On the Versatility of the Nesterov Acceleration Scheme

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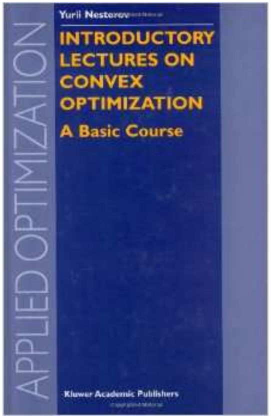
Tribute to Y. Nesterov's teaching and research


APPLIED OPTIMIZATION
Yurii Nesterov
INTRODUCTORY LECTURES ON CONVEX OPTIMIZATION
A Basic Course
Kluwer Academic Publishers

Book

Y. Nesterov. Introductory Lectures on Convex Optimization: a basic course. Kluwer Academic Publishers.

Collaborators

Hongzhou
Lin

Julien
Mairal

Publication

H. Lin, J. Mairal and Z. Harchaoui. A Universal Catalyst for First-Order Optimization. Adv. NIPS 2015.

Focus of this work

Minimizing large finite sums

Consider the minimization of a large sum of convex functions

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},$$

where each f_i is **smooth and convex** and ψ is a convex but not necessarily differentiable penalty.

Goal of this work

- Design accelerated methods for minimizing large finite sums.
- Give a generic acceleration scheme which can apply to previously un-accelerated algorithms.

Why do large finite sums matter?

Empirical risk minimization

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},$$

- Typically, x represents **model parameters**.
- Each function f_i measures the **fidelity** of x to a data point.
- ψ is a **regularization function** to prevent overfitting.

For instance, given training data $(y_i, z_i)_{i=1,\ldots,n}$ with features z_i in $\mathbb{R}^p$ and labels y_i in $\{-1, +1\}$, we may want to predict y_i by $\text{sign}(\langle z_i, x \rangle)$. Functions f_i measures how far the prediction is from the true label.

This would be a **classification problem with a linear model**.

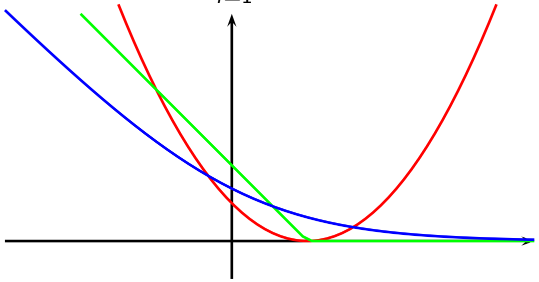
Why large finite sums matter?

A few examples

Ridge regression: $$\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2}(y_i - \langle x, z_i\rangle)^2 + \frac{\lambda}{2}\|x\|_2^2.$$

Linear SVM: $$\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} \max(0, 1 - y_i\langle x, z_i\rangle) + \frac{\lambda}{2}\|x\|_2^2.$$

Logistic regression: $$\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} \log\left(1 + e^{-y_i\langle x, z_i\rangle}\right) + \frac{\lambda}{2}\|x\|_2^2.$$



Why does the composite problem matter?

A few examples

Ridge regression: $$\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2}(y_i - \langle x, z_i\rangle)^2 + \frac{\lambda}{2}\|x\|_2^2.$$

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The **squared ℓ_2-norm** penalizes large entries in x.

Why does the composite problem matter?

A few examples

Ridge regression:
$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2}(y_i - \langle x, z_i \rangle)^2 + \lambda \|x\|_1.$$

Linear SVM:
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Logistic regression:
$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \log \left(1 + e^{-y_i \langle x, z_i \rangle}\right) + \lambda \|x\|_1.$$

When one knows in advance that x should be sparse, one should use a **sparsity-inducing** regularization such as the ℓ_1-norm.

[Chen et al., 1999, Tibshirani, 1996].

How to minimize a large sum composite problem?

Two major challenges

- **Non-differentiable regularization penalty.**
  Exclude existing solver such as MOSEK, CPLEX, etc.
- **Large-scale and high-dimensionality**
  Exclude higher-order (Newton) methods.

This leads us to first-order gradient-based methods.

Gradient descent methods

Let us consider the composite problem

$$\min_{x \in \mathbb{R}^p} f(x) + \psi(x),$$

where f is convex, differentiable with L-Lipschitz continuous gradient and ψ is convex, but not necessarily differentiable.

The classical forward-backward/ISTA algorithm

$$x_k \leftarrow \arg\min_{x \in \mathbb{R}^p} \frac{1}{2} \left\| x - \left(x_{k-1} - \frac{1}{L} \nabla f(x_{k-1}) \right) \right\|_2^2 + \frac{1}{L} \psi(x).$$

- $f(x_k) - f^\star = O(1/k)$ for **convex** problems;
- $f(x_k) - f^\star = O((1 - \mu/L)^k)$ for μ-**strongly convex** problems;

[Nowak and Figueiredo, 2001, Daubechies et al., 2004, Combettes and Wajs, 2006, Beck and Teboulle, 2009, Wright et al., 2009, Nesterov, 2013]...

Accelerated gradient descent methods

Nesterov introduced in 1983 an acceleration scheme for the gradient descent algorithm. It was generalized later to the composite setting [Nesterov, 1983, 2004, 2013].

FISTA [Beck and Teboulle, 2009]

$$x_k \leftarrow \arg\min_{x \in \mathbb{R}^p} \frac{1}{2}\left\| x - \left(y_{k-1} - \frac{1}{L}\nabla f(y_{k-1}) \right) \right\|_2^2 + \frac{1}{L}\psi(x);$$

$$\text{Find } \alpha_k > 0 \ \text{ s.t. } \ \alpha_k^2 = (1-\alpha_k)\alpha_{k-1}^2 + \frac{\mu}{L}\alpha_k;$$

$$y_k \leftarrow x_k + \beta_k(x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1-\alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}.$$

- $f(x_k) - f^\star = O(1/k^2)$ for **convex** problems;
- $f(x_k) - f^\star = O((1-\sqrt{\mu/L})^k)$ for **μ-strongly convex** problems;
- Acceleration works in many practical cases.

see also [Nesterov, 1983, 2004, 2013]

What do we mean by “acceleration”?

