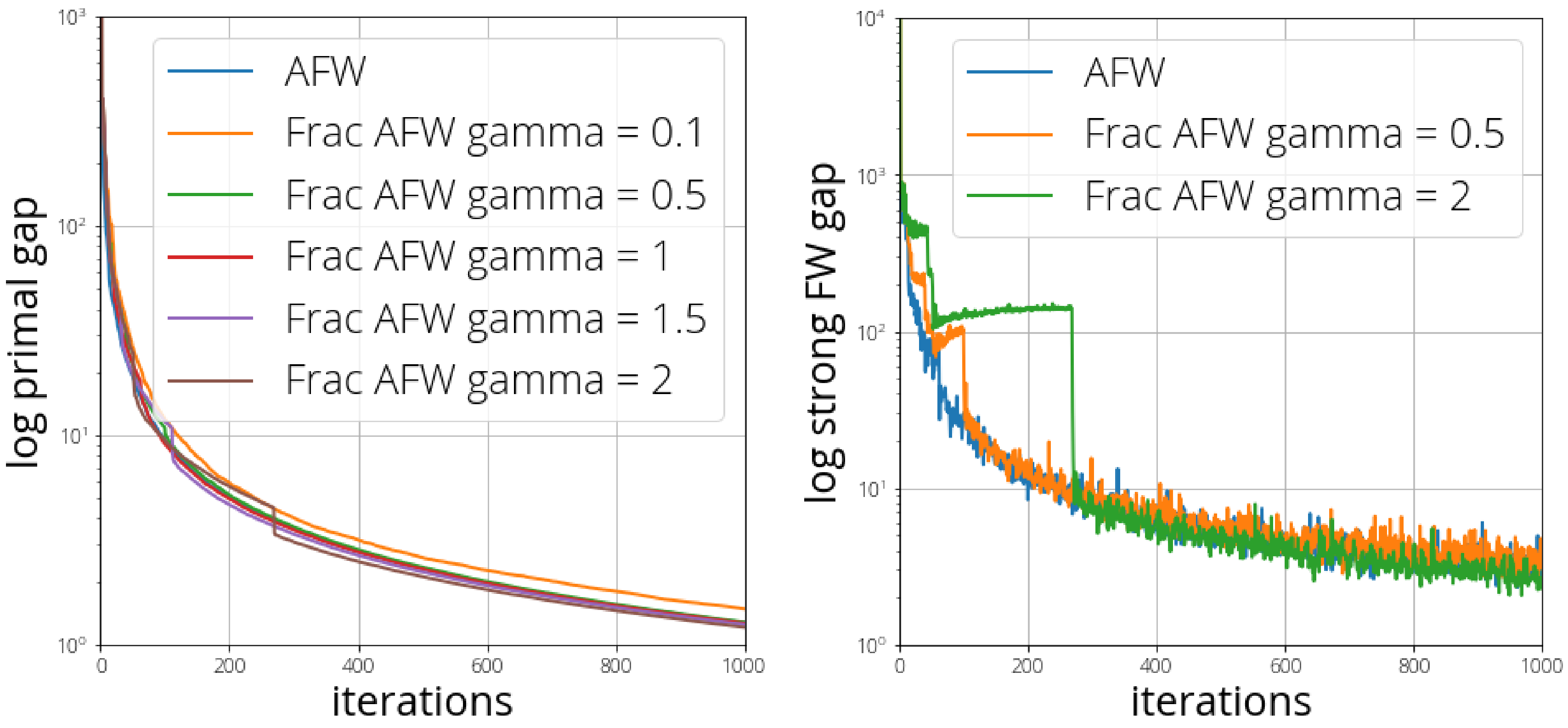
Numerical Results



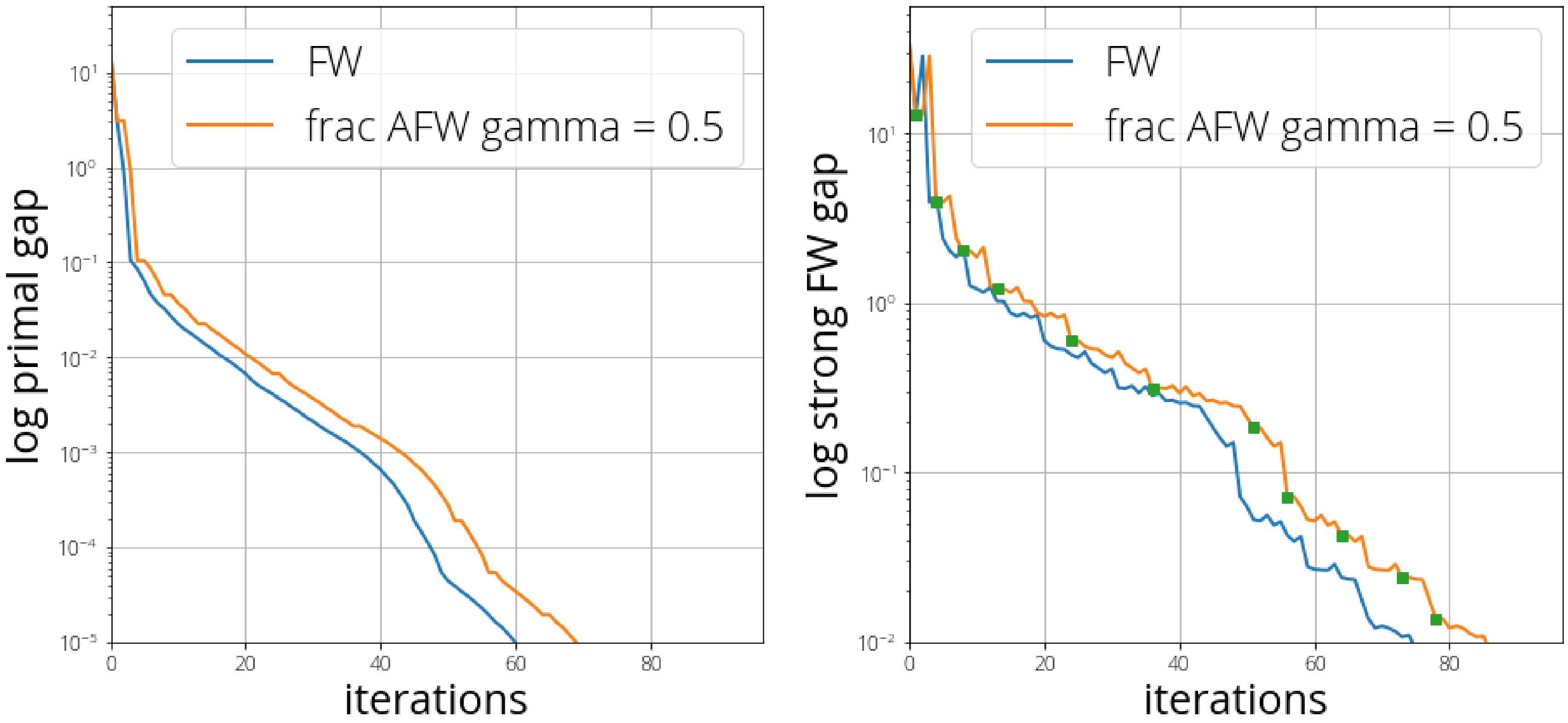
Comparing classical FW and FAFW with $\gamma = 0.5$ on a regression problem with loss power $\alpha = 1.5$, so that the classical geometric strong convexity condition does not hold. Green squares indicate restart times.

Numerical Results



Representative examples on Lasso with various values of γ in restart schemes of algorithm FAFW.

Numerical Results



Comparing classical FW and FAFW with $\gamma = 0.5$ on logistic regression with ℓ_1 constraint, where the constrained minimum lies in the interior of the ball. Here AFW and FW share the very same curve.

Conclusion

- Restarting Frank-Wolfe yields generically faster rates.
- Performance gains controlled by sharpness.
- Restart scheme is robust to growth condition/sharpness parameters.

Open problems.

- Fully adaptive bounds? Restart means we're missing something in the FAFW convergence proof. . .

*

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