Quadratic transformations: feasibility and convexity

B. Polyak

with P. Shcherbakov, E. Gryazina

Institute for Control Science and
SkolTech Center for Energy Systems, Moscow

Workshop "Optimization Without Borders",
February 7 - 12, 2016, Les Houches, France

Quadratic maps

Have $f\colon\ \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$f(x) = \big(f_1(x), \ldots, f_m(x)\big)^\top, \qquad f_i(x) = (A_i x, x) + 2(b_i, x), \quad i = 1, \ldots, m \le n$$

$$A_i = A_i^\top \in \mathbb{R}^{n\times n}, \quad b_i \in \mathbb{R}^n,$$

or $f\colon\ \mathbb{C}^n \to \mathbb{R}^m$ of the form

$$f(x) = \big(f_1(x), \ldots, f_m(x)\big)^\top, \qquad f_i(x) = (A_i x, x) + (b_i^*, x) + (b_i, x^*), \quad i = 1, \ldots, m \le n$$

$$A_i = A_i^* \in \mathbb{C}^{n\times n}, \quad b_i \in \mathbb{C}^n.$$

Image sets in $\mathbb{R}^m$:

$$F = \{f(x)\colon\ x \in \mathbb{R}^n\}$$

or

$$F = \{f(x)\colon\ x \in \mathbb{C}^n\}$$

and

$$F_r = \{f(x)\colon\ x \in \mathbb{R}^n,\ \|x\| \le r\}$$

Problems

Convexity/nonconvexity Is F (or F_r) convex or not?
If F is convex, all related optimization problems are "good".

Our approach: check convexity/nonconvexity for individual transformation.

Membership Oracle (= Feasibility problem). Given $y \in \mathbb{R}^m$, check if $y \in F$
— Solvability of system of quadratic equations.

Applications — Optimization

- General quadratic programming:

$$\min f_0(x)$$

$$\text{s.t.} \qquad f_i(x) \le 0,\ i \in I, \quad f_i(x) = 0,\ i \in J$$

If F is convex + regularity conditions $\Longrightarrow$ duality theory holds. Fradkov-Yakubovich, Vestnik LGU, 1973; Fradkov, Siberian Math. J., 1973

- Boolean programming

$$x_i = \{-1, +1\} \iff x_i^2 = 1$$

- Convex relaxation for F can be easily written: When is it tight? Shor 1986, Nesterov, Beck, Teboulle ...

- Pareto optimization: objective functions are linear/quadratic.

Applications — Control

- S-theorem: When do the two quadratic inequalities imply the third one?

  Originally — absolute stability. Lurie-Postnikov, 1944, Aizerman-Gantmacher, 1963; solution — Yakubovich 1971

  Now S-theorem plays significant role in LMI techniques, in robustness analysis, in quadratically constrained linear-quadratic theory.

- Structured singular value (μ-analysis and synthesis.) Doyle, 1982, Packard-Doyle, Automatica, 1993. Complex μ, real μ — different properties due to convexity/nonconvexity of quadratic images.

Applications — Physics

- Quantum systems. Detectability depends on convexity properties of quadratic images.

- Power flow (PF) — feasibility of the desired regime; Optimal power flow (OPF): Power network with n buses connected to loads or generators.

  Variables: Active and reactive powers generated at buses and complex voltages

  Constraints: Active and reactive loads

  Cost functions: Quadratic functions of variables

  Result: Zero duality gap under some conditions (J. Lavaei, S.H. Low, 2012)

Convexity vs Nonconvexity

- Simplest example:

$$\min(Ax, x) \quad \text{s.t.} \quad \|x\| = 1$$

This problem is nonconvex! However the closed-form solution is straightforward:

$$x^* = e_1,$$

where e_1 is the eigenvector associated with the minimal eigenvalue of A

- Titles of papers:

— *Hidden convexity in some nonconvex quadratically constrained quadratic programming* [Ben-Tal, Teboulle, 1996]

— *Permanently going back and forth between the "quadratic world" and the "convexity world" in optimization* [J.-B. Hiriart-Urruty, M. Tork, 2002]

- When the images of quadratic maps are convex?