Complexity analysis for large finite sums

Since f is a sum of n functions, computing ∇f requires computing n gradients ∇f_i. The complexity to reach an $\varepsilon-$solution is given below

	$\mu > 0$	$\mu = 0$
ISTA	$O\left(n\frac{L}{\mu}\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(\frac{nL}{\varepsilon}\right)$
FISTA	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(\frac{nL}{\sqrt{\varepsilon}}\right)$

Remarks

- ε-solution means here $f(x_k) - f^\star \leq \varepsilon$.
- For $n = 1$, the rates of FISTA are optimal for a “first-order local black box” [Nesterov, 2004].
- For $n > 1$, the sum structure of f is not exploited.

Can we do better for large finite sums?

Several **randomized** algorithms are designed with one ∇f_i computed per iteration, which yields a better **expected** **computational complexity**.

	$\mu > 0$
FISTA	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$
SVRG, SAG, SAGA, SDCA, MISO, Finito	$O\left(\max\left(n, \frac{L}{\mu}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$

SVRG, SAG, SAGA, SDCA, MISO, Finito improve upon FISTA when

$$\max\left(n, \frac{L}{\mu}\right) \leq n\sqrt{\frac{L}{\mu}} \quad \Leftrightarrow \sqrt{\frac{L}{\mu}} \leq n,$$

but they are not "accelerated" in the sense of Nesterov.

[Schmidt et al., 2013, Xiao and Zhang, 2014, Defazio et al., 2014a,b, Shalev-Shwartz and Zhang, 2012, Mairal, 2015, Zhang and Xiao, 2015]

Can we do even better for large finite sums?

Without vs with acceleration

	$\mu > 0$
FISTA	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$
SVRG, SAG, SAGA, SDCA, MISO, Finito	$O\left(\max\left(n, \frac{L}{\mu}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$
Acc-SDCA	$\tilde{O}\left(\max\left(n, \sqrt{n\frac{L}{\mu}}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$

Acc-SDCA is due to Shalev-Shwartz and Zhang [2014].

- Acceleration occurs when $n \leq \frac{L}{\mu}$.
- see [Agarwal and Bottou, 2015] for discussions about optimality.

Challenge: can we accelerate these algorithms by a universal scheme for both convex and strongly convex objectives ?

Catalyst is coming

Main idea

Catalyst, a meta-algorithm

Given an algorithm $\mathcal{M}$ that can solve a convex problem "appropriately".

- At iteration k, rather than minimizing F, we use $\mathcal{M}$ to minimize a function G_k, defined as follows,

$$G_k(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2,$$

  up to accuracy ε_k, i.e., such that $G_k(x_k) - G_k^\star \leq \varepsilon_k$.

- Then compute the next prox-center y_k using an extrapolation step

$$y_k = x_k + \beta_k(x_k - x_{k-1}).$$

The choices of $\beta_k, \epsilon_k, \kappa$ are driven by the theoretical analysis.

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The choices of $\beta_k, \epsilon_k, \kappa$ are driven by the theoretical analysis.

Catalyst is a wrapper of $\mathcal{M}$ that yields an **accelerated** algorithm $\mathcal{A}$.

Sources of inspiration

In addition to accelerated proximal algorithms [Beck and Teboulle, 2009, Nesterov, 2013], several works have inspired Catalyst.

The inexact accelerated proximal point algorithm of Güler [1992].

- Catalyst is a variant of inexact accelerated PPA.
- Complexity analysis for **outer-loop only** with non practical inexactness criterium.

Accelerated SDCA of Shalev-Shwartz and Zhang [2014].

- Accelerated SDCA is an instance of inexact accelerated PPA.
- Complexity analysis **limited to μ-strongly convex objectives**.

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Other related work

[Frostig et al., 2015, Schmidt et al., 2011, Salzo and Villa, 2012, He and Yuan, 2012, Lan, 2015]. $+$ Chambolle and Pock, 15.

This work

Contributions

- **Generic acceleration scheme**, which applies to previously unaccelerated algorithms such as SVRG, SAG, SAGA, SDCA, MISO, or Finito, and which is not taylored to finite sums.
- Provides explicit **support to non-strongly convex objectives**.
- Complexity analysis for μ-strongly convex objectives.
- Complexity analysis for non-strongly convex objectives.

Example of application

Garber and Hazan [2015] have used Catalyst to accelerate new principal component analysis algorithms based on convex optimization.

Appropriate $\mathcal{M}$ = Linear convergence rate when $\mu > 0$

Linear convergence rate

Consider a **strongly convex** minimization problem

$$\min_{z \in \mathbb{R}^p} H(z).$$

We say that an algorithm $\mathcal{M}$ has a **linear convergence rate** if $\mathcal{M}$ generates a sequence of iterates $(z_t)_{t \in \mathbb{N}}$ such that there exists $\tau_{\mathcal{M},H}$ in $(0,1)$ and a constant $C_{\mathcal{M},H}$ in $\mathbb{R}$ satisfying

$$H(z_t) - H^\star \leq C_{\mathcal{M},H}(1 - \tau_{\mathcal{M},H})^t. \tag{1}$$

- $\tau_{\mathcal{M},H}$ depends usually on the **condition number** L/μ, e.g., $\tau_{\mathcal{M},H} = \mu/L$ for ISTA and $\tau_{\mathcal{M},H} = \sqrt{\mu/L}$ for FISTA.
- $C_{\mathcal{M},H}$ depends usually on $H(z_0) - H^*$.

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Important message: we do not make any assumption for non strongly convex objectives. It is possible that $\mathcal{M}$ is not even defined for $\mu = 0$.

