FINITE-DIFFERENCE APPROXIMATIONS AND OPTIMAL CONTROL OF THE SWEEPING PROCESS

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THE CONTROLLED SWEEPING PROCESS

is described by the dissipative differential inclusion

$$\begin{cases} -\dot{x}(t) & \in N\Big(x(t);C(t)\Big) \text{ a.e. } t \in [0,T] \\ x(0) & = x_0 \in C(0), \end{cases}$$

where $N(\cdot;\Omega)$ stands for the usual normal cone of convex analysis, and where $t \mapsto C(t)$ is a Lipschitzian set-valued mapping (moving set). Classical theory of the sweeping process establishes the existence and uniqueness of Lipschitzian solutions for a given moving set $C(t)$, and so doesn't allow any room for optimization. We suggest to control the sweeping set $C(t)$ by some forces and thus formulate and study new classes of optimal control problems for controlled sweeping process with various applications; in particular, to quasistatic elastoplasticity, magnetic hysteresis, social-economic modeling, etc.

OPTIMAL CONTROL PROBLEM

Given a terminal cost function φ and a running cost ℓ, consider the optimal control problem (P): minimize

$$J[x,u,b] = \varphi\Big(x(T)\Big) + \int_0^T \ell\Big(t, x(t), u(t), b(t), \dot{x}(t), \dot{u}(t), \dot{b}(t)\Big)\, dt$$

over the controlled sweeping dynamics governed by the so-called play-and-stop operator appearing, e.g., in hysteresis

$$\begin{cases} \dot{x}(t) \in -N\Big(x(t); C(t)\Big) + f\Big(x(t), b(t)\Big) \\ \text{for a.e. } \; t \in [0,T], \quad x(0) = x_0 \in C(0) \subset I\!R^n \\ \text{with } \; C(t) = C + u(t), \; C = \Big\{x \in I\!R^n \Big| \; \langle a_i, x\rangle \le 0, \; i = 1, \ldots, m\Big\} \\ \|u(t)\| = 1 \; \text{ for all } \; t \in [0,T] \end{cases}$$

where the trajectory $x(t)$ and control $u(t) = (u_1(t), \ldots, u_n(t)$, $b(t) = (b_1(t), \ldots, b_n(t))$ functions are absolutely continuous on the fixed interval $[0, T]$

Observe that we have the intrinsic/hidden state constraints

$$\Big\langle a_i, x(t) - u(t) \Big\rangle \le 0 \quad \text{for all} \quad t \in [0, T],\ i = 1, \ldots, m$$

due to the construction of the normal cone to $C(t) = C + u(t)$

THEOREM Problem (P) admits a feasible (absolutely continuous) solution under natural and mild assumptions

DISCUSSION ON OPTIMAL CONTROL

The formulated optimal control problem for the sweeping process is not an optimization problem over a differential inclusion of the type $\dot{x} \in F(t,x)$. In our case the velocity set $F(t,x) = -N\big(x; C(t)\big) + f(x, b(t))$ is not fixed since the sweeping set $C(t) = C_{u(t)}(t)$ and the perturbation $f(x, b(t))$ are different for each control (u, b). Thus we optimize in the shape of $F(t,x)$ which somehow relates this problem to dynamic shape optimization. In fact there is no sense to formulate any optimization problem for the differential inclusion

$$\dot{x} \in F(t,x) := -N\Big(x; C(t)\Big) + f\Big(x, b(t)\Big), \quad t \in [0,T]$$

when $C(t)$ is fixed since, in major cases, the sweeping inclusion admits a unique solution for every initial point $x(0) = x_0 \in C(0)$

REFORMULATION

Denote $z := (x, u, b) \in I\!R^{3n}$, $z(0) := (x_0, u(0), b(0))$

$F(z) := -N\Big(x; C(u)\Big) + f(x, b)$ with $C(u) := \Big\{x \mid \langle a_i, x \rangle \le 0,\ i = 1, \ldots, m\Big\}$