Simple Illustrations

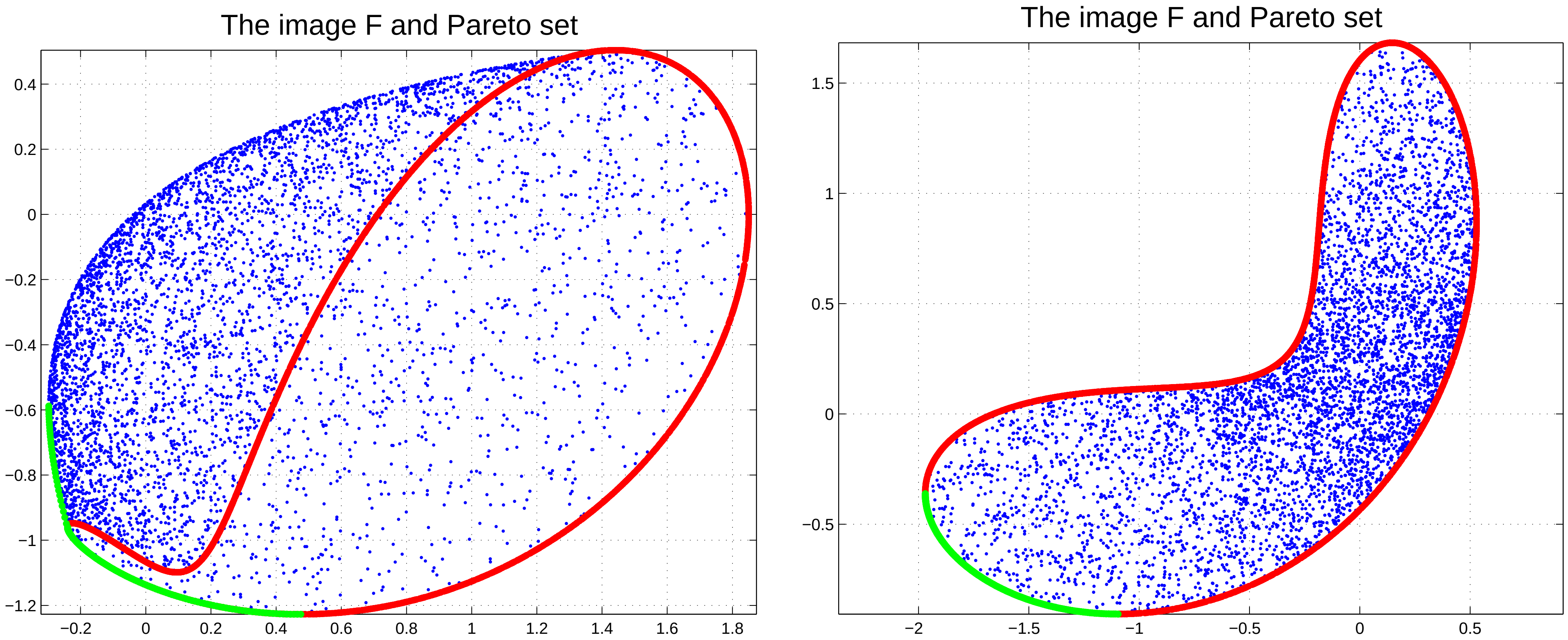


Figure 1: $n = m = 2$: Image of unit circle (red) and of unit disk (blue), Pareto boundary (green)

Known Facts (Homogeneous forms)

Complex case — [Toeplitz, 1918; Hausdorff, 1919]: F_1 is convex for $m = 2$ (numerical range); [Au-Yeng, Tsing 1983] same for $m = 3$.

Real case:

- $m = 2, \quad \Longrightarrow \quad F$ is convex [Dines, 1941]
- $m = 2,\ n \geq 3, \quad \Longrightarrow \quad F_1$ is convex [Brickman, 1961]
- $m = 3,\ n \geq 3; \sum c_i A_i \succ 0 \implies F$ is convex [Calabi, 1982; Polyak, 1998]
- m is arbitrary, A_i commute $\implies F$ is convex [Fradkov, 1973].

Known Facts (Nonhomogeneous functions)

Complex case — F is convex for $m = 2$.

Real case:

- $m = 2, c_1 A_1 + c_2 A_2 \succ 0 \implies F$ is convex [Polyak, 1998]
- m is arbitrary, A_i have nonpositive off-diagonal entries, $b_i \leq 0 \implies$ Pareto set of F is convex ($F + \mathbb{R}^m_+$ is convex) [Zhang, Kim-Kojima, Jeyakumar a.o.]
- m is arbitrary, b_i are linearly independent $\implies$ F_r is convex for r small enough [Polyak, 2001] — "Small ball" theorem.

Convex Hull (i)

The idea of convex relaxations for quadratic problems goes back to [Shor, 1986]; also see [Nesterov 1998], [Zhang 2000], [Beck and Teboulle, 2005].

Recent survey:

Luo, Ma, So, Ye, Zhang, *Semidefinite relaxation of quadratic optimization problems*, IEEE Sig. Proc. Magazine, 2010.