Catalyst action

Catalyst action

$$G_k(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2,$$

- G_k is always strongly convex as long as F is convex.
- When F is strongly convex, the condition number of G_k is better than that of F since $\frac{L+\kappa}{\mu+\kappa} < \frac{L}{\mu}$.

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Minimizing G_k is easier than minimizing F !

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- When F is strongly convex, the condition number of G_k is better than that of F since $\frac{L+\kappa}{\mu+\kappa} < \frac{L}{\mu}$.

Minimizing G_k is easier than minimizing F !

- If $\kappa \gg 1$, then minimizing G_k is easy;
- If $\kappa \approx 0$, then G_k is a good approximation of F.

We will choose κ to optimize the computational complexity.

Convergence analysis

An analysis in two stages

$$G_k(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2,$$

x_k is a approximate minimizer of G_k such that $G_k(x_k) - G_k^* \leq \epsilon_k$.

- Outer loop: once we obtain the sequence $(x_k)_{k \in \mathbb{N}}$, what can we say about the convergence rate of $F(x_k) - F^*$?
  $\Rightarrow$ Wisely choose (y_k) and control the accumulation of errors.
- Inner loop: how much effort do we need to obtain a x_k with accuracy ϵ_k?
  $\Rightarrow$ Wisely choose the starting point.

Choice of $(y_k)_{k\in\mathbb{N}}$

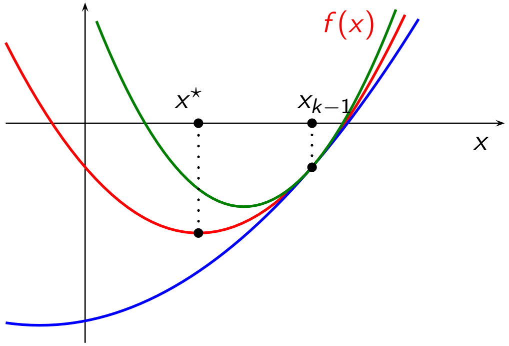
Extrapolation

$$y_k = x_k + \beta_k(x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1-\alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}.$$

- This update is identical to Nesterov's accelerated gradient descent or FISTA.
- Unfortunately, the literature does not provide any simple geometric explanation why it yields an acceleration...
- The construction is purely theoretical by using a mechanism introduced by Nesterov, called **“estimate sequence”**.

How does “acceleration” work?

If f is μ-strongly convex and ∇f is L-Lipschitz continuous



- $f(x) \leq f(x_{k-1}) + \nabla f(x_{k-1})^\top (x - x_{k-1}) + \frac{L}{2}\|x - x_{k-1}\|_2^2$;
- $f(x) \geq f(x_{k-1}) + \nabla f(x_{k-1})^\top (x - x_{k-1}) + \frac{\mu}{2}\|x - x_{k-1}\|_2^2$;

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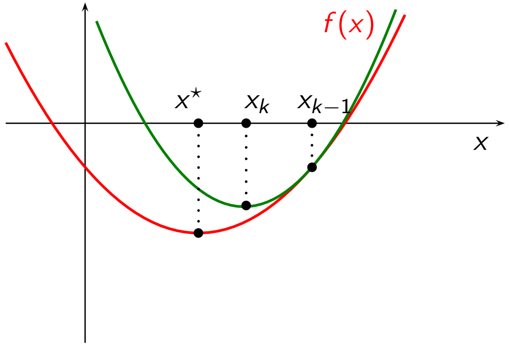

- $f(x) \leq f(x_{k-1}) + \nabla f(x_{k-1})^\top (x - x_{k-1}) + \frac{L}{2}\|x - x_{k-1}\|_2^2$;
- $\boxed{x_k = x_{k-1} - \frac{1}{L}\nabla f(x_{k-1}) \text{ (gradient descent step).}}$

How does “acceleration” work?

Definition of estimate sequence [Nesterov].

A pair of sequences $(\varphi_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$, with $\varphi_k : \mathbb{R}^p \to \mathbb{R}$ and $\lambda_k \geq 0$, is called an **estimate sequence** of function F if

- $\lambda_k \to 0$;
- $\varphi_k(x) \leq (1-\lambda_k)F(x) + \lambda_k\varphi_0(x)$, for any k, x;
- There exists a sequence $(x_k)_{k\geq 0}$ such that

$$F(x_k) \leq \varphi_k^\star \overset{\triangle}{=} \min_{x\in\mathbb{R}^p} \varphi_k(x).$$

Remarks

- φ_k is neither an upper-bound, nor a lower-bound;
- Finding the right estimate sequence is often nontrivial.

Convergence of outer-loop algorithm

Analysis for μ-strongly convex objective functions

Choose $\alpha_0 = \sqrt{q}$ with $q = \mu/(\mu+\kappa)$ and

$$\epsilon_k = \frac{2}{9}(F(x_0) - F^*)(1-\rho)^k \quad \text{with} \quad \rho < \sqrt{q}.$$

Then, the algorithm generates iterates $(x_k)_{k\geq 0}$ such that

$$F(x_k) - F^* \leq C(1-\rho)^{k+1}(F(x_0) - F^*) \quad \text{with} \quad C = \frac{8}{(\sqrt{q}-\rho)^2}.$$

In practice

- Choice of ρ can safely be set to $\rho = 0.9\sqrt{q}$.
- Choice of $(\varepsilon_k)_{k\geq 0}$ typically follows from a duality gap at x_0. When F is non-negative, we can set $\varepsilon_k = (2/9)F(x_0)(1-\rho)^k$.

Convergence of outer-loop algorithm

Analysis for non-strongly convex objective functions, $\mu = 0$

Choose $\alpha_0 = (\sqrt{5} - 1)/2$ and

$$\epsilon_k = \frac{2(F(x_0) - F^*)}{9(k+2)^{4+\eta}} \quad \text{with } \eta > 0.$$

Then, the meta-algorithm generates iterates $(x_k)_{k \geq 0}$ such that

$$F(x_k) - F^* \leq \frac{8}{(k+2)^2} \left(\left(1 + \frac{2}{\eta}\right)^2 (F(x_0) - F^*) + \frac{\kappa}{2} \|x_0 - x^*\|^2 \right). \tag{2}$$

In practice

- Choice of η can be set to $\eta = 0.1$.