Problem (P) can be reformulated as: minimize

$$J[z] = \varphi\Big(z(T)\Big) + \int_0^T \ell\Big(t, z(t), \dot{z}(t)\Big)\, dt \ \text{ s.t.}$$

$$\dot{z}(t) \in G\Big(z(t)\Big) := F\Big(z(t)\Big) \times I\!R^n \times I\!R^n \ \text{ a.e. } \ t \in [0, T]$$

$$\Big\langle a_i, x(t) - u(t) \Big\rangle \le 0 \ \text{ for all } \ t \in [0, T],\ i = 1, \ldots, m$$

$$\|u(t)\| = 1 \ \text{ for all } \ t \in [0, T]$$

$G(z)$ is unbounded and highly non-Lipschitzian

DISCRETE APPROXIMATIONS OF SWEEPING TRAJECTORIES

THEOREM Fix an arbitrary feasible solution $\bar{z}(\cdot)$ to (P) and consider discrete partitions

$$\Delta_k := \left\{0 = t_0^k < t_1^k < \ldots < t_k^k = T\right\} \ \textit{with}\ h_k := \max_{0 \le j \le k-1} \{t_{j+1}^k - t_j^k\} \downarrow 0$$

Then there is a sequence of piecewise linear functions $z^k(t) := (x^k(t), u^k(t), b^k(t))$ on $[0, T]$ with $\|u_i^k(t_j^k)\| = 1$ for $i = 1, \ldots, m$ satisfying the discretized inclusions

$$x^k(t) = x^k(t_j) + (t - t_j)v_j^k,\ x(0) = x_0,\ t_j^k \le t \le t_{j+1}^k, \quad j = 0, \ldots, k-1$$

with $v_j^k \in F(z^k(t_j^k))$ on Δ_k and such that

$$z^k(t) \to \bar{z}(t) \ \textit{uniformly on}\ [0, T],\ \int_0^T \|\dot{z}^k(t) - \dot{\bar{z}}(t)\|^2 \, dt \to 0$$

The latter implies the a.e. pointwise on $[0, T]$ convergence of some subsequence of the derivatives $\dot{z}^k(t) \to \dot{\bar{z}}(t)$

DISCRETE CONTROL PROBLEMS

Let $\bar{z}(\cdot) = \big(\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot)\big)$ be a local optimal solution to (P). Consider discrete approximation problems (P_k) : minimize

$$J_k[z^k] := \varphi(x_k^k) + h_k \sum_{j=0}^{k-1} \ell\Big(z_j^k, \frac{z_{j+1}^k - z_j^k}{h_k}\Big)$$

$$+ \sum_{j=0}^{k-1} \int_{t_j^k}^{t_{j+1}^k} \Big\| \frac{z_{j+1}^k - z_j^k}{h_k} - \dot{\bar{z}}(t) \Big\|^2 dt$$

over $z^k := (x_0^k, \ldots, x_k^k, u_0^k, \ldots, u_k^k, b_0^k, \ldots, b_k^k)$ satisfying

$$x_{j+1}^k \in x_j^k + h_k F(x_j^k, u_j^k, b_j^k),\ j = 0, \ldots, k-1,\ \ x_0^k = x_0$$

$$\Big\langle a_i, x_j^k - u_j^k \Big\rangle \le 0,\ \ \Big\| u_j^k \Big\| = 1,\ j = 0, \ldots, k-1,\ \ i = 1, \ldots, m$$

EXISTENCE OF DISCRETE OPTIMAL SOLUTIONS

THEOREM Let φ and ℓ be lower semicontinuous around $\bar{z}(\cdot)$. Then each problem (P_k) admits an optimal solution

PROOF employs the Attouch theorem on subdifferential convergence for convex extended-real valued functions

RELAXATION AND HIDDEN CONVEXITY

Relaxed Sweeping Control Problem (R): minimize

$$\widehat{J}[z] := \varphi\Big(x(T)\Big) + \int_0^T \widehat{\ell}\Big(t, x(t), u(t), b(t), \dot{x}(t), \dot{u}(t), \dot{b}(t)\Big)dt$$

subject to convexified inclusion

$$\dot{x}(t) \in \text{co}\, F\Big(x(t), u(t), b(t)\Big)$$

under the same constraints, where $\widehat{\ell}$ stands for the convexification of ℓ with respect to velocity variables