Two typical results:

Lemma 1. *For* $b_i = 0$ *have*

$$Conv(F_r) = \{\mathcal{A}(X) \colon \; X \succcurlyeq 0, \; TrX \le r^2\},$$

where $X = X^\top \in \mathbb{R}^{n \times n}$, $\mathcal{A}(X) = \big(\langle A_1, X\rangle, \ldots, \langle A_m, X\rangle\big)^\top$, *and* $\langle A, X\rangle = TrAX$.

Convex Hull (ii)

Lemma 2. *In the general case* $(b_i \neq 0)$ *have*

$$G = Conv(F) = \{\mathcal{H}(X)\colon\ X \succcurlyeq 0,\ \ X_{n+1,n+1} = 1\}$$

where $X = X^\top \in \mathbb{R}^{(n+1)\times(n+1)}$, $\quad \mathcal{H}(X) = \big(\langle H_1, X\rangle, \ldots, \langle H_m, X\rangle\big)^\top$,

and $H_i = \begin{bmatrix} A_i & b_i \\ b_i^T & 0 \end{bmatrix}$.

Idea of proof: $(A_i x, x) = \langle A_i, xx^\top\rangle = \langle A_i, X\rangle,\ X \succcurlyeq 0,\ \text{rank}X = 1,\ \ TrX = \|x\|^2.$

For $z = (x;t) \in \mathbb{R}^{n+1}$ have $(H_i z, z) = (A_i x, x) + 2(b_i, x)t = f_i(x)$ if $t = 1$.

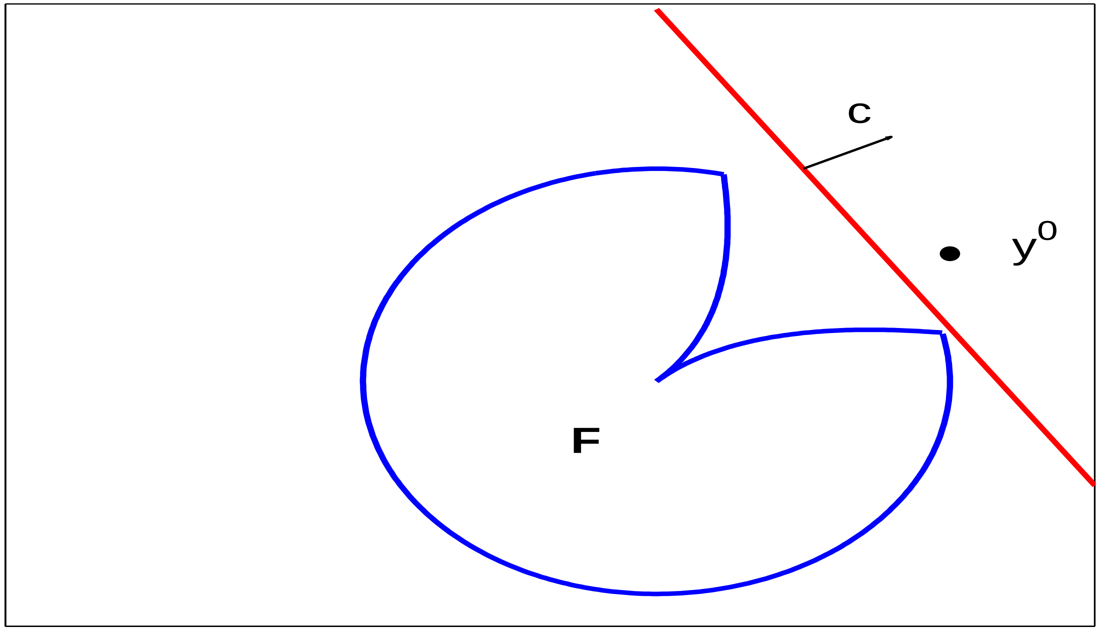
Convexity/nonconvexity certificates

We focus on real nonhomogeneous case. Our goal is to provide convexity/nonconvexity certificates for image of the individual quadratic map and feasibility/infeasibility certificate for the map and the point y. Notation:

$$c \in \mathbb{R}^m, y \in \mathbb{R}^m, A(c) = \sum c_i A_i, b(c) = \sum c_i b_i, y(c) = \sum c_i y_i$$

$$H_i = \begin{bmatrix} A_i & b_i \\ b_i^T & 0 \end{bmatrix}, \quad H(c) = \begin{bmatrix} A(c) & b(c) \\ b(c)^T & 0 \end{bmatrix}.$$

Separating F and y



Strict separation is possible if $\min_{f\in F}(c,f) = \min_x[(A(c)x,x) + 2(b(c),x)] > (y,c)$

for some c. This is equivalent to LMI $\begin{bmatrix} A(c) & b(c) \\ b(c)^T & -1-(y,c) \end{bmatrix} \succcurlyeq 0.$

Nonconvexity Certificate NC1

If LMI

$$A(c) \succcurlyeq 0$$

has no solutions in $c \neq 0$ and $F \neq \mathbb{R}^m$, then F is nonconvex.