How many iterates of $\mathcal{M}$ do we need to obtain x_k?

Control of inner-loop complexity

For minimizing G_k, consider a method $\mathcal{M}$ generating iterates $(z_t)_{t\geq 0}$ with linear convergence rate

$$G_k(z_t) - G_k^\star \leq A(1-\tau_{\mathcal{M}})^t (G_k(z_0) - G_k^\star).$$

Then by choosing $z_0 = x_{k-1}$, the precision ε_k is reached with at most

- A constant number of iterations $T_{\mathcal{M}}$ when $\mu > 0$;
- A logarithmic increasing number of iterations $T_{\mathcal{M}} \log(k+2)$ when $\mu = 0$.

where $T_{\mathcal{M}} = \tilde{O}(1/\tau_{\mathcal{M}})$ is independent of k.

Global computational complexity

Analysis for μ-strongly convex objective functions

The global convergence rate of the accelerated algorithm $\mathcal{A}$ is

$$F_s - F^\star \leq C \left(1 - \frac{\rho}{T_{\mathcal{M}}}\right)^s (F(x_0) - F^*). \quad (3)$$

where F_s is the objective function value obtained after performing $s = kT_{\mathcal{M}}$ iterations of the method $\mathcal{M}$. As a result,

$$\tau_{\mathcal{A},F} = \frac{\rho}{T_{\mathcal{M}}} = \tilde{O}(\tau_{\mathcal{M}}\sqrt{\mu}/\sqrt{\mu+\kappa}),$$

where $\tau_{\mathcal{M}}$ typically depends on κ (the greater, the faster).

$$\boxed{\kappa \text{ will be chosen to maximize the ratio } \tau_{\mathcal{M}}/\sqrt{\mu+\kappa}.}$$

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where $\tau_\mathcal{M}$ typically depends on κ (the greater, the faster).

$$\boxed{\text{e.g., } \kappa = L - 2\mu \text{ when } \tau_\mathcal{M} = \tfrac{\mu+\kappa}{L+\kappa} \Rightarrow \tau_\mathcal{A} = \tilde{O}\left(\sqrt{\tfrac{\mu}{L}}\right).}$$

Global computational complexity

Analysis for non-strongly convex objective functions

The global convergence rate of the accelerated algorithm $\mathcal{A}$ is

$$F_s - F^* \leq \frac{8 T_{\mathcal{M}}^2 \log^2(s)}{s^2} \left(\left(1 + \frac{2}{\eta}\right)^2 (F(x_0) - F^*) + \frac{\kappa}{2}\|x_0 - x^*\|^2 \right) .$$

If $\mathcal{M}$ is a first-order method, this rate is **near-optimal**, up to a logarithmic factor, when compared to the optimal rate $O(1/s^2)$, which may be the price to pay for using a generic acceleration scheme.

$$\boxed{\kappa \text{ will be chosen to maximize the ratio } \tau_{\mathcal{M}}/\sqrt{L+\kappa}}$$

Applications

Expected computational complexity in the regime $n \leq L/\mu$ when $\mu > 0$,

	$\mu > 0$	$\mu = 0$	Catalyst $\mu > 0$	Cat. $\mu = 0$
FG	$O\left(n\left(\frac{L}{\mu}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(n\frac{L}{\varepsilon}\right)$	$\tilde{O}\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	$\tilde{O}\left(n\frac{L}{\sqrt{\varepsilon}}\right)$
SAG	$O\left(\frac{L}{\mu}\log\left(\frac{1}{\varepsilon}\right)\right)$		$\tilde{O}\left(\sqrt{\frac{nL}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	
SAGA				
Finito/MISO		NA		
SDCA				
SVRG	$O\left(\frac{L'}{\mu}\log\left(\frac{1}{\varepsilon}\right)\right)$		$\tilde{O}\left(\sqrt{\frac{nL'}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	
Acc-FG	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(n\frac{L}{\sqrt{\varepsilon}}\right)$	no acceleration	
Acc-SDCA	$\tilde{O}\left(\sqrt{\frac{nL}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	NA		

Experiments with MISO/SAG/SAGA

ℓ_2-logistic regression formulation

Given some data (y_i, z_i), with y_i in $\{-1, +1\}$ and z_i in $\mathbb{R}^p$, minimize

$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \log(1 + e^{-y_i x^\top z_i}) + \frac{\mu}{2} \|x\|_2^2,$$

μ is the regularization parameter and the strong convexity modulus.

Datasets

name	rcv1	real-sim	covtype	ocr	alpha
n	781 265	72 309	581 012	2 500 000	250 000
p	47 152	20 958	54	1 155	500

Experiments with MISO/SAG/SAGA

The complexity analysis is not just a theoretical exercise since it provides the values of $\kappa, \varepsilon_k, \beta_k$, which are required in concrete implementations.

Here, **theoretical values match practical ones**.

Restarting

The theory tells us to restart $\mathcal{M}$ with x_{k-1}. For SDCA/Finito/MISO, the theory tells us to use instead $x_{k-1} + \frac{\kappa}{\mu+\kappa}(y_{k-1} - y_{k-2})$. We also tried this as a heuristic for SAG and SAGA.

One-pass heuristic

constrain $\mathcal{M}$ to always perform at most n iterations in inner loop; we call this variant AMISO2 for MISO, whereas AMISO1 refers to the regular "vanilla" accelerated variant; idem to accelerate SAG and SAGA.