Relaxation Stability: Optimal solution $\bar{z}(\cdot)$ to (R) exists, min(R)=inf(P), and $\bar{z}(\cdot)$ can be strongly approximated by feasible solutions to (P)

THEOREM The sweeping control problem (P) enjoys relaxation stability under the standing assumptions

STRONG CONVERGENCE OF DISCRETE APPROXIMATIONS

THEOREM Let $\bar{z}(\cdot) = (\bar{x}(\cdot), \bar{u}(\cdot), \bar{b}(\cdot))$ be a given optimal solution to (P). Then any sequence of piecewise linearly extended to $[0, T]$ optimal solutions $\bar{z}^k(t)$ of the discrete problems (P_k) strongly converges to $\bar{z}(t)$ in the Sobolev space $W^{1,2}[0, T]$

PROOF Using the above result on the strong approximation of trajectories for the sweeping inclusion and relaxation stability

GENERALIZED DIFFERENTIATION

Normal Cone to a closed set $\Omega \subset I\!R^n$ at $\bar{x} \in \Omega$

$$N(\bar{x};\Omega) := \Big\{ v \Big| \; \exists x_k \to \bar{x}, w_k \in \Pi(x_k;\Omega), \alpha_k \geq 0, \alpha_k(x_k - w_k) \to v \Big\}$$

Subdifferential of an l.s.c. function $\varphi\colon I\!R^n \to (-\infty,\infty]$ at $\bar{x}$

$$\partial\varphi(\bar{x}) := \Big\{ v \Big| \; (v,-1) \in N((\bar{x},\varphi(\bar{x})); \text{epi}\,\varphi) \Big\}, \quad \bar{x} \in \text{dom}\,\varphi$$

Coderivative of a set-valued mapping G

$$D^*G(\bar{x},\bar{y})(u) := \Big\{ v \Big| \; (v,-u) \in N((\bar{x},\bar{y}) : \text{gph}\,G) \Big\}, \quad \bar{y} \in G(\bar{x})$$

Generalized Hessian of φ at $\bar{x}$

$$\partial^2\varphi(\bar{x}) := D^*(\partial\varphi)(\bar{x},\bar{v}), \quad \bar{v} \in \partial\varphi(\bar{x})$$

FURTHER STRATEGY

• For each k reduce problem (P_k) to a problem of mathematical programming (MP) with functional and increasingly many geometric constraints. The latter are given by graphs of the mapping $F(z) := -N(x; C(u)) + f(x, b)$, and so (MP) is intrinsically nonsmooth and nonconvex even for smooth initial data

• Use variational analysis analysis and generalized differentiation (first- and second-order) to derive necessary optimality conditions for (MP) and then discrete control problems (P_k)

• Explicitly calculate the coderivative of $F(z)$ entirely in terms of the initial data of (P)

• By passing to the limit as $k \to \infty$, to derive necessary optimality conditions for the sweeping control problem (P)

NECESSARY OPTIMALITY CONDITIONS FOR (P)

For simplicity consider the case of smooth φ, ℓ

THEOREM Let $\bar{z}(\cdot)$ be an optimal solution to (P) such that the vectors $\{a_i\}$ for active constraint $i \in I(\bar{x}(t) - \bar{u}(t))$ indices are linearly independent. Then there exist a multiplier $\lambda \geq 0$, an adjoint arc $p(t) = (p_x, p_u, p_b)(t)$ absolutely continuous on $[0, T]$, and regular Borel measures measures $\gamma \in C^*_+([0, T]; I\!R^m)$ and $\xi \in C^*([0, T]; I\!R)$ satisfying the following conditions:

the **primal-dual relationships** for a.e. $t \in [0, T]$

$$\Big\langle a_i, \bar{x}(t) - \bar{u}(t) \Big\rangle < 0 \Longrightarrow \eta_i(t) = 0$$

$$\eta_i(t) > 0 \Longrightarrow \Big\langle a_i, \lambda \nabla_{\dot{x}} \ell(\bar{z}(t), \dot{z}(t)) - q_x(t) \Big\rangle = 0, \quad i = 1, \ldots, m$$