Indeed a convex set either has a supporting hyperplane or coincides with the entire space.

Example. $\operatorname{tr} A_i = 0, A_i$ are linearly independent. Then either $F = \mathbb{R}^m$, or F is nonconvex.

Infeasibility Certificate NF1

If LMI in c

$$\begin{bmatrix} A(c) & b(c) \\ b(c)^\top & -1 - y(c) \end{bmatrix} \succcurlyeq 0$$

is solvable, then equation $f(x) = y$ has no solution.

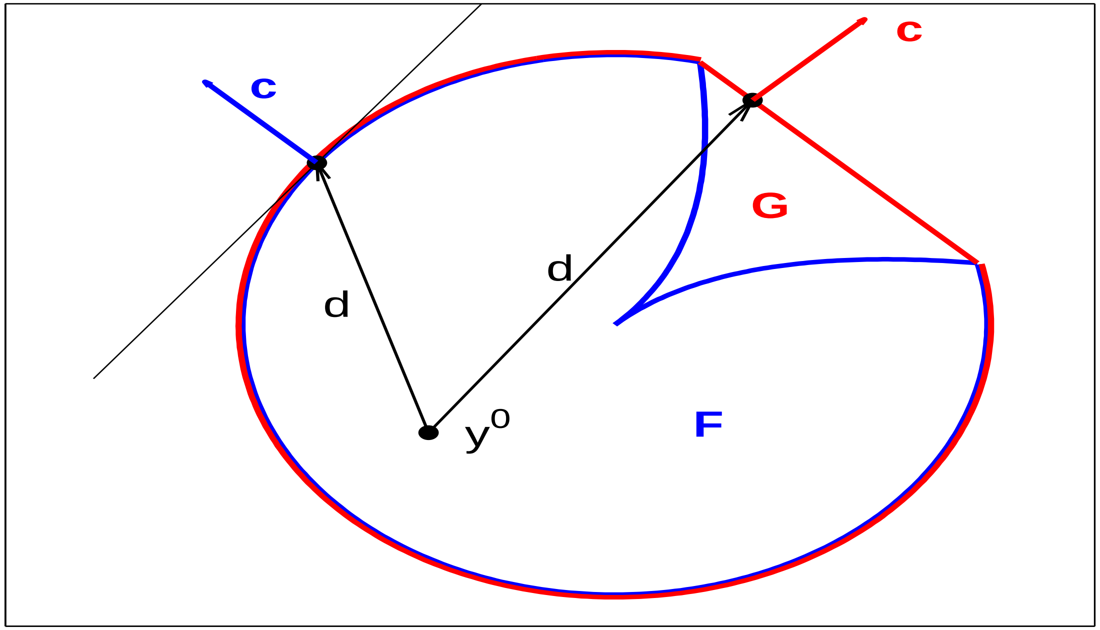
Remark. If F is convex, this is necessary and sufficient condition.

Nonconvexity Certificate NC1

Let $m \geq 3$, $n \geq 3$, *and let for some* c*, the matrix* $A(c)$ *has simple zero eigenvalue and eigenvector* e *such that* $A(c)e = 0, (b(c), e) = 0$. *Denote* $d = -A(c)^{+}b(c)$, $x_\alpha = \alpha e + d$, $f^\alpha = f(x^\alpha) = f^0 + f^1\alpha + f^2\alpha^2$. *If* $|(f^1, f^2)| < \|f^1\| \cdot \|f^2\|$, *then* F *is nonconvex.*

Proof: $\operatorname{Arg}\min_{f \in F}(c, f) = f(x^\alpha)$, where $f(x^\alpha)$ is 2-D parabola, which is nondegenerate due to the assumptions. Hence, the intersection of F and the supporting hyperplane $(c, f) = Const$ is nonconvex

How to find such c?



Given $y^0 \in F$ and direction d, to find boundary oracle for $y^0 + td \in Conv(F)$ solve

$$\min(t + (c, y^0))$$

$$\begin{bmatrix} \sum A(c) & \sum b(c) \\ \sum b(c)^T & t \end{bmatrix} \succeq 0, (c, d) = -1.$$

For d^k random find "flat" part of the boundary w.p.1.