Experiments without strong convexity, $\mu = 0$

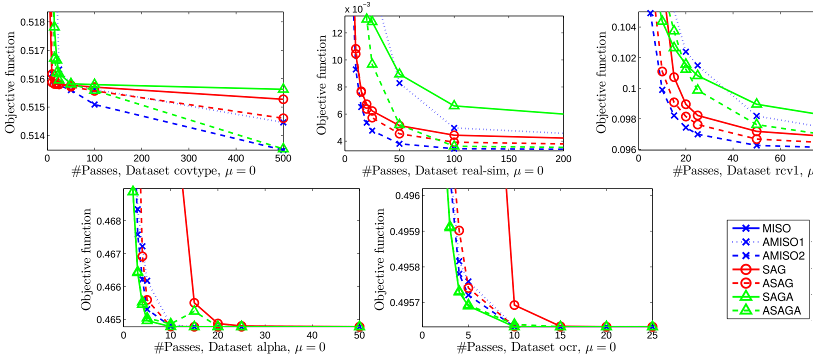


Figure: Objective function value for different number of passes on data.

Conclusions

- SAG, SAGA are accelerated when they do not perform well already;
- AMISO2 $\geq$ AMISO1 (vanilla), MISO does not apply.

Experiments without strong convexity, $\mu = 10^{-1}/n$

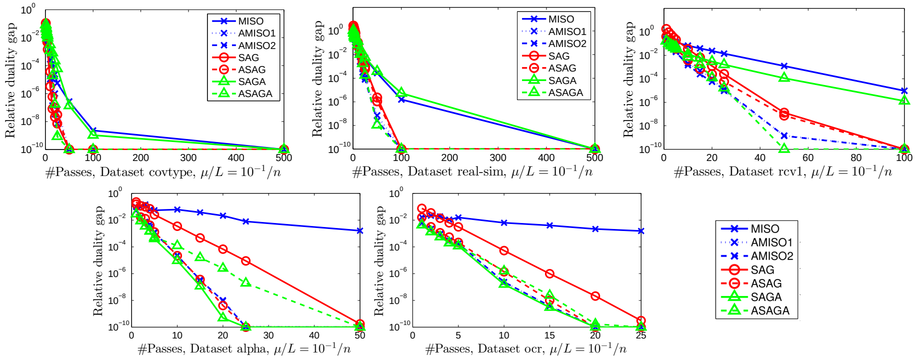


Figure: Relative duality gap (log-scale) for different number of passes on data.

Conclusions

- SAG, SAGA are not always accelerated, but often.
- AMISO2,AMISO1 $\gg$ MISO.

Experiments without strong convexity, $\mu = 10^{-3}/n$

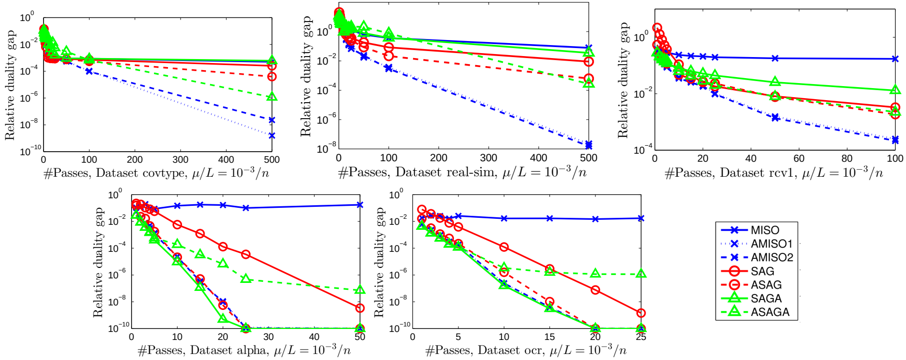


Figure: Relative duality gap (log-scale) for different number of passes on data.

Conclusions

- same conclusions as $\mu = 10^{-1}/n$;
- μ is so small that (unaccelerated) MISO becomes numerically unstable.

General conclusions about Catalyst

Summary: lots of nice features

- Simple acceleration scheme with broad application range.
- Recover near-optimal rates for known algorithms.
- Effortless implementation.

... but also lots of unsolved problems

- Acceleration occurs when $n \leq L/\mu$; otherwise, the "unaccelerated" complexity $O(n \log(1/\varepsilon))$ seems unbeatable.
- μ is an estimate of the true strong convexity parameter $\mu' \geq \mu$.
- μ is the global strong convexity parameter, not a local one $\mu^\star \geq \mu$.
- When $n \leq L/\mu$, but $n \geq L/(\mu'$ or $\mu^\star)$, a method $\mathcal{M}$ that adapts to the unknown strong convexity may be impossible to accelerate.
- The optimal restart for $\mathcal{M}$ is not yet fully understood.

Happy birthday!

Catalyst, the algorithm

Algorithm 1 Catalyst

```
input  initial estimate $x_0 \in \mathbb{R}^p$, parameters $\kappa$ and $\alpha_0$, sequence $(\varepsilon_k)_{k\geq 0}$,
    optimization method $\mathcal{M}$; initialize $q = \mu/(\mu+\kappa)$ and $y_0 = x_0$;
1: while the desired stopping criterion is not satisfied do
2:     Find an approx. solution $x_k$ using $\mathcal{M}$ s.t. $G_k(x_k) - G_k^\star \leq \varepsilon_k$
           $$x_k \approx \arg\min_{x\in\mathbb{R}^p} \left\{ G_t(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|^2 \right\}$$
3:     Compute $\alpha_k \in (0,1)$ from equation $\alpha_k^2 = (1-\alpha_k)\alpha_{k-1}^2 + q\alpha_k$;
4:     Compute
           $$y_k = x_k + \beta_k(x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1-\alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}.$$
5: end while
output  $x_k$ (final estimate).
```

Ideas of the proofs

Main theorem

Let us denote

$$\lambda_k = \prod_{i=0}^{k-1}(1-\alpha_i), \tag{4}$$

where the α_i's are defined in Catalyst. Then, the sequence $(x_k)_{k\geq 0}$ satisfies

$$F(x_k) - F^* \leq \lambda_k \left(\sqrt{S_k} + 2\sum_{i=1}^{k}\sqrt{\frac{\epsilon_i}{\lambda_i}}\right)^2, \tag{5}$$

where $F^\star$ is the minimum value of F and

$$S_k = F(x_0) - F^* + \frac{\gamma_0}{2}\|x_0 - x^*\|^2 + \sum_{i=1}^{k}\frac{\epsilon_i}{\lambda_i} \quad \text{where} \quad \gamma_0 = \frac{\alpha_0\left((\kappa+\mu)\alpha_0 - \mu\right)}{1-\alpha_0}, \tag{6}$$

where $x^\star$ is a minimizer of F.