where the function $\eta = (\eta_1, \ldots, \eta_m) \in L^\infty([0,T]; \mathbb{R}^m_+)$ is uniquely defined by

$$\dot{\bar{x}}(t) = -\sum_{i=1}^{m} \eta_i(t) a_i + f\Big(\bar{x}(t), \bar{b}(t)\Big)$$

and where $q(t) = (q_x, q_u, q_b)$ is of bounded variation given by

$$q(t) = p(t) - \int_t^T \Big(d\gamma(s), 2\bar{u}(s)d\xi(s) - d\gamma(s), 0\Big), \quad t \in [0,T)$$

$$q_u(t) = \lambda \nabla_{\dot{u}} \ell\Big(t, \bar{z}(t), \dot{\bar{z}}(t)\Big), \quad q_b(t) = \lambda \nabla_{\dot{b}} \ell\Big(t, \bar{z}(t), \dot{\bar{z}}(t)\Big), \quad t \in [0,T]$$

along the **adjoint inclusion**

$$\dot{p}(t) \in \text{co}\Big\{\lambda \nabla_z \ell(\bar{z}(t), \dot{\bar{z}}(t) + D^*F\Big(\bar{x}(t), \bar{u}(t), \bar{b}(t), -\dot{\bar{x}}(t)\Big)\Big(\lambda \nabla_{\dot{x}}(t) - q_x(t)\Big)\Big\}$$

where the coderivative D^*F is calculate via the problem data

Furthermore, we have the **transversality conditions**

$$\Big(-p_x(T), p_u(T)\Big) \in \Big(\lambda \nabla \varphi\Big(\bar{x}(T), 0\Big) + \Big(0, \lambda \nabla_{\dot{u}} \ell(T, \bar{z}(T), \dot{\bar{z}}(T))\Big) + N\Big(\bar{x}(T) - \bar{u}(T); C\Big)$$

$$p_b(T) = \nabla_b \ell\Big(T, \bar{z}(T), \dot{\bar{z}}(T)\Big) = 0$$

and the **nontriviality condition**

$$\lambda + \|q_u(0)\| + \|p(T)\| \neq 0$$

CROWD MOTION MODEL

The model is designed to deal with local interactions between individuals in order to describe the dynamics of the pedestrian traffic. This microscopic model for crowd motion rests on the two principles: A spontaneous velocity corresponding to the velocity each individual would like to have in the absence of others; the actual velocity is then computed as the projection of the spontaneous velocity onto the set of admissible velocities which do not violate a certain non-overlapping constraint. We consider $N (N \geq 2)$ individuals identified to rigid disks with the same radius r in a corridor (see Fig. 1)

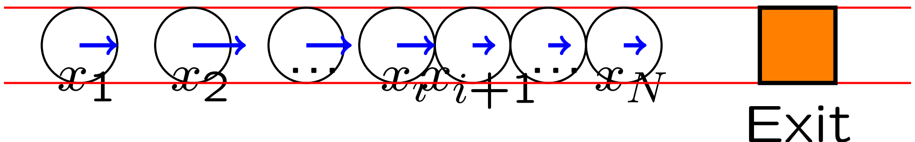

Fig 1 Crowd model motion in a corridor

All the individuals have the same behavior: they want to reach the exist by following the shortest part and minimal control energy. This problem can be modeled as a sweeping process

$$\begin{cases} -\dot{x}(t) \in N(x(t); C(t)) + f(x(t), b(t)), \ x(0) = x_0 \in C(0) \\ C(t) = C + u(t), \ u_{i+1}(t) - u_i(t) = 2r, \ u_1(t) = 0, \|u(t)\| = 1 \\ f(x(t), b(t)) = (s_1 b_1, \ldots, s_N b_N) \\ x_{i+1}(T) - x_i(T) > 2r, \quad i = 1, \ldots, N-1 \end{cases}$$

with controls in perturbations and the cost function

$$\text{minimize } \ J[x, b] = \frac{\|x(T)\|^2}{2} + \int_0^T \frac{\|b(t)\|^2}{2} dt$$

The obtained necessary optimality conditions allow us to determine the optimal strategy

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