Feasibility Certificate F1

Suppose $y \in Conv(F)$*. Solve SDP in* $c, \lambda \geq 0$ *with parameter* r^2

$$\min(c, y)$$

$$\begin{bmatrix} A(c) + \lambda I & b(c) \\ b(c)^\top & (c, y) - \lambda r^2 \end{bmatrix} \succeq 0$$

Assume that the minimal eigenvalue of the matrix $A(c^*) + \lambda^* I$ *is positive. Calculate* $p(r) = ||(A(c^*) + \lambda^* I)^{-1} b(c^*)||$ *and find minimal root of* $p(r) = r$*. If it exists,* $y \in F$*.*

Indeed, for this $r > 0$ the point $y \in \partial Conv(F_r)$ and it is the unique minimizer of (c, f) on this set.

Hence, the supporting hyperplane has the unique intersection point both with F_r and its convex hull.

Convexity certificate

Suppose matrix B with columns $b_i, i = 1, \ldots, m$ is full-rank and its smallest singular value is $\sigma > 0$. Denote $L = \sqrt{\sum_i ||A_i||^2}, R = \sigma/(2L)$. Then F_r is strictly convex for any $0 < r < R$.

This is "small ball" theorem, [Polyak 2001]. There are better estimates for R — [Dymarsky, 2016], [Xia, 2014].

If for some r in the previous test $p(r) < r$ and $r < R$, then $y \in F$.

Possible extensions

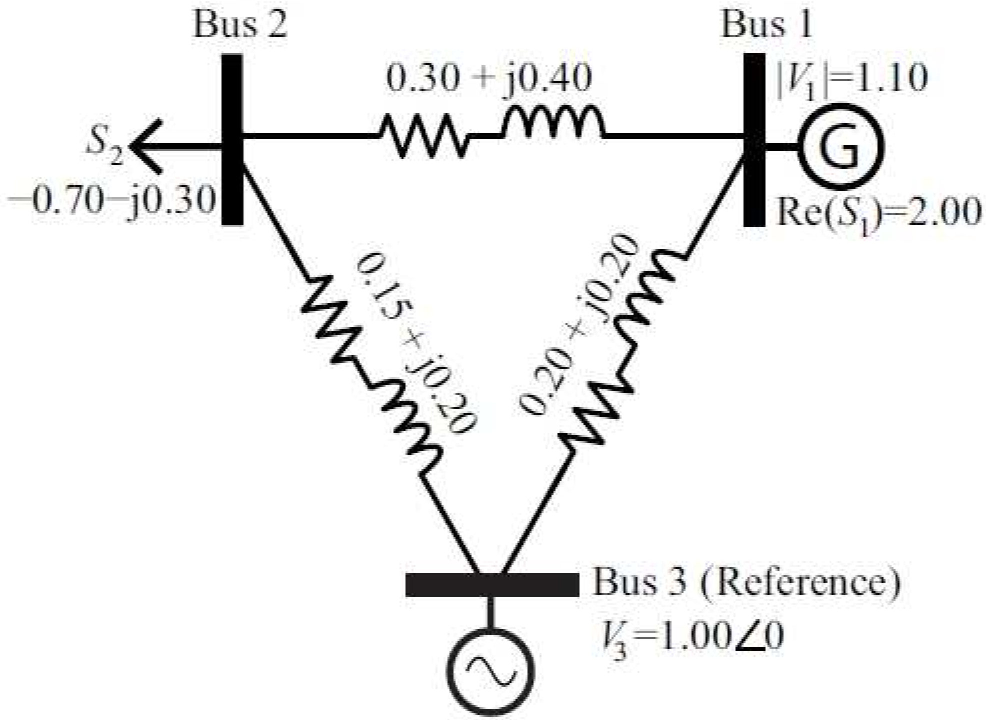
- Some of functions are linear

$$F = \{f(x) : Cx = d\}.$$

- Complex case (important for power systems).
- Homogenous case (e.g. nonconvexity certificate for F_r can be specified — intersection of supporting hyperplane and F_r is 2-D ellipse).

Example

3 buses (slack, PV, PQ), $n = m = 4$, borrowed from literature



Nonconvexity detected!

Other examples

Intensive numerical testing for checking convexity. For all examples were images were known to be nonconvex, nonconvexity has been detected. For random data nonconvexity is typical.

Future Work

- From images to optimization
- Algorithms for high dimensions
- Feasibility problems more deeply
- “The best” inner convex approximation of F
- Cutting off “convex parts” of F.