Ideas of the proofs

Then, the theorem will be used with the following lemma to control the convergence rate of the sequence $(\lambda_k)_{k\geq 0}$, whose definition follows the classical use of estimate sequences. This will provide us convergence rates both for the strongly convex and non-strongly convex cases.

Lemma 2.2.4 from Nesterov [2004]

If in the quantity γ_0 defined in (6) satisfies $\gamma_0 \geq \mu$, then the sequence $(\lambda_k)_{k\geq 0}$ from (4) satisfies

$$\lambda_k \leq \min\left\{ (1-\sqrt{q})^k, \frac{4}{\left(2 + k\sqrt{\frac{\gamma_0}{\kappa+\mu}}\right)^2} \right\}, \tag{7}$$

where $q \overset{\triangle}{=} \mu/(\mu+\kappa)$.

Ideas of proofs

Step 1: build an approximate estimate sequence

- Remember that in general, we build φ_k from φ_{k-1} as follows

$$\varphi_k(x) \triangleq (1-\alpha_k)\varphi_{k-1}(x) + \alpha_k d_k(x),$$

  where d_k is a lower bound.

- Here, a natural lower bound would be

$$F(x) \geq d_k(x) \triangleq F(x_k^*) + \langle \kappa(y_{k-1} - x_k^*), x - x_k^* \rangle + \frac{\mu}{2}\|x - x_k^*\|^2,$$

  where $x_k^\star \triangleq \arg\min_{x\in\mathbb{R}^p} \left\{ G_k(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2 \right\}$.

- But $x_k^\star$ is unknown! Then, use instead $d_k'(x)$ defined as

$$d_k'(x) \triangleq F(x_k) + \langle \kappa(y_{k-1} - x_k), x - x_k \rangle + \frac{\mu}{2}\|x - x_k\|^2.$$

Ideas of proofs

Step 2: Relax the condition $F(x_k) \leq \varphi_k^\star$.

- We can show that Catalyst generates iterates $(x_k)_{k\geq 0}$ such that

$$F(x_k) \leq \phi_k^* + \xi_k,$$

where the sequence $(\xi_k)_{k\geq 0}$ is defined by $\xi_0 = 0$ and

$$\xi_k = (1-\alpha_{k-1})(\xi_{k-1} + \varepsilon_k - (\kappa+\mu)\langle x_k - x_k^*, x_{k-1} - x_k\rangle).$$

- The sequences $(\alpha_k)_{k\geq 0}$ and $(y_k)_{k\geq 0}$ are chosen in such a way that all the terms involving $y_{k-1} - x_k$ are cancelled.
- We will control later the quantity $x_k - x_k^*$ by strong convexity of G_k:

$$\frac{\kappa+\mu}{2}\|x_k - x_k^*\|_2^2 \leq G_k(x_k) - G_k^\star \leq \varepsilon_k.$$

Ideas of proofs

Step 3: Control how this errors sum up together.

- Do cumbersome calculus.

Catalyst in practice

General strategy and application to randomized algorithms

Calculating the iteration-complexity decomposes into three steps:

1. When F is μ-strongly convex, find κ that maximizes the ratio $\tau_{\mathcal{M},G_k}/\sqrt{\mu+\kappa}$ for algorithm $\mathcal{M}$. When F is non-strongly convex, maximize instead the ratio $\tau_{\mathcal{M},G_k}/\sqrt{L+\kappa}$.
2. Compute the upper-bound of the number of outer iterations k_{out} using the theorems.
3. Compute the upper-bound of the expected number of inner iterations
$$\max_{k=1,\ldots,k_{\text{out}}} \mathbb{E}[T_{\mathcal{M},G_k}(\varepsilon_k)] \leq k_{\text{in}},$$

Then, the expected iteration-complexity denoted Comp. is given by

$$\text{Comp} \leq k_{\text{in}} \times k_{\text{out}} \ .$$

Applications

Deterministic and Randomized Incremental Gradient methods

- Stochastic Average Gradient (SAG and SAGA) [Schmidt et al., 2013, Defazio et al., 2014a];
- Finito and MISO [Mairal, 2015, Defazio et al., 2014b];
- Semi-Stochastic/Mixed Gradient [Konečnỳ et al., 2014, Zhang et al., 2013];
- Stochastic Dual coordinate Ascent [Shalev-Shwartz and Zhang, 2012];
- Stochastic Variance Reduced Gradient [Xiao and Zhang, 2014].

But also, randomized coordinate descent methods, and more generally first-order methods with linear convergence rates.

Appendix on proximal MISO

Original motivation

Given some data, learn some model parameters x in $\mathbb{R}^p$ by minimizing

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right\},$$

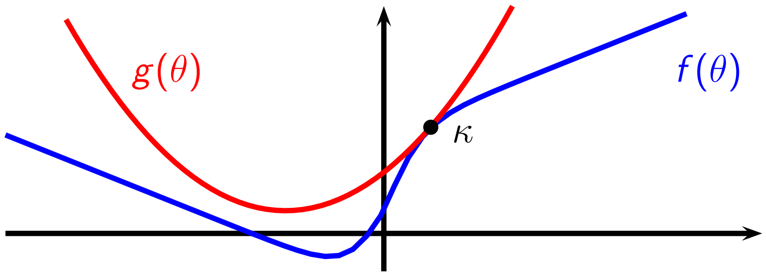
where each f_i may be **nonsmooth and nonconvex**.

The original MISO algorithm is an incremental extension of the **majorization-minimization** principle [Lange et al., 2000].

Paper

- J. Mairal. Incremental Majorization-Minimization Optimization with Application to Large-Scale Machine Learning. SIAM Journal on Optimization. 2015.
- J. Mairal. Optimization with First-Order Surrogate Functions. ICML. 2013.

Majorization-minimization principle



- Iteratively minimize locally tight upper bounds of the objective.
- The objective monotonically decreases.
- Under some assumptions, we get similar convergence rates as gradient-based approaches for convex problems.

Incremental optimization: MISO

Algorithm 2 Incremental scheme MISO

```
input $x_0 \in \mathbb{R}^p$; $T$ (number of iterations).
 1: Choose surrogates $g_i^0$ of $f_i$ near $x_0$ for all $i$;
 2: for $k = 1, \dots, K$ do
 3:     Randomly pick up one index $\hat{\imath}_k$ and choose a surrogate $g_{\hat{\imath}_k}^k$ of $f_{\hat{\imath}_k}$
        near $x_{k-1}$. Set $g_i^k \triangleq g_i^{k-1}$ for $i \neq \hat{\imath}_k$;
 4:     Update the solution:
            $$x_k \in \arg\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} g_i^k(x).$$
 5: end for
output $x_K$ (final estimate);
```

Incremental Optimization: MISO

Update rule with basic upper bounds

We want to minimize $\frac{1}{n}\sum_{i=1}^{n} f_i(x)$, where the f_i's are smooth.

$$x_k \leftarrow \operatorname*{arg\,min}_{x \in \mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} f_i(y_i^k) + \nabla f_i(y_i^k)^\top (x - y_i^k) + \frac{L}{2}\|x - y_i^k\|_2^2$$
$$= \frac{1}{n}\sum_{i=1}^{n} y_i^k - \frac{1}{Ln}\sum_{i=1}^{n} \nabla f_i(y_i^k).$$

At iteration k, randomly draw one index $\hat{\imath}_k$, and update $y_{\hat{\imath}_k}^k \leftarrow x_k$.

Remarks

- replace $(1/n)\sum_{i=1}^{n} y_i^k$ by x_{k-1} yields SAG [Schmidt et al., 2013].
- replace $(1/L)$ by $(1/\mu)$ for strongly convex problems is close to a variant of SDCA [Shalev-Shwartz and Zhang, 2012].

Incremental Optimization: MISOμ.

Update rule with lower bounds???

We want to minimize $\frac{1}{n}\sum_{i=1}^n f_i(x)$, where the f_i's are smooth.

$$x_k = \arg\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^n f_i(y_i^k) + \nabla f_i(y_i^k)^\top (x - y_i^k) + \frac{\mu}{2}\|x - y_i^k\|_2^2$$
$$= \frac{1}{n}\sum_{i=1}^n y_i^k - \frac{1}{\mu n}\sum_{i=1}^n \nabla f_i(y_i^k).$$

Remarks

- Requires strong convexity.
- Use a counter-intuitive minorization-minimization principle.
- Close to a variant of SDCA [Shalev-Shwartz and Zhang, 2012].
- Much faster than the basic MISO (faster rate).

Incremental Optimization: MISOμ.

In the first part of this presentation, what we have called MISO is the algorithm that uses $1/(\mu n)$ step-sizes (sorry for the confusion).
To minimize $F(x) \triangleq \frac{1}{n}\sum_{i=1}^{n} f_i(x)$, MISO$\mu$ has the following guarantees

Proposition [Mairal, 2015]

When the functions f_i are μ-strongly convex, differentiable with L-Lipschitz gradient, and non-negative, MISOμ satisfies

$$\mathbb{E}[F(x_k) - F^\star] \leq \left(1 - \frac{1}{3n}\right)^k nf^\star,$$

under the condition $n \geq 2L/\mu$.

Remarks

- When $n \leq 2L/\mu$, the algorithm may diverge;
- When μ is very small, numerical stability is an issue.
- The condition $f_i \geq 0$ does not really matter.

Proximal MISO [Lin, Mairal, and Harchaoui, 2015]

Main goals

- Remove the condition $n \leq 2L/\mu$;
- Allow a composite term ψ:

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},$$

Starting points

MISOμ is iteratively updating/minimizing a lower-bound of F

$$x_k \leftarrow \arg\min_{x \in \mathbb{R}^p} \left\{ D_k(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} d_i^k(x) \right\},$$

[Lin, Mairal, and Harchaoui, 2015].

Proximal MISO

Adding the proximal term

$$x_t \leftarrow \arg\min_{x \in \mathbb{R}^p} \left\{ D_k(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} d_i^k(x) + \psi(x) \right\},$$

Remove the condition $n \geq 2L/\mu$

For $i = \hat{\imath}_k$,

$$d_i^k(x) = (1-\delta) d_i^{k-1}(x) + \delta \left(f_i(x_{k-1}) + \langle \nabla f_i(x_{k-1}), x - x_{k-1} \rangle + \frac{\mu}{2} \|x - x_{k-1}\|^2 \right)$$

Remarks

- the original MISOμ uses $\delta = 1$. To get rid of the condition $n \geq 2L/\mu$, proximal MISO uses instead $\delta = \min\left(1, \frac{\mu n}{2(L-\mu)}\right)$.
- variant "5" of SDCA [Shalev-Shwartz and Zhang, 2012] is identical with another value $\delta = \frac{\mu n}{L + \mu n}$ in $(0, 1)$.

Proximal MISO

Convergence of MISO-Prox

Let $(x_k)_{k\geq 0}$ be obtained by MISO-Prox, then

$$\mathbb{E}[F(x_k)]-F^* \leq \frac{1}{\tau}(1-\tau)^{k+1}\left(F(x_0) - D_0(x_0)\right) \text{ with } \tau \geq \min\left\{\frac{\mu}{4L}, \frac{1}{2n}\right\}. \tag{8}$$

Furthermore, we also have fast convergence of the certificate

$$\mathbb{E}[F(x_k) - D_k(x_k)] \leq \frac{1}{\tau}(1-\tau)^k\left(F^* - D_0(x_0)\right).$$

Differences with SDCA

- The construction is **primal**. The proof of convergence and the algorithm do not use duality, while SDCA is a dual ascent technique.
- $D_k(x_k)$ is a lower-bound of $F^\star$; it plays the same role as the dual in SDCA, but is **easier to evaluate**.

Conclusions

- Relatively simple algorithm, with simple convergence proof, and simple optimality certificate.
- Catalyst not only accelerates it, but also **stabilizes** it numerically, with the parameter $\delta = 1$.
- Close to SDCA, but without duality.

References I

A. Agarwal and L. Bottou. A lower bound for the optimization of finite sums. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2015.

A. Beck and M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. *SIAM Journal on Imaging Sciences*, 2(1):183–202, 2009.

S. S. Chen, D. L. Donoho, and M. A. Saunders. Atomic decomposition by basis pursuit. *SIAM Journal on Scientific Computing*, 20:33–61, 1999.

P. L. Combettes and V. R. Wajs. Signal recovery by proximal forward-backward splitting. *SIAM Multiscale Modeling and Simulation*, 4(4):1168–1200, 2006.

I. Daubechies, M. Defrise, and C. De Mol. An iterative thresholding algorithm for linear inverse problems with a sparsity constraint. *Communications on Pure and Applied Mathematics*, 57(11):1413–1457, 2004.

A. Defazio, F. Bach, and S. Lacoste-Julien. SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives. In *Advances in Neural Information Processing Systems (NIPS)*, 2014a.

References II

A. J. Defazio, T. S. Caetano, and J. Domke. Finito: A faster, permutable incremental gradient method for big data problems. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2014b.

R. Frostig, R. Ge, S. M. Kakade, and A. Sidford. Un-regularizing: approximate proximal point algorithms for empirical risk minimization. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2015.

Dan Garber and Elad Hazan. Fast and simple pca via convex optimization. *arXiv preprint arXiv:1509.05647*, 2015.

O. Güler. New proximal point algorithms for convex minimization. *SIAM Journal on Optimization*, 2(4):649–664, 1992.

B. He and X. Yuan. An accelerated inexact proximal point algorithm for convex minimization. *Journal of Optimization Theory and Applications*, 154(2): 536–548, 2012.

Jakub Konečnỳ, Jie Liu, Peter Richtárik, and Martin Takáč. ms2gd: Mini-batch semi-stochastic gradient descent in the proximal setting. *arXiv preprint arXiv:1410.4744*, 2014.

References III

Guanghui Lan. An optimal randomized incremental gradient method. *arXiv preprint arXiv:1507.02000*, 2015.

K. Lange, D. R. Hunter, and I. Yang. Optimization transfer using surrogate objective functions. *Journal of Computational and Graphical Statistics*, 9(1): 1–20, 2000.

H. Lin, J. Mairal, and Z. Harchaoui. A universal catalyst for first-order optimization. In *Advances in Neural Information Processing Systems*, 2015.

J. Mairal. Incremental majorization-minimization optimization with application to large-scale machine learning. *SIAM Journal on Optimization*, 25(2): 829–855, 2015.

Y. Nesterov. *Introductory lectures on convex optimization: a basic course*. Kluwer Academic Publishers, 2004.

Y. Nesterov. Gradient methods for minimizing composite objective function. *Mathematical Programming*, 140(1):125–161, 2013.

Yurii Nesterov. A method for unconstrained convex minimization problem with the rate of convergence o (1/k2). In *Doklady an SSSR*, volume 269, pages 543–547, 1983.

References IV

R. D. Nowak and M. A. T. Figueiredo. Fast wavelet-based image deconvolution using the EM algorithm. In *Conference Record of the Thirty-Fifth Asilomar Conference on Signals, Systems and Computers.*, 2001.

S. Salzo and S. Villa. Inexact and accelerated proximal point algorithms. *Journal of Convex Analysis*, 19(4):1167–1192, 2012.

M. Schmidt, N. Le Roux, and F. Bach. Convergence rates of inexact proximal-gradient methods for convex optimization. In *Advances in Neural Information Processing Systems (NIPS)*, 2011.

M. Schmidt, N. Le Roux, and F. Bach. Minimizing finite sums with the stochastic average gradient. *arXiv:1309.2388*, 2013.

S. Shalev-Shwartz and T. Zhang. Proximal stochastic dual coordinate ascent. *arXiv:1211.2717*, 2012.

S. Shalev-Shwartz and T. Zhang. Accelerated proximal stochastic dual coordinate ascent for regularized loss minimization. *Mathematical Programming*, pages 1–41, 2014.

R. Tibshirani. Regression shrinkage and selection via the Lasso. *Journal of the Royal Statistical Society Series B*, 58(1):267–288, 1996.

References V

S.J. Wright, R.D. Nowak, and M.A.T. Figueiredo. Sparse reconstruction by separable approximation. *IEEE Transactions on Signal Processing*, 57(7): 2479–2493, 2009.

L. Xiao and T. Zhang. A proximal stochastic gradient method with progressive variance reduction. *SIAM Journal on Optimization*, 24(4):2057–2075, 2014.

Lijun Zhang, Mehrdad Mahdavi, and Rong Jin. Linear convergence with condition number independent access of full gradients. In *Advances in Neural Information Processing Systems*, pages 980–988, 2013.

Y. Zhang and L. Xiao. Stochastic primal-dual coordinate method for regularized empirical risk minimization. In *Proceedings of the International Conference on Machine Learning (ICML)*, 2